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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH BOUNDARY CONDITIONS

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Abstract. This article concerns the existence of solutions for fractional-order differential inclusions with boundary-value conditions. The main tools are based on fixed point theorems due to Bohnerblust-Karlin and Leray-Schauder together with a continuous selection theorem for upper semi-continuous multi-valued maps.

1. Introduction

This article concerns the existence of solutions to the fractional-order differential inclusions with boundary-value conditions

$$ {}^cD_{0+}^{\alpha} y(t) \in F(t, y(t)), \quad t \in [0,1],\ \alpha \in (1,2), \tag{1.1}$$

$$ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i), \tag{1.2}$$

where ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map defined on $[0,1]$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $k_i > 0$, $\xi_i \in [0,1]$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Fractional differential equations play a important role in understanding many phenomena in science and engineering. Such as electrochemistry, control, viscoelasticity, porousmedia, electromagnetic and so on. For details two wonderful books [12, 14] on the subject of fractional differential equations, summarizing much of fractional calculus and its applications. In recent years, much attention has been paid to the existence of solutions fractional differential equations with boundary value conditions. For instance, Bai and Lü[1], Bai [2], Stojanović[15], Yu and Gao [16], Zhang [17]. Following this trend, fractional differential inclusion has got focus. In 2007 , Ouahab [11] investigated the existence of solutions for α-fractional differential inclusions by means of selection theorem together with a fixed point theorem. Very recently, Chang and Nieto [4] established some new existence results for fractional differential inclusions due to fixed point theorem of multi-valued maps. About other results on fractional differential inclusions, we refer the reader to [9]. To the best of our knowledge, for fractional differential inclusions , very few results

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are obtained. In order to fill this gap, motivated by the above mentioned works, existence of solutions criterion for fractional differential inclusions are given for (1.1) and (1.2). This paper is organized as follows. In next section, we present some basic definitions and notations about fractional calculus and multi-valued maps. Section 3 is devoted to the existence results for fractional differential inclusions. In the last section, an example is given to illustrate our main result.

2. Preliminaries

In this section, we recall some notation, definitions and preliminaries about fractional calculus (see [8, 12, 14]) and multi-valued maps (see[5, 6, 13]) that will be used in the remainder.

Definition 2.1. The αth fractional order integral of the function $u:(0,\infty)\mapsto R$ is defined by

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

where $\alpha>0$, Γ is the gamma function, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The αth fractional order derivative of a continuous function $u:(0,\infty)\mapsto R$ is defined by

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $\alpha>0$, $n=[\alpha]+1$, provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. Caputo fractional derivative of order $\alpha>0$ for a function u defined on $[0,\infty)$ is given by

$${}^cD_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

where $n=[\alpha]+1$, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.4 (citedu)**.** *Let ε, η are two positive constants, then*

(i) $I_{0^+}^{\varepsilon}:L^1(J,R)\to L^1(J,R)$.
(ii) $I_{0^+}^{\varepsilon}I_{0^+}^{\eta}f(t)=I_{0^+}^{\varepsilon+\eta}f(t)$, $f(t)\in L^1(J,R)$.
(iii) $\lim_{\varepsilon\to n}I_{0^+}^{\varepsilon}f(t)=I_{0^+}^{n}f(t)$, $n=1,2,\dots$, $I_{0^+}^{1}f(t)=\int_0^t f(s)ds$.

Let $C([0,1],\mathbb{R})$ be the Banach space consisting of continuous functions y from $[0,1]$ to $\mathbb{R}$ with the norm

$$\|y\|_\infty:=\sup\{|y|:t\in[0,1]\}.$$

and $L^1([0,1],\mathbb{R})$ represent the functions $y:[0,1]\to X$ which are Lebesgue integrable and

$$\|y\|_{L^1}=\int_0^1|y(t)|dt.$$

Let $(X,|\cdot|)$ be a Banach space. Then a multi-valued map $\Theta:X\to\mathcal{P}(X)$ is convex (closed) value if $\Theta(x)$ is convex (closed) for all $x\in X$. Θ is bounded on bounded sets if $\Theta(B)=\bigcup_{x\in B}\Theta(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y|:y\in\Theta(x)\}\}<\infty$).

