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NON-LOCAL PROBLEMS FOR PARABOLIC-HYPERBOLIC EQUATIONS WITH DEVIATION FROM THE CHARACTERISTICS AND THREE TYPE-CHANGING LINES

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Abstract. We prove the existence and uniqueness of solutions for a partial differential equations of mixed type (parabolic-hyperbolic type). We use energy integrals and methods from integral equations to study a problem that has deviation from the characteristics and three lines where the type changes.

1. Introduction and formulation of the problem

The need for studying boundary-value problems of parabolic-hyperbolic type was emphasized by Gelfand [7] in 1959. Later Zolina [12] considered several of these problems and gave some physical interpretations. Among the applications of these problems, we have irrigation models found in the monograph by Serbina [11]. Omitting many works for local and nonlocal problems, we mention some recent works that are closely related to the present investigation. Berdyshev [1] studied the unique solvability of Bitsadze-Samarskii type problem with deviation from the characteristics for parabolic-hyperbolic equations with one line where type changes In [2, 5, 8, 9], the unique solvability of nonlocal problems for parabolic-hyperbolic type equations with continuous and special gluing conditions were studied. Eleev and Lesev [4] studied Parabolic-hyperbolic type equations with lines where the type changes. Boundary value problems with nonlocal conditions for parabolic-hyperbolic equations with three lines where the type changes were studied in [3].

We use energy integrals and methods from integral equations, to prove the unique solvability of a boundary-value problems with nonlocal conditions. This conditions relate values of the unknown function on the line where the type changes, with values of its derivatives on curves lying inside of hyperbolic part of the domain.

Consider the equation

$$u_{xx} + \frac{\operatorname{sgn}(xy(1-x)) - 1}{2} u_{yy} + \frac{\operatorname{sgn}(xy(x-1)) - 1}{2} u_y = 0 \tag{1.1}$$

on a domain $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup AB \cup AA_0 \cup BB_0$. Here $\Omega_0 = \{(x,y) : 0 < x < 1, 0 < y < 1\}$ and Ω_1, Ω_2, Ω_3 are characteristic triangles with endpoints $A(0,0), B(1,0), C(\frac{1}{2}, -\frac{1}{2})$; $D(-\frac{1}{2}, \frac{1}{2}), A_0(0,1)$; $E(\frac{3}{2}, \frac{1}{2}), B_0(1,1)$, respectively.

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Problem NP. Find a regular solution of the (1.1), satisfying conditions

$$[u_x - u_y]\theta_1(t) + \mu_1(t)[u_x - u_y]\theta_1^*(t) = \varphi_1(t), \tag{1.2}$$
$$[u_x - u_y]\theta_2(t) + \mu_2(t)[u_x - u_y]\theta_2^*(t) = \varphi_2(t), \tag{1.3}$$
$$[u_x + u_y]\theta_3(t) + \mu_3(t)[u_x + u_y]\theta_3^*(t) = \varphi_3(t), \tag{1.4}$$

and $u(A) = 0$, $u(B) = 0$.

Here $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$, $[\theta_1^*(t), \theta_2^*(t), \theta_3^*(t)]$ are affixes of intersection's points of characteristics, outgoing from the points $(x,0) \in AB$; $(0,y) \in AA_0$; $(1,y) \in BB_0$ with $AC; AD; BE$ $[AN; AK; BM]$, respectively. $AN : y = -\gamma_1(x)$, $0 \le x \le l_1$, $1/2 \le l_1 \le 1$, $AK : x = -\gamma_2(y)$, $0 \le y \le l_2$, $1/2 \le l_2 \le 1$, $BM : x = -\gamma_3(y)$, $0 \le y \le l_3$, $1/2 \le l_3 \le 1$; $\mu_i(t)$ and $\varphi_i(t)(i = \overline{1,3})$ are given functions.

