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OSCILLATION OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract. This article is devoted to studying the solutions to the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \quad k \geq 2,$$

where coefficients $A_j(z)$ are entire functions of integer order. We obtain estimates on the orders and the hyper orders of the solutions to the above equation.

1. Introduction and main results

In this note, we apply standard notation of the Nevanlinna theory, see [6]. Let $f(z)$ be a nonconstant meromorphic function. As usual, $\sigma(f)$ denote the order. In addition, we use the notation $\sigma_2(f)$ to denote the hyper-order of $f(z)$,

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Chen [1] studied this differential equation when all the coefficients are of order 1.

Theorem 1.1. *Let a, b be nonzero complex numbers and $a \neq b$, $Q(z)$ be a nonconstant polynomial or $Q(z) = h(z)e^{bz}$ where $h(z)$ is nonzero polynomial. Then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{az}f' + Q(z)f = 0 \tag{1.1}$$

is of infinite order.

Later on, Li and Huang[7], Chen and Shon[2] extended this result to the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0, \quad k \geq 2. \tag{1.2}$$

Chen and Shon[2] obtained the following result.

Theorem 1.2. *Let $A_j(z) = B_j(z)e^{P_j(z)}(0 \leq j \leq k-1)$, where $B_j(z)$ are entire functions with $\sigma(B_j) < 1$ and $P_j(z) = a_j z$ with a_j are complex numbers. Suppose that there exists a_s such that $B_s \not\equiv 0$, and for $j \neq s$, if $B_j \not\equiv 0$, $a_j = c_j a_s$, $0 < c_j < 1$; If $B_j \equiv 0$, we define $c_j = 0$. Then every transcendental solution f of*

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the (1.2) *satisfies* $\sigma(f) = \infty$. *Furthermore, if* $\max\{c_1, \dots, c_{s-1}\} < c_0$, *then every solution* $f(\not\equiv 0)$ *of (1) is of infinite order.*

Theorem 1.3. *Let* P_j *be polynomials,* s *and* A_j,a_j, B_j *satisfy the other additional hypotheses of Theorem 1.2. Then every transcendental solution* f *of the* (1.2) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) = 1$. *Furthermore, if max*$\{c_1, \dots, c_{s-1}\} < c_0$, *then every solution* $f(\not\equiv 0)$ *of* (1.2) *is of infinite order and* $\sigma_2(f) = 1$.

The aim of this paper is to improve Theorems 1.2 and 1.3.

Theorem 1.4. *Let* $A_j(z) = B_j(z)e^{P_j(z)}$ $(j = 0, 1, \dots, k-1)$, *where* $B_j(z)$ *are entire functions,* $P_j(z)$ *are non-constant polynomials with* $\deg(P_j(z) - P_i(z)) \geq 1$ *and* $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_i - P_j)(i \neq j)$. *Then every transcendental solution* f *of* (1.2) *satisfies* $\sigma(f) = \infty$.

Theorem 1.5. *Let* $P_j(z) = a_{j,n}z^n + a_{j,n-1}z^{n-1} + \dots + a_{j,0}(0 \leq j \leq k-1)$ *be non-constant polynomials, where* $a_{j,n} \neq 0$ *and* $\deg(P_j(z)-P_i(z)) = n$, *and let* $Q_j(z)$ *and* $B_j(z)(0 \leq j \leq k-1)$ *be entire functions with* $\max\{\sigma(B_j), \sigma(Q_j), 0 \leq j \leq k-1\} < n$. *Set* $A_j(z) = B_j(z)e^{P_j(z)} + Q_j(z)$. *Suppose that one of the following occurs:*

(1) *There exist* $t, s \in \{0, 1, \dots, k-1\}$, *such that* $\frac{a_{t,n}}{a_{s,n}} < 0$;

(2) $\arg a_{0,n} \neq \arg a_{1,n}$ *and* $a_{j,n} = c_j a_{1,n}(c_j > 0,\ j = 2, 3, \dots, k-1)$.

Then every transcendental solution f *of* (1.2) *satisfies* $\sigma(f) = \infty$.

Theorem 1.6. *Let* $A_j = P_j(e^{R(z)}) + Q_j(e^{-R(z)})$ *for* $j = 1, 2, \dots, k-1$ *where* $P_j(z), Q_j(z)$ *and* $R(z) = c_s z^s + \dots + c_1 z + c_0(s(\geq 1)$ *is an integer) are polynomials. Suppose that* $P_0(z) + Q_0(z) \not\equiv 0$ *and there exists* $d(0 \leq d \leq k-1)$, *such that for* $j \neq d$, $\deg P_d > \deg P_j$ *and* $\deg Q_d > \deg Q_j$. *Then every solution* $f(z)$ *of* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = s$.