We call Θ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $\Theta(x_0)$ is a nonempty closed subset of X, and if for each open set B of X containing $\Theta(x_0)$, there exists an open neighborhood V of x_0 such that $\Theta(V) \subseteq B$. Θ is said to be completely continuous if Θ is u.s.c. if and only if Θ has a closed graph, i.e.,

$$x_n \to x_*, \quad y_n \to y_*, \quad y_n \in \Theta x_n \quad \text{imply } y_* \in \Theta x_*.$$

Let $CC(X)$ be the set of all nonempty compact-convex subsets of X. For each $y \in C([0,1],\mathbb{R})$, let $S_{F,y}$ be the set of selections of F defined by

$$S_{F,y} = \{f \in L^1([0,1],\mathbb{R}) : f \in F(t,y(t)) \quad \text{a.e. } t \in [0,1]\}.$$

Definition 2.5. A function $y \in C([0,1],\mathbb{R})$ is said to be a solution of (1.1) and (1.2) if y satisfies the fractional differential inclusion (1.1) on $[0,1]$ and the boundary value condition (1.2).

To set the frame for our main results, we introduce the following lemmas.

Lemma 2.6 (Bohnerblust-Karlin, [3])**.** *Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : D \to \mathcal{P}(X) \setminus \{0\}$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $\overline{G(D)}$ compact. Then G has a fixed point.*

Lemma 2.7 (Leray-Schauder Nonlinear Alternative, [7])**.** *Let Let X be a Banach space, with $C \subset X$ convex. Assume V is a relatively open subset of C with $0 \in V$ and $G : \overline{V} \to \mathcal{P}(C)$ is a compact multivalued map, u.s.c. with convex closed values. Then either*

(i) *G has a fixed point in $\overline{V}$; or*
(ii) *there exists a point $v \in \partial V$ such that $v \in \lambda G(v)$ for some $\lambda \in (0,1)$.*

Lemma 2.8 ([10])**.** *Let X be a Banach space. Let $F : [a,b] \times X \to CC(X); (t,y) \mapsto F(t,y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for a.e. $t \in [a,b]$ and $S_{F,y} \neq \emptyset$ for any $y \in C([a,b],X)$ and let Λ be a linear continuous mapping from $L^1([a,b],X)$ to $C([a,b],X)$, then the operator $\Lambda \circ S_F : C([a,b],X) \to CC(C([a,b],X))$ $y \mapsto (\Lambda \circ S_F)(y) := \Lambda(S_{F,y})$ is a closed graph operator in $C([a,b],X) \times C([a,b],X)$.*

Now we are in the position to state and prove our main results.

3. Main results

Let us list the following assumptions:

(A1) $\sum_{i=1}^{m-2} k_i\xi_i \neq 1$.
(A2) $F : [0,1] \times \mathbb{R} \to CC(\mathbb{R})$, $t \mapsto F(t,y)$ is measurable for each $y \in \mathbb{R}$, $y \mapsto F(t,y)$ is u.s.c. for a.e. $t \in [0,1]$.
(A3) For each $r > 0$, there exists a function $\varphi_r \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,y)\| = \sup\{|f| : f \in F(t,y)\} \leq \varphi_r(t),$$

for $(t,y) \in [0,1] \times \mathbb{R}$ with $|y| \leq r$, and

$$\liminf_{r\to\infty} \frac{1}{r}\int_0^1 \varphi_r(t)dt = \mu.$$

(A4) There exist a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$, a function $q \in L^1([0,1],\mathbb{R}_+)$ and a positive constant M such that
$$\|F(t,y)\| \le q(t)\phi(|y|)$$
for each $(t,y) \in [0,1]\times\mathbb{R}$, and
$$\frac{M}{\Big(1+\frac{1}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}+\frac{1\sum_{i=1}^{m-2}k_i\xi_i}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}\Big)\phi(M)\int_0^1 q(s)ds} > 1.$$

We notice that, for each $y \in C([0,1],\mathbb{R})$, by [13], the set $S_{F,y}$ is nonempty. The following lemmas are basic results for the fractional differential equations.