Regarding to curves $\gamma_i(t)$ we assume the following conditions:

- $\gamma_i(0) = 0$, $l_i + \gamma_i(l_i) = 1$, $0 < \gamma_i'(0) < 1$, $\gamma_i(t) > 0$, $t > 0$;
- $t - \gamma_i(t)$, $t + \gamma_i(t)$ are monotonically increasing;
- $\gamma_i(t)$ are twice continuously differentiable functions.

2. Unique solvability of the problem

Theorem 2.1. *If $\mu_i(t) \neq -1$ and $\mu_i(t), \varphi_i(t) \in C^1[0,1]$, for $i = \overline{1,3}$, $0 \le t \le 1$, then problem NP has unique solution.*

We introduce the following notation

$$u(x, \pm 0) = \tau_1^{\pm}(x), \quad u_y(x, \pm 0) = \nu_1^{\pm}(x), \tag{2.1}$$
$$u(\pm 0, y) = \tau_2^{\pm}(y), \quad u_x(\pm 0, y) = \nu_2^{\pm}(y), \tag{2.2}$$
$$u(1 \pm 0, y) = \tau_3^{\pm}(y), \quad u_x(1 \pm 0, y) = \nu_3^{\pm}(y)\,. \tag{2.3}$$

It is known [10] that the solution of Cauchy's problem of (1.1) in the domain Ω_1 has the form

$$u(x,y) = \frac{1}{2}\{\tau_1^-(x+y) + \tau_1^-(x-y) + \int_{x-y}^{x+y} \nu_1^-(t)dt\}. \tag{2.4}$$

Calculating derivatives, we have

$$u_x = \frac{1}{2}\{\tau_1^{-\prime}(x+y) + \tau_1^{-\prime}(x-y) + \nu_1^-(x+y) - \nu_1^-(x-y)\},$$
$$u_y = \frac{1}{2}\{\tau_1^{-\prime}(x+y) - \tau_1^{-\prime}(x-y) + \nu_1^-(x+y) + \nu_1^-(x-y)\},$$
$$u_x - u_y = \tau_1^{-\prime}(x-y) - \nu_1^-(x-y).$$

By the conditions on the function $\gamma_1(x)$, an equation of the curve AN in characteristic coordinates $\xi = x+y$, $\eta = x-y$ can be given as $\xi = \lambda_1(\eta)$, $0 \le \eta \le 1$, moreover $0 < \lambda_1{}'(0) < 1$, $\lambda_1(\eta) < \eta$. Then

$$\theta_1(t) = \big(\frac{t}{2}; -\frac{t}{2}\big), \quad \theta_1^*(t) = \big(\frac{\lambda_1(t)+t}{2}; \frac{\lambda_1(t)-t}{2}\big).$$

Then we calculate

$$[u_x - u_y]\theta_1(t) = \tau_1^{-\prime}(t) - \nu_1^-(t), \quad [u_x - u_y]\theta_1^*(t) = \tau_1^{-\prime}(t) - \nu_1^-(t).$$

Using condition (1.2), we find

$$\nu_1^-(t) = \tau_1^{-\prime}(t) - \frac{\varphi_1(t)}{1+\mu_1(t)}, \quad \mu_1(t) \neq -1. \tag{2.5}$$

Similarly, using conditions (1.3) and (1.4), we obtain functional relations on lines AA_0 and BB_0, reduced from the domains Ω_2, Ω_3, respectively:

$$\nu_2^-(t) = \tau_2^{-\prime}(t) - \frac{\varphi_2(t)}{1+\mu_2(t)}, \quad \mu_2(t) \neq -1, \tag{2.6}$$

$$\nu_3^+(t) = -\tau_3^{+\prime}(t) + \frac{\varphi_3(t)}{1+\mu_3(t)}, \quad \mu_3(t) \neq -1. \tag{2.7}$$

On the domain Ω_0 we obtain the equality

$$\begin{aligned} &\iint_{\Omega_0} u_x^2(x,y)\,dx\,dy + \int_0^1 \tau_2^+(y)\nu_2^+(y)dy - \int_0^1 \tau_3^-(y)\nu_3^-(y)dy \\ &+ \frac{1}{2}\int_0^1 u^2(x,1)dx - \frac{1}{2}\int_0^1 [\tau_1^+(x)]^2 dx = 0. \end{aligned} \tag{2.8}$$

To obtain this equality, first we multiplied (1.1) by $u(x,y)$ and then integrated along the domain Ω_0. Then apply the Green's formula [10] and use the introduced notation to obtain (2.8).