We remark that many authors have studied the order and the hyper order of solutions of (1.2). But, they always require that there exists some coefficient A_j ($j \in \{0, 1, \dots, k-1\}$) such that the order of A_j is greater than the order of other coefficients. We note that our theorems do not need the hypothesis. Our hypothesis of Theorem 1.6 are partly motivated by [3].

2. Preliminary lemmas

Assume that $R(z) = c_s z^s + \dots + c_1 z + c_0(s(\geq 1)$ is a polynomial. Below, for $\theta \in [0, 2\pi)$, we denote $\delta_j(R, \theta) = \mathrm{Re}(c_j(e^{i\theta})^j)$ for $j \in \{1, 2, \dots, s\}$. Especially, we write $\delta(R, \theta) = \delta_s(R, \theta)$.

For $j \in \{0, 1, \dots, k-1\}$, let

$$P_j(e^{R(z)}) = a_{jm_j}e^{m_j R(z)} + a_{j(m_j-1)}e^{(m_j-1)R(z)} + \dots + a_{j1}e^{R(z)} + a_{j0}$$

and

$$Q_j(e^{-R(z)}) = b_{jt_j}e^{-t_j R(z)} + b_{j(t_j-1)}e^{-(t_j-1)R(z)} + \dots + b_{j1}e^{-R(z)} + b_{j0},$$

where $a_{jm_j}, \dots, a_{j0}$ and $b_{jt_j}, \dots, b_{j0}$ are constants, $m_j \geq 0$ and $t_j \geq 0$ are integers, $a_{jm_j} \neq 0$, $b_{jt_j} \neq 0$. So we have

$$\begin{aligned} &|P_j(e^{R(z)}) + Q_j(e^{-R(z)})| \\ &= \begin{cases} |a_{jm_j}|e^{m_j r^s \delta(R,\theta)}(1 + o(1)), & \arg z = \theta, \delta(R, \theta) > 0, r \to \infty, \\ |b_{jt_j}|e^{-t_j r^s \delta(R,\theta)}(1 + o(1)), & \arg z = \theta, \delta(R, \theta) < 0, r \to \infty; \end{cases} \end{aligned} \tag{2.1}$$

To prove our results, some lemmas are needed.

Lemma 2.1. *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$. Let $\Gamma = \{(k_1, j_1), \dots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \dots, m$. Also let $\epsilon > 0$ be a given constant. Then, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Gamma$, we have*

$$\frac{|w^{(k)}(z)|}{|w^{(j)}(z)|} \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

The above lemma is [5, Corollary 1]. We also need the following lemma given in Chen [2].

Lemma 2.2. *Suppose that $P(z)$ is a non-constant polynomial, $w(z)$ is a meromorphic function with $\sigma(w) < \deg P(z) = n$. Let $g(z) = w(z)e^{P(z)}$, then there exists a set $H_1 \subset [0, 2\pi)$ that has linear measure zero, such that for $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$ and arbitrary constant $\epsilon (0 < \epsilon < 1)$, when $r > r_0(\theta, \epsilon)$, we have*

(1) *if $\delta(P, \theta) < 0$, then $\exp((1+\epsilon)\delta(P, \theta)r^n) \leq |g(re^{i\theta})| \leq \exp((1-\epsilon)\delta(P, \theta)r^n)$,*
(2) *if $\delta(P, \theta) > 0$, then $\exp((1-\epsilon)\delta(P, \theta)r^n) \leq |g(re^{i\theta})| \leq \exp((1+\epsilon)\delta(P, \theta)r^n)$, where $H_2 = \{\theta : \delta(P, \theta) = 0, 0 \leq \theta < 2\pi\}$ is a finite set.*

We shall use a special version of Phragmén-Lindelöf-type theorem to prove our results. We refer to Titchmarsh [8, p.177].

Lemma 2.3. *Let $f(z)$ be an analytic function of $z = re^{i\theta}$, regular in the region D between two straight lines making an angle $\frac{\pi}{\beta-\alpha}$ at the origin and on the lines themselves. Suppose that $|f(z)| \leq M$ on the lines, and for any given $\epsilon > 0$, as $r \to \infty$, $|f(z)| < O(e^{\epsilon r^{\frac{\pi}{\beta-\alpha}}})$, uniformly in the angle. Then actually the inequality $|f(z)| \leq M$ holds throughout the region D.*