Lemma 3.1 ([8]). *Let $\alpha > 0$. then the fractional differential equation*
$$^cD^{\alpha}_{0+}y(t) = 0$$
has a solution
$$y(t) = c_0 + c_1t + c_2t^2 + \dots + c_nt^{n-1},$$
and $c_i \in \mathbb{R}$, $i = 1,2,\dots,n$, $n = [\alpha]+1$.

Lemma 3.2 ([8]). *Let $\alpha > 0$. Then*
$$I^{\alpha c}_{0+}D^{\alpha}_{0+}y(t) = y(t) + c_0 + c_1t + c_2t^2 + \dots + c_nt^{n-1},$$
for some $c_i \in \mathbb{R}$, $i = 1,2,\dots,n$, and $n = [\alpha]+1$.

By Lemma 3.1 and Lemma 3.2, it is easy to obtain the following lemma.

Lemma 3.3. *Suppose* (A1) *holds, and $g \in C([0,1],\mathbb{R})$. Then $y(t)$ is a solution of the problem*
$$^cD^{\alpha}_{0+}y(t) = g(t), \quad t \in [0,1], \quad 1<\alpha<2. \tag{3.1}$$
$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2}k_iy(\xi_i), \tag{3.2}$$
if and only if
$$\begin{aligned} y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\int_0^1 (1-s)^{\alpha-1}g(s)ds \\ + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}g(s)ds. \end{aligned} \tag{3.3}$$

Proof. If $y(t)$ is a solution of (3.1)-(3.2), then
$$^cD^{\alpha}_{0+}y(t) = g(t), \tag{3.4}$$
Lemma 3.2 implies
$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds + c_1 + c_2t. \tag{3.5}$$
By the boundary condition $y(0)=0$, we have
$$c_1 = 0. \tag{3.6}$$

Furthermore, by $y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i)$ and (3.5), we obtain

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)ds + c_2 = \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} g(s)ds + \sum_{i=1}^{m-2} k_i \xi_i c_2. \tag{3.7}$$

After a rearrangement of (3.7), we obtain

$$(1 - \sum_{i=1}^{m-2} k_i \xi_i) c_2 = \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} g(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)ds. \tag{3.8}$$

That is,

$$\begin{aligned} c_2 = &\frac{1}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} g(s)ds \\ &- \frac{1}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \int_0^1 (1-s)^{\alpha-1} g(s)ds. \end{aligned} \tag{3.9}$$

Substituting (3.6) and (3.9) into (3.5), we have that (3.1)-(3.2) has a unique solution

$$\begin{aligned} y(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds - \frac{t}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \int_0^1 (1-s)^{\alpha-1} g(s)ds \\ &+ \frac{t}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} g(s)ds. \end{aligned} \tag{3.10}$$

If $y(t)$ is defined as in (3.3), it is easy to check that $y(t)$ satisfies (3.1)-(3.2), which completes the proof. □

Next, we shall present and prove our main results on the existence of solutions to fractional differential inclusion (1.1)-(1.2).

Theorem 3.4. *Assume* (A1)–(A3) *hold. Furthermore, if*

$$\frac{1}{\Gamma(\alpha)} \Big(1 + \frac{1}{|1 - \sum_{i=1}^{m-2} k_i \xi_i|} + \frac{\sum_{i=1}^{m-2} k_i \xi_i}{|1 - \sum_{i=1}^{m-2} k_i \xi_i|} \Big) \mu < 1, \tag{3.11}$$

Then problem (1.1)-(1.2) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $N : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ defined by

$$\begin{aligned} N(y) = \{ h \in C([0,1], \mathbb{R}) : h(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y(s))ds \\ &- \frac{t}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \int_0^1 (1-s)^{\alpha-1} f(y(s))ds \\ &+ \frac{t}{\Gamma(\alpha)(1 - \sum_{i=1}^{m-2} k_i \xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(y(s))ds, \quad f \in S_{F,y} \}. \end{aligned} \tag{3.12}$$

It is obvious that the fixed points of N are solutions to the problem (1.1)-(1.2). Then, we shall prove N satisfies all the assumptions of Lemma 2.6, which is broken into several steps.

