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UNIQUENESS AND PARAMETER DEPENDENCE OF SOLUTIONS OF FOURTH-ORDER FOUR-POINT NONHOMOGENEOUS BVPS

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Abstract. In this article, we investigate the fourth-order four-point nonhomogeneous Sturm-Liouville boundary-value problem

$$u^{(4)}(t) = f(t, u(t)), \quad t \in [0, 1],$$
$$\alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0,$$
$$au''(\xi_1) - bu'''(\xi_1) = -\lambda, \quad cu''(\xi_2) + du'''(\xi_2) = -\mu,$$

where $0 \le \xi_1 < \xi_2 \le 1$ and λ and μ are nonnegative parameters. We obtain sufficient conditions for the existence and uniqueness of positive solutions. The dependence of the solution on the parameters λ and μ is also studied.

1. Introduction

Recently, nonhomogeneous boundary-value problems (BVPs for short) have received much attention from many authors. For example, Ma [5, 6] and Kong and Kong [2, 3, 4] studied some second-order multi-point nonhomogeneous BVPs. In particular, Kong and Kong [4] considered the following second-order BVP with nonhomogeneous multi-point boundary condition

$$u'' + a(t)f(u) = 0, \quad t \in (0, 1),$$
$$u(0) = \sum_{i=1}^{m} a_i u(t_i) + \lambda, \quad u(1) = \sum_{i=1}^{m} b_i u(t_i) + \mu,$$

where λ and μ are nonnegative parameters. They derived some conditions for the above BVP to have a unique solution and then studied the dependence of this solution on the parameters λ and μ. Sun [8] discussed the existence and nonexistence of positive solutions to a class of third-order three-point nonhomogeneous BVP. However, to the best of our knowledge, fewer results on fourth-order nonhomogeneous BVPs can be found in the literature. It is worth mentioning that the authors in [7]

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studied the multiplicity of positive solutions for some fourth-order two-point nonhomogeneous BVP by using a fixed point theorem of cone expansion/compression type.

Being directly inspired by [4], in this paper we are concerned with the nonhomogeneous Sturm-Liouville BVP consisting of the fourth-order differential equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1] \tag{1.1}$$

and the four-point boundary conditions

$$\alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0, \tag{1.2}$$

$$au''(\xi_1) - bu'''(\xi_1) = -\lambda, \quad cu''(\xi_2) + du'''(\xi_2) = -\mu, \tag{1.3}$$

where $0 \le \xi_1 < \xi_2 \le 1$ and λ and μ are nonnegative parameters. We will use the following assumptions:

(A1) $\alpha, \beta, \gamma, \delta, a, b, c$ and d are nonnegative constants with $\beta > 0$, $\delta > 0$, $\rho_1 := \alpha\gamma + \alpha\delta + \gamma\beta > 0$, $\rho_2 := ad + bc + ac(\xi_2 - \xi_1) > 0$, $-a\xi_1 + b > 0$ and $c(\xi_2 - 1) + d > 0$;

(A2) $f(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and monotone increasing in u;

(A3) There exists $0 \le \theta < 1$ such that
$$f(t, ku) \ge k^\theta f(t,u) \quad \text{for all } t \in [0,1],\ k \in (0,1),\ u \in [0,+\infty).$$

We prove the existence and uniqueness of a positive solution for the BVP (1.1)–(1.3) and study the dependence of this solution on the parameters λ and μ.

2. Preliminary lemmas

First, we recall some fundamental definitions.

Definition 2.1. Let X be a Banach space with a norm $\|\cdot\|$.

(1) A nonempty closed convex set $P \subseteq X$ is said to be a cone if $\lambda P \subseteq P$ for all $\lambda \ge 0$ and $P \cap (-P) = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero element of X;

(2) Every cone P in X defines a partial ordering in X by $u \le v \Leftrightarrow v - u \in P$;

(3) A cone P is said to be normal if there exists $M > 0$ such that $\mathbf{0} \le u \le v$ implies $\|u\| \le M\|v\|$;

(4) A cone P is said to be solid if the interior P^0 of P is nonempty.

Let P be a solid cone in a real Banach space X, $T : P^0 \to P^0$ be an operator and $0 \le \theta < 1$. Then T is called a θ-concave operator if

$$T(ku) \ge k^\theta Tu \quad \text{for all } k \in (0,1),\ u \in P^0.$$

Next, we state a fixed point theorem, which is our main tool.

Lemma 2.2 ([1]). *Assume that P is a normal solid cone in a real Banach space X, $0 \le \theta < 1$ and $T : P^0 \to P^0$ is a θ-concave increasing operator. Then T has a unique fixed point in P^0.*

The following two lemmas are crucial for our main results.