Lemma 2.4. *Let $n \geq 2$ and $A_j(z) = B_j(z)e^{P_j(z)} (1 \leq j \leq n)$ where each $B_j(z)$ is an entire function , and $P_j(z)$ is a non-constant polynomial. Suppose that $\deg(P_j(z) - P_i(z)) \geq 1$, $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_i - P_j)$ for $i \neq j$. Then there exists a set $H_1 \subset [0, 2\pi)$ that has linear measure zero such that for any given constant $M > 0$ and $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, we have some $s = s(\theta) \in \{1, \dots, n\}$, for $j \neq s$,*

$$\frac{|A_j(re^{i\theta})||z|^M}{|A_s(re^{i\theta})|} \to 0, \quad \text{as } r \to \infty,$$

where $H_2 = \{\theta : \delta(P_j, \theta) = 0 \text{ or } \delta(P_i, \theta) = \delta(P_j, \theta),\ i, j \in \{1, 2, \dots, n\},\ i \neq j,\ 0 \leq \theta < 2\pi\}$ is a finite set.

Proof. We use mathematical induction. For $n = 2$, Lemma 2.4 can be proved by applying Lemma 2.2 to $\frac{A_1}{A_2}$ or $\frac{A_2}{A_1}$.

Assume that Lemma 2.4 holds for $n \leq k - 1$. For the case $n = k$. Take $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there exists some $t = t(\theta) \in \{1, 2, \dots, k-1\}$, such that $\frac{|A_j(re^{i\theta})||z|^M}{|A_t(re^{i\theta})|} \to 0$ for $j \in \{1, \dots, t-1, t+1, \dots k-1\}$. Now we compare $A_t(re^{i\theta})$ with $A_k(re^{i\theta})$. If $\delta(P_t - P_k, \theta,) < 0$, from Lemma 2.2, $\deg(P_t - P_k) \geq 1$ and $max\{\sigma(B_t), \sigma(B_k)\} < \deg(P_t - P_k)$, for any given $1 > \epsilon > 0$, we have

$$|\frac{A_t(re^{i\theta})}{A_k(re^{i\theta})}| \leq e^{(1-\epsilon)\delta(P_t-P_k,\theta)r^{\deg(P_t-P_k)}} \leq e^{(1-\epsilon)\delta(P_t-P_k,\theta)r},$$

thus $|\frac{A_t(re^{i\theta})}{A_k(re^{i\theta})}||z^M| \to 0$ as $r \to \infty$. Therefore, for $j \neq k$,

$$|\frac{A_j(re^{i\theta})}{A_k(re^{i\theta})}||z^M| = |\frac{A_j(re^{i\theta})}{A_t(re^{i\theta})}||z^M||\frac{A_t(re^{i\theta})}{A_k(re^{i\theta})}| \to 0.$$

If $\delta(P_t - P_k, \theta) > 0$, then $\delta(P_k - P_t, \theta) < 0$, by the similar discussion as above, we have $|\frac{A_k(re^{i\theta})}{A_t(re^{i\theta})}||z^M| \to 0$ as $r \to \infty$. The proof is now complete. $\square$

Observe that (1) For $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, set $v = \deg P_s$ and $\delta = \delta(P_s, \theta)$ as in Lemma 2.4. Since for $j \neq s$, $\frac{|A_j(re^{i\theta})||z|^M}{|A_s(re^{i\theta})|} \to 0$, as $r \to \infty$. Then if $\deg P_j > v, j \neq s$, we have $\deg(P_t - P_s) = \deg P_t$, so $\delta(P_j, \theta) < 0$. If $\deg P_j = v, j \neq s$, then $\delta(P_j, \theta) < \delta$. If $\deg P_j < v, j \neq t$, $\frac{|A_j(re^{i\theta})||z|^M}{|A_s(re^{i\theta})|} \to 0$ no matter that $\delta(P_j, \theta)$ is positive or negative.

(2) From the proof of Lemma 2.4, if there exist a polynomial $P_v(z)$ which is a constant ($v \in \{1, 2, \dots, n\}$), then the lemma is also true. In fact, the hypothesis $\deg(P_j(z) - P_i(z)) \geq 1 (j \neq i)$ implies that there is at most one polynomial which can be a constant.

From the proof of Lemma 2.4, we can easily obtain the following lemma.

Lemma 2.5. *Let $P_j(z)(1 \leq j \leq m)$ be non-constant polynomial with degree n. Let $B_j(z)$ and $Q_j(z)(1 \leq j \leq m)$ be entire functions with $\max\{\sigma(B_j), \sigma(Q_j), 1 \leq j \leq m\} < n$. Set $A_j(z) = B_j(z)e^{P_j(z)} + Q_j(z)$. For $\theta \in [0, 2\pi)$, suppose that not all $\delta(P_j, \theta)(1 \leq j \leq n)$ are negative. Then there exists some $s = s(\theta) \in \{1, \dots, n\}$, for $j \neq s$, as $r \to \infty$,