Step 1. $N(y)$ is convex for each $y \in C([0,1],\mathbb{R})$. In fact, if $h_1, h_2 \in N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in [0,1]$ we have

$$\begin{aligned}
&h_\eta(t)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_\eta(y(s))ds - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\int_0^1 (1-s)^{\alpha-1} f_\eta(y(s))ds\\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f_\eta(y(s))ds, \quad \eta = 1,2.
\end{aligned}$$

Let $0 \le \varepsilon \le 1$, for $t \in [0,1]$. We have

$$\begin{aligned}
&(\varepsilon h_1 + (1-\varepsilon)h_2)(t)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\varepsilon f_1 + (1-\varepsilon) f_2)(y(s))ds\\
&\quad - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\int_0^1 (1-s)^{\alpha-1}(\varepsilon f_1 + (1-\varepsilon) f_2)(y(s))ds\\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}(\varepsilon f_1 + (1-\varepsilon) f_2)(y(s))ds,
\end{aligned}$$

Since $S_{F,y}$ is convex (F has convex values), we have

$$\varepsilon h_1 + (1-\varepsilon)h_2 \in N(y).$$

Step 2. N maps bounded sets into bounded sets. Let $B_r = \{y \in C([0,1],\mathbb{R}) : \|y\| \le r\}$. Then B_r is a bounded closed convex set in $C([0,1],\mathbb{R})$. We shall prove that there exists a positive number r' such that $N(B'_r) \subseteq B'_r$. If not, for each positive number r, there exists a function $y_r(\cdot) \in B_r$, however, $\|N(y_r)\| > r$ for some $t \in [0,1]$, and

$$\begin{aligned}
&h_r(t)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_r(y(s))ds - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\int_0^1 (1-s)^{\alpha-1} f_r(y(s))ds\\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f_r(y(s))ds.
\end{aligned}$$

for some $f_r \in S_{F,y_r}$. On the other hand, we have

$$\begin{aligned}
r &< \|N(y_r)\|\\
&\le \frac{1}{\Gamma(\alpha)}\Big(\int_0^1 \varphi_r(s)ds + \frac{1}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}\int_0^1 \varphi_r(s)ds\\
&\quad + \frac{1}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}\sum_{i=1}^{m-2} k_i \int_0^1 \xi_i\varphi_r(s)ds\Big)\\
&\le \frac{1}{\Gamma(\alpha)}\Big(1 + \frac{1}{|1-\sum_{i=1}^{m-2}k_i\xi_i|} + \frac{1\sum_{i=1}^{m-2}k_i\xi_i}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}\Big)\int_0^1 \varphi_r(s)ds.
\end{aligned} \tag{3.13}$$

Dividing both sides of (3.13) by r, then taking the lower limit as $r \to \infty$, we obtain

$$\frac{1}{\Gamma(\alpha)}\Big(1+\frac{1}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}+\frac{\sum_{i=1}^{m-2}k_i\xi_i}{|1-\sum_{i=1}^{m-2}k_i\xi_i|}\Big)\mu \geq 1,$$

which contradicts (3.11). It implies for some positive number r', we conclude that $N(B_{r'}) \subseteq B_{r'}$.