Lemma 2.3. *Let $\rho_1 \ne 0$ and $\rho_2 \ne 0$. Then for any $h \in C[0,1]$, the BVP consisting of the equation*

$$u^{(4)}(t) = h(t), \quad t \in [0,1]$$

and the boundary conditions (1.2)–(1.3) *has a unique solution*

$$u(t)=\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)h(\tau)d\tau ds+\lambda\Phi(t)+\mu\Psi(t),\quad t\in[0,1],$$

where

$$G_1(t,s)=\frac{1}{\rho_1}\begin{cases}(\alpha s+\beta)(\gamma+\delta-\gamma t), & 0\le s\le t\le 1,\\ (\alpha t+\beta)(\gamma+\delta-\gamma s), & 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s)=\frac{1}{\rho_2}\begin{cases}(a(s-\xi_1)+b)(c(\xi_2-t)+d), & s\le t,\ \xi_1\le s\le \xi_2,\\ (a(t-\xi_1)+b)(c(\xi_2-s)+d), & t\le s,\ \xi_1\le s\le \xi_2,\end{cases}$$

$$\Phi(t)=\frac{1}{\rho_2}\int_0^1 (c(\xi_2-s)+d)G_1(t,s)ds,\quad t\in[0,1],$$

$$\Psi(t)=\frac{1}{\rho_2}\int_0^1 (a(s-\xi_1)+b)G_1(t,s)ds,\quad t\in[0,1].$$

Proof. Let

$$u''(t)=v(t),\quad t\in[0,1]. \tag{2.1}$$

Then

$$v''(t)=h(t),\quad t\in[0,1]. \tag{2.2}$$

By (2.1) and (1.2), we know that

$$u(t)=-\int_0^1 G_1(t,s)v(s)ds,\quad t\in[0,1]. \tag{2.3}$$

On the other hand, in view of (2.1) and (1.3), we have

$$av(\xi_1)-bv'(\xi_1)=-\lambda,\ cv(\xi_2)+dv'(\xi_2)=-\mu. \tag{2.4}$$

So, it follows from (2.2) and (2.4) that

$$v(t)=-\int_{\xi_1}^{\xi_2} G_2(t,s)h(s)ds+\frac{1}{\rho_2}(c\lambda-a\mu)t+\frac{1}{\rho_2}((a\xi_1-b)\mu-(c\xi_2+d)\lambda),\quad t\in[0,1],$$

which together with (2.3) implies

$$u(t)=\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)h(\tau)d\tau ds+\lambda\Phi(t)+\mu\Psi(t),\quad t\in[0,1].$$

□

Lemma 2.4. *Assume* (A1). *Then*

(1) $G_1(t,s)>0$ *for* $t,s\in[0,1]$*;*
(2) $G_2(t,s)>0$ *for* $t\in[0,1]$ *and* $s\in[\xi_1,\xi_2]$*;*
(3) $\Phi(t)>0$ *and* $\Psi(t)>0$ *for* $t\in[0,1]$.

3. Main result

In the remainder of this article, the following notation will be used:

(1) $(\lambda,\mu)\to\infty$ if at least one of λ and μ approaches ∞;
(2) $(\lambda_1,\mu_1)>(\lambda_2,\mu_2)$ if $\lambda_1\ge\lambda_2$ and $\mu_1\ge\mu_2$ and at least one of them is strict;
(3) $(\lambda_1,\mu_1)<(\lambda_2,\mu_2)$ if $\lambda_1\le\lambda_2$ and $\mu_1\le\mu_2$ and at least one of them is strict;
(4) $(\lambda,\mu)\to(\lambda_0,\mu_0)$ if $\lambda\to\lambda_0$ and $\mu\to\mu_0$.

Our main result is the following theorem. Here, for any $u \in C[0,1]$, we write $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Theorem 3.1. *Assume* (A1)-(A3)*. Then the BVP* (1.1)-(1.3) *has a unique positive solution* $u_{\lambda,\mu}(t)$ *for any* $(\lambda,\mu) > (0,0)$*. Furthermore, such a solution* $u_{\lambda,\mu}(t)$ *satisfies the following three properties:*

(P1) $lim_{(\lambda,\mu)\to\infty}\|u_{\lambda,\mu}\| = \infty$;

(P2) $u_{\lambda,\mu}(t)$ *is strictly increasing in* λ *and* μ*; i.e.,*
$$(\lambda_1,\mu_1) > (\lambda_2,\mu_2) > (0,0) \Longrightarrow u_{\lambda_1,\mu_1}(t) > u_{\lambda_2,\mu_2}(t) \text{ on } [0,1];$$