2.1. **Uniqueness of the solution.** To prove the uniqueness, as usual we suppose that the problem has two solutions u_1 and u_2. Taking difference of these solution we obtain a homogeneous problem regarding for the new function $u = u_1 - u_2$. Below we prove that homogeneous problem NP has only the trivial solution. Consequently, given functions $\varphi_i(t)$ are equal to zero.

Let us to prove that $u(x, \pm 0) = \tau_1^+(x) = \tau_1^-(x) = 0$. Passing to the limit in the domain Ω_0, at $y \to +0$ from the equation $u_{xx} - u_y = 0$, we obtain

$$\tau_1^{+\prime\prime}(x) - \nu_1^+(x) = 0. \tag{2.9}$$

Consider the integral $I_1 = \int_0^1 \tau_1^+(x)\nu_1^+(x)dx$. Taking (2.9) into account, we have

$$I_1 = \int_0^1 \tau_1^+(x)\tau_1^{+\prime\prime}(x)dx = -\int_0^1 (\tau_1^{+\prime}(x))^2 dx.$$

It is obvious that $I_1 \leq 0$.

From relation (2.5) we obtain $\nu_1^-(x) = \tau_1^{-\prime}(x)$. Considering $\tau_1^-(x) = \tau_1^+(x)$, $\nu_1^-(x) = \nu_1^+(x)$, $\varphi_1(t) = 0$, we obtain

$$I_1 = \int_0^1 \tau_1^+(x)\tau_1^{+\prime}(x)dx = \frac{1}{2}(\tau_1^+(x))^2\Big|_0^1 = 0. \tag{2.10}$$

Further, consider the integrals

$$I_2 = \int_0^1 \tau_2^+(y)\nu_2^+(y)dy \geq 0, \quad I_3 = \int_0^1 \tau_3^-(y)\nu_3^-(y)dy \leq 0.$$

Using the functional relation $\nu_2^+(y) = \tau_2^{+\prime}(y)$, we have

$$I_2 = \int_0^1 \tau_2^+(y)\tau_2^{+\prime}(y)dy = \frac{1}{2}(\tau_2^+(1))^2 \geq 0, \tag{2.11}$$

Similarly, we obtain

$$I_3 = -\int_0^1 \tau_3^-(y)\tau_3^{-\prime}(y)dy = -\frac{1}{2}(\tau_3^-(1))^2 \leq 0. \tag{2.12}$$

Taking (2.11), (2.12) and $\tau_1(x) = 0$ into account, from (2.8), we obtain $u_x(x, y) = 0$. Since $u(x, y) \in C(\overline{\Omega}), u(x, y) \equiv 0$ in the domain Ω. The uniqueness of the solution for the problem NP is proved.

2.2. **Existence of the solution.** Excluding $\nu_1^+(x) = \nu_1^-(x)$ from (2.5) and (2.9), we have

$$\tau_1^{+\prime\prime}(x) - \tau_1^{+\prime}(x) = -\frac{\varphi_1(x)}{1+\mu_1(x)}.$$

from here and considering $\tau_1^+(0) = \tau_1^+(1) = 0$, we obtain

$$\tau_1^+(x) = \int_0^x \frac{\varphi_1(t)[1-e^{x-t}]}{1+\mu_1(t)}dt + \frac{e^x-1}{e-1}\int_0^1 \frac{\varphi_1(t)[e^{1-t}-1]}{1+\mu_1(t)}dt.$$