Step 3. The family $\{Ny : y \in B_{r'}\}$ is a family of equicontinuous functions. Let $t_1, t_2 \in [0,1]$, $t_1 \leq t_2$ and $y \in B_{r'}$ for each $h \in N(y)$, we have

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}f(y(s))ds\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(y(s))ds\Big| \\
&\quad + \frac{t_2-t_1}{\Gamma(\alpha)|1-\sum_{i=1}^{m-2}k_i\xi_i|}\int_0^1(1-s)^{\alpha-1}f(y(s))ds \\
&\quad + \frac{t_2-t_1}{\Gamma(\alpha)|1-\sum_{i=1}^{m-2}k_i\xi_i|}\sum_{i=1}^{m-2}k_i\Big(\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}f(y(s))\Big)ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|\varphi(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\varphi(s)ds + \frac{t_2-t_1}{\Gamma(\alpha)|1-\sum_{i=1}^{m-2}k_i\xi_i|}\int_0^1\varphi(s)ds \\
&\quad + \frac{t_2-t_1}{\Gamma(\alpha)|1-\sum_{i=1}^{m-2}k_i\xi_i|}\sum_{i=1}^{m-2}k_i\xi_i\Big(\int_0^{\xi_i}\varphi(s)\Big)ds
\end{aligned} \tag{3.14}$$

The right hand of (3.14) tends to 0 as $t_2 \to t_1$. Therefore, the set $\{Ny : y \in B_{r'}\}$ is equicontinuous.

Combining Steps 1, 2 and 3 with Ascoli-Arzela theorem, we claim that N is a compact valued map.

Step 4. $N(y)$ is closed for each $y \in C([0,1],\mathbb{R})$. Let $\{h_n\}_{n\geq 0} \in N(y)$ be such that $h_n \to h_*(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then, $h_* \in C([0,1],\mathbb{R})$ and there exist $f_n \in S_{F,y_n}$, such that for each $t \in [0,1]$,

$$\begin{aligned}
&h_n(t) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_n(y(s))ds - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\int_0^1(1-s)^{\alpha-1}f_n(y(s))ds \\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2}k_i\xi_i)}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}f_n(y(s))ds.
\end{aligned}$$

Using the fact that N has compact values, we shall pass to a subsequence if necessary to obtain that $f_n \to f$ in $L^1([0,1],\mathbb{R})$ and therefore $f \in S_{F,y}$, then we have for each $t \in [0,1]$,

$$h_n \to h_*(t) = \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_*(y(s))ds$$

$$
\begin{aligned}
&- \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \int_0^1 (1-s)^{\alpha-1} f_*(y(s))ds \\
&+ \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} f_*(y(s))ds.
\end{aligned}
$$

Thus, $h_* \in N(y)$.

Step 5. N has closed graph. Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$ as $n \to \infty$. Consider the continuous linear operator $\Gamma : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$,

$$
\begin{aligned}
f \mapsto \Gamma(f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y(s))ds \\
&\quad - \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \int_0^1 (1-s)^{\alpha-1} f(y(s))ds \\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} f(y(s))ds.
\end{aligned}
$$

From Lemma 2.8, then $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have $h_n \in \Gamma(S_{F,y_n})$. Since $y_n \to y_*$ as $n \to \infty$. Lemma 2.8 implies there exists h_* such that

$$
\begin{aligned}
&h_*(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(y(s))ds - \frac{t}{\Gamma(1-\sum_{i=1}^{m-2} k_i\xi_i)} \int_0^1 (1-s)^{\alpha-1} f_*(y(s))ds \\
&\quad + \frac{t}{\Gamma(1-\sum_{i=1}^{m-2} k_i\xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} f_*(y(s))ds
\end{aligned}
$$

for some $f_* \in S_{F,y_*}$. Hence, we conclude that N is a compact multi-valued map, u.s.c. with convex closed values. In view of Lemma 2.6, we deduce that N has a fixed point which is a solution to problem (1.1)-(1.2). □

Theorem 3.5. *Assume that* (A1), (A2), (A4) *hold. Then the problem* (1.1) *and* (1.2) *has at least one solution on* $[0,1]$.