(P3) $u_{\lambda,\mu}(t)$ *is continuous in* λ *and* μ*; i.e., for any* $(\lambda_0,\mu_0) > (0,0)$*,*
$$(\lambda,\mu) \to (\lambda_0,\mu_0) \Longrightarrow \|u_{\lambda,\mu} - u_{\lambda_0,\mu_0}\| \to 0.$$

Proof. Let $X = C[0,1]$. Then $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined as usual by the sup norm. Denote $P = \{u \in X : u(t) \geq 0,\ t \in [0,1]\}$. Then P is a normal solid cone in X with $P^0 = \{u \in X \mid u(t) > 0,\ t \in [0,1]\}$. For any $(\lambda,\mu) > (0,0)$, if we define an operator $T_{\lambda,\mu} : P^0 \to X$ as follows
$$T_{\lambda,\mu}u(t) = \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t), \tag{3.1}$$
then it is not difficult to verify that u is a positive solution of the BVP (1.1)-(1.3) if and only if u is a fixed point of $T_{\lambda,\mu}$.

Now, we prove that $T_{\lambda,\mu}$ has a unique fixed point by using Lemma 2.2

First, in view of Lemma 2.4, we know that $T_{\lambda,\mu} : P^0 \to P^0$. Next, we claim that $T_{\lambda,\mu} : P^0 \to P^0$ is a θ-concave operator.

In fact, for any $k \in (0,1)$ and $u \in P^0$, it follows from (3.1) and (A3) that
$$\begin{aligned}
T_{\lambda,\mu}(ku)(t) &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,ku(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t)\\
&\geq k^\theta \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t)\\
&\geq k^\theta \Big(\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t)\Big)\\
&= k^\theta T_{\lambda,\mu}u(t), \quad t \in [0,1],
\end{aligned}$$
which shows that $T_{\lambda,\mu}$ is θ-concave.

Finally, we assert that $T_{\lambda,\mu} : P^0 \to P^0$ is an increasing operator. Suppose $u, v \in P^0$ and $u \leq v$. By (3.1) and (A2), we have
$$\begin{aligned}
T_{\lambda,\mu}u(t) &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t)\\
&\leq \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,v(\tau))d\tau ds + \lambda\Phi(t) + \mu\Psi(t)\\
&= T_{\lambda,\mu}v(t),\ t \in [0,1],
\end{aligned}$$
which indicates that $T_{\lambda,\mu}$ is increasing.

Therefore, it follows from Lemma 2.2 that $T_{\lambda,\mu}$ has a unique fixed point $u_{\lambda,\mu} \in P^0$, which is the unique positive solution of the BVP (1.1)-(1.3). The first part of the theorem is proved.

In the rest of the proof, we prove that the solution $u_{\lambda,\mu}$ satisfies the properties (P1), (P2) and (P3). First, for $t\in[0,1]$,

$$\begin{aligned} u_{\lambda,\mu}(t) &= T_{\lambda,\mu}u_{\lambda,\mu}(t)\\ &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u_{\lambda,\mu}(\tau))d\tau ds + \lambda\Phi(t)+\mu\Psi(t), \end{aligned}$$

which together with $\Phi(t)>0$ and $\Psi(t)>0$ for $t\in[0,1]$ implies (P1).

Next, we show (P2). Assume $(\lambda_1,\mu_1)>(\lambda_2,\mu_2)>(0,0)$. Let

$$\overline{\chi}=\sup\big\{\chi>0: u_{\lambda_1,\mu_1}(t)\ge \chi u_{\lambda_2,\mu_2}(t),\ t\in[0,1]\big\}.$$

Then $u_{\lambda_1,\mu_1}(t)\ge\overline{\chi}u_{\lambda_2,\mu_2}(t)$ for $t\in[0,1]$. We assert that $\overline{\chi}\ge 1$. Suppose on the contrary that $0<\overline{\chi}<1$. Since $T_{\lambda,\mu}$ is a θ-concave increasing operator, and for given $u\in P^0$, $T_{\lambda,\mu}u$ is strictly increasing in λ and μ, we have

$$\begin{aligned} u_{\lambda_1,\mu_1}(t) &= T_{\lambda_1,\mu_1}u_{\lambda_1,\mu_1}(t)\ge T_{\lambda_1,\mu_1}(\overline{\chi}u_{\lambda_2,\mu_2})(t)\\ &> T_{\lambda_2,\mu_2}(\overline{\chi}u_{\lambda_2,\mu_2})(t)\\ &\ge (\overline{\chi})^\theta T_{\lambda_2,\mu_2}u_{\lambda_2,\mu_2}(t)=(\overline{\chi})^\theta u_{\lambda_2,\mu_2}(t)\\ &> \overline{\chi}u_{\lambda_2,\mu_2}(t),\quad t\in[0,1], \end{aligned}$$