By the unique solvability of the first boundary problem for the heat equation [6], the solution of (1.1) in the domain Ω_0 is represented as

$$\begin{aligned} u(x,y) = \int_0^1 \tau_1^+(x)G(x,y,x_1,0)dx_1 + \int_0^y \tau_2^+(y_1)G_{x_1}(x,y,0,y_1)dy_1 \\ - \int_0^y \tau_3^-(y_1)G_{x_1}(x,y,1,y_1)dy_1, \end{aligned} \tag{2.13}$$

where $G(x, y, x_1, y_1)$ is Green's function of the first boundary problem for the heat equation [6].

Differentiating (2.13) once by x, considering (2.1)-(2.2), we obtain

$$\nu_2^+(y) = F_1(y) - \int_0^y \tau_2^{+\prime}(y_1)K_1(y,y_1)dy_1 + \int_0^y \tau_3^{-\prime}(y_1)K_2(y,y_1)dy_1, \tag{2.14}$$

$$\nu_3^-(y) = F_2(y) - \int_0^y \tau_2^{+\prime}(y_1)K_3(y,y_1)dy_1 + \int_0^y \tau_3^{-\prime}(y_1)K_4(y,y_1)dy_1, \tag{2.15}$$

where

$$F_1(y) = -\int_0^1 \tau_1^{+\prime}(y_1)K_2(y,y_1)dy_1, \quad F_2(y) = -\int_0^1 \tau_1^{+\prime}(y_1)K_4(y,y_1)dy_1,$$

$$K_1(y,y_1) = \frac{1}{\sqrt{\pi(y-y_1)}}\Big[1 + \sum_{n=-\infty,\, n\neq 0}^{+\infty} e^{-\frac{n^2}{y-y_1}}\Big],$$

$$K_2(y,y_1) = \frac{1}{2\sqrt{\pi(y-y_1)}}\Big[2e^{-\frac{1}{4(y-y_1)}} + \sum_{n=-\infty,\, n\neq 0}^{+\infty} (e^{-\frac{(2n-1)^2}{4(y-y_1)}} + e^{-\frac{(2n+1)^2}{4(y-y_1)}})\Big],$$

$$K_3(y,y_1) = \frac{1}{\sqrt{\pi(y-y_1)}}\Big[e^{-\frac{1}{4(y-y_1)}} + \sum_{n=-\infty,\, n\neq 0}^{+\infty} e^{-\frac{(2n+1)^2}{4(y-y_1)}}\Big],$$

$$K_4(y,y_1) = \frac{1}{2\sqrt{\pi(y-y_1)}}\Big[1 + e^{-\frac{1}{y-y_1}} + \sum_{n=-\infty,\, n\neq 0}^{+\infty} (e^{-\frac{n^2}{y-y_1}} + e^{-\frac{(n+1)^2}{y-y_1}})\Big].$$

From (2.6) and (2.14), (2.7) and (2.15), excluding $\nu_2^+(y), \nu_3^-(y)$, we have

$$\tau_2^{+\prime}(y) + \int_0^y \tau_2^{+\prime}(y_1)K_1(y,y_1)dy_1 = F_3(y), \tag{2.16}$$

$$\tau_3^{-\prime}(y) + \int_0^y \tau_3^{-\prime}(y_1)K_4(y,y_1)dy_1 = F_4(y), \tag{2.17}$$

where

$$F_3(y) = F_1(y) + \frac{\varphi_2(y)}{1+\mu_2(y)} + \int_0^y \tau_3^{-\prime}(y_1) K_2(y, y_1) dy_1,$$
$$F_4(y) = -F_2(y) + \frac{\varphi_3(y)}{1+\mu_3(y)} + \int_0^y \tau_2^{+\prime}(y_1) K_3(y, y_1) dy_1.$$