Proof. Define the operator $N : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ as (3.12). Let $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then there exists a function $f \in S_{F,y}$ such that for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
y(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(y(s))ds \\
&\quad - \frac{\lambda t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \int_0^1 (1-s)^{\alpha-1} f(y(s))ds \\
&\quad + \frac{\lambda t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} f(y(s))ds.
\end{aligned}
\tag{3.15}
$$

It from (A4), for each $t \in [0,1]$,

$$\begin{aligned}
|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s)|ds + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \int_0^1 (1-s)^{\alpha-1}|f(s)|ds \\
&\quad + \frac{t}{\Gamma(\alpha)(1-\sum_{i=1}^{m-2} k_i\xi_i)} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}|f(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(1 + \frac{1}{|1-\sum_{i=1}^{m-2} k_i\xi_i|} + \frac{\sum_{i=1}^{m-2} k_i\xi_i}{|1-\sum_{i=1}^{m-2} k_i\xi_i|}\Big) \int_0^1 |f(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(1 + \frac{1}{|1-\sum_{i=1}^{m-2} k_i\xi_i|} + \frac{\sum_{i=1}^{m-2} k_i\xi_i}{|1-\sum_{i=1}^{m-2} k_i\xi_i|}\Big)\phi(\|y\|) \int_0^1 q(s)ds.
\end{aligned} \tag{3.16}$$

Hence,

$$\frac{\|y\|}{\Big(1 + \frac{1}{|1-\sum_{i=1}^{m-2} k_i\xi_i|} + \frac{\sum_{i=1}^{m-2} k_i\xi_i}{|1-\sum_{i=1}^{m-2} k_i\xi_i|}\Big)\phi(\|y\|)\int_0^1 q(s)ds} \leq 1.$$

Then by (A4), there exists M such that $\|y\| \neq M$. Define

$$V = \{y \in C([0,1],\mathbb{R}) : \|y\| < M\}.$$

Proceed as the proof of Theorem 3.4, we claim that the operator $N : \overline{V} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of V, there is no $y \in \partial V$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of Lemma2.7, we conclude that N has a fixed point y which is a solution of the problem (1.1) and (1.2). □

4. Applications

In this section, we present an example to illustrate our main results. Consider the fractional differential inclusions with boundary-value conditions

$$y^{6/5}(t) \in F(t, y(t)), \quad t \in [0,1], \tag{4.1}$$

$$y(0) = 0, \quad y(1) = \frac{1}{3}y(\frac{1}{5}) + \frac{1}{9}y(\frac{1}{25}), \tag{4.2}$$

where $k_1 = \frac{1}{3}$, $k_2 = \frac{1}{9}$, $\xi_1 = \frac{1}{25}$, $\xi_2 = \frac{1}{5}$, $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map defined by

$$u \to F(t,u) := [\frac{u^5}{u^5+3} + t^5 + 3, \frac{u}{u+1} + t + 1]. \tag{4.3}$$

It is clear that (A1) is satisfied, and F satisfies (A2). let $f \in F$, then

$$|f| \leq \max\big(\frac{u^5}{u^5+3} + t^5 + 3, \frac{u}{u+1} + t + 1\big) \leq 5, \quad u \in \mathbb{R}.$$

Thus,

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq 5 := q(t)\phi(|u|), \quad u \in \mathbb{R},$$

where $q(t) = 1$, $\phi(|u|) = 5$. We could find a positive real number M such that

$$\frac{M}{\Gamma(\alpha)\Big(1 + \frac{1}{|1-(k_1\xi_1+k_2\xi_2)|} + \frac{(k_1\xi_1+k_2\xi_2)}{|1-(k_1\xi_1+k_2\xi_2)|}\Big)\phi(M)\int_0^1 q(s)ds} > 1,$$

$$\frac{M}{\Gamma(6/5)5\left(1+\frac{1}{|1-\frac{3}{25}|}+\frac{3}{25|1-\frac{3}{25}|}\right)}>1;$$

that is, $M > 10.43$. Thus, all the assumptions of Theorem 3.5 are satisfied. We conclude that fractional differential inclusion (4.1)-(4.2) has at least one solution.

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