which contradicts the definition of $\overline{\chi}$. Thus, we get $u_{\lambda_1,\mu_1}(t)\ge u_{\lambda_2,\mu_2}(t)$ for $t\in[0,1]$. And so,

$$\begin{aligned} u_{\lambda_1,\mu_1}(t) &= T_{\lambda_1,\mu_1}u_{\lambda_1,\mu_1}(t)\ge T_{\lambda_1,\mu_1}u_{\lambda_2,\mu_2}(t)\\ &> T_{\lambda_2,\mu_2}u_{\lambda_2,\mu_2}(t)=u_{\lambda_2,\mu_2}(t),\quad t\in[0,1], \end{aligned}$$

which indicates that $u_{\lambda,\mu}(t)$ is strictly increasing in λ and μ.

Finally, we show (P3). For any given $(\lambda_0,\mu_0)>(0,0)$, we first suppose $(\lambda,\mu)\to(\lambda_0,\mu_0)$ with $(\lambda_0/2,\mu_0/2)<(\lambda,\mu)<(\lambda_0,\mu_0)$. From (P2), we have

$$u_{\lambda,\mu}(t)<u_{\lambda_0,\mu_0}(t),\quad t\in[0,1]. \tag{3.2}$$

Let

$$\overline{\sigma}=\sup\{\sigma>0: u_{\lambda,\mu}(t)\ge\sigma u_{\lambda_0,\mu_0}(t),\quad t\in[0,1]\}.$$

Then $0<\overline{\sigma}<1$ and $u_{\lambda,\mu}(t)\ge\overline{\sigma}u_{\lambda_0,\mu_0}(t)$ for $t\in[0,1]$. Define

$$\omega(\lambda,\mu)=\begin{cases}\min\{\frac{\lambda}{\lambda_0},\frac{\mu}{\mu_0}\}, & \text{if } \lambda_0\ne 0 \text{ and } \mu_0\ne 0,\\ \frac{\mu}{\mu_0}, & \text{if } \lambda_0=0,\\ \frac{\lambda}{\lambda_0}, & \text{if } \mu_0=0,\end{cases}$$

then $0<\omega(\lambda,\mu)<1$ and

$$\begin{aligned} u_{\lambda,\mu}(t) &= T_{\lambda,\mu}u_{\lambda,\mu}(t)\ge T_{\lambda,\mu}(\overline{\sigma}u_{\lambda_0,\mu_0})(t)\\ &> \omega(\lambda,\mu)T_{\lambda_0,\mu_0}(\overline{\sigma}u_{\lambda_0,\mu_0})(t)\\ &\ge \omega(\lambda,\mu)(\overline{\sigma})^\theta T_{\lambda_0,\mu_0}u_{\lambda_0,\mu_0}(t)\\ &= \omega(\lambda,\mu)(\overline{\sigma})^\theta u_{\lambda_0,\mu_0}(t),\quad t\in[0,1], \end{aligned}$$

which together with the definition of $\overline{\sigma}$ implies

$$\omega(\lambda,\mu)(\overline{\sigma})^\theta\le\overline{\sigma}.$$

Thus $\overline{\sigma}\ge(\omega(\lambda,\mu))^{\frac{1}{1-\theta}}$. And so,

$$u_{\lambda,\mu}(t)\ge\overline{\sigma}u_{\lambda_0,\mu_0}(t)\ge(\omega(\lambda,\mu))^{\frac{1}{1-\theta}}u_{\lambda_0,\mu_0}(t),\quad t\in[0,1]. \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$\|u_{\lambda_0,\mu_0} - u_{\lambda,\mu}\| \le (1 - (\omega(\lambda,\mu))^{\frac{1}{1-\theta}})\|u_{\lambda_0,\mu_0}\|,$$

which together with the fact that $\omega(\lambda,\mu) \to 1$ as $(\lambda,\mu) \to (\lambda_0,\mu_0)$ shows that

$$\|u_{\lambda_0,\mu_0} - u_{\lambda,\mu}\| \to 0 \text{ as } (\lambda,\mu) \to (\lambda_0,\mu_0).$$

Similarly, we can also prove that

$$\|u_{\lambda_0,\mu_0} - u_{\lambda,\mu}\| \to 0$$

as $(\lambda,\mu) \to (\lambda_0,\mu_0)$ with $(\lambda,\mu) > (\lambda_0,\mu_0)$. Hence, (P3) holds. The proof is complete. □

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