Equations (2.16)-(2.17) can be considered as a system of equations regarding unknown functions $\tau_2^{+\prime}(y)$ and $\tau_3^{-\prime}(y)$. First we solve equation (2.16) considering function $\tau_3^{-\prime}(y)$ as known. Equation (2.16) is a Volterra type integral equation regarding to the function $\tau_2^{+\prime}(y)$ with continuous right-hand side $F_3(y)$. Since the kernel $K_1(y, y_1)$ has a weak singularity, one can represent solution of this equation via the resolvent,

$$\tau_2^{+\prime}(y) = F_3(y) + \int_0^y R_1(y, y_1) F_3(y_1) dy_1, \tag{2.18}$$

where $R_1(y, y_1)$ is the resolvent of the kernel $K_1(y, y_1)$.

Integrating once, from (2.18), we obtain

$$\tau_2^{+}(y) = \int_0^y F_3(t) dt + \int_0^y \Big(\int_0^t R_1(t, y_1) F_3(y_1) dy_1 \Big) dt.$$

Substituting $\tau_2^{+}(y)$ into (2.17), we obtain the the second kind type Volterra integral equation regarding $\tau_3^{-\prime}(y)$, which has unique solution [6].

Since, functions $\tau_1^{\pm}(x)$, $\tau_2^{\pm}(y)$, $\tau_3^{\pm}(y)$ are known, using (2.5), (2.6), (2.7) we find the functions $\nu_1^{\pm}(x)$, $\nu_2^{\pm}(y)$, $\nu_3^{\pm}(y)$.

Finally, one can obtain solution to problem NP in the domain Ω_0 by the formula (2.13), and in domains Ω_i, $(i = \overline{1,3})$ as a solution of the Cauchy's problem, for instance (2.4).

References

[1] , A. S. Berdyshev: "On Volterness of some problems with Bitsadze-Samarskii type conditions for mixed parabolic-hyperbolic equations". *Siberian mathematical journal*, Vol. 46, No. 3 (2005), pp. 500-510.

[2] A. S. Berdyshev and E. T. Karimov: "Some non-local problems for the parabolic-hyperbolic type equation with non-characteristic line of changing type", CEJM, 4(2), pp. 183-193. 2006.

[3] A. S. Berdyshev and N. A. Rakhmatullaeva: "Nonlocal Problems with Special Gluing for a Parabolic-Hyperbolic Equation", "Further Progress in Analysis" Proceedings of the 6th International ISAAC Congress. pp.727-734. Ankara, Turkey, 13-18 August 2007.

[4] V. A. Eleev and V. N. Lesev: "On two boundary-value problems for mixed equations with perpendicular lines if type changing", Vladicaucasian mathematical journal, 3(4), pp. 10-22, 2001.

[5] B. E. Eshmatov and E. T. Karimov: "Boundary-value problems with continuous and special gluing conditions for parabolic-hyperbolic type equations", CEJM, 5(4), pp. 741-750. 2007.

[6] A. Friedman: *Partial differential equations of parabolic type*, Prentice-Hall, 1964.

[7] I. M. Gel'fand: "Some questions of analysis and differential equations", UMN, ser.3 (87), Vol. XIV, (1959), pp. 3–19.

[8] E. T. Karimov: "Non-local problems with special gluing condition for the parabolic-hyperbolic type equation", PMJ, 17(2), pp. 11-20. 2007.

[9] E. T. Karimov: "Some non-local problems for the parabolic-hyperbolic type equation with complex spectral parameter", Mathematische Nachrichten, 281(7), pp. 959-970.

[10] J. M. Rassias: *Lecture Notes on Mixed Type Partial Differential Equations*,World Sci., Singapore. 1990.

[11] L. I. Serbina: Nonlocal mathematical models of movement in watertransit systems, Moscow: Nauka, 2007.

[12] L. A. Zolina: "On a boundary-value problem for hyperbolic-parabolic equation", *Computational mathematics and mathematical physics of USSR*. Vol. 6, No. 6, (1966), pp. 63-78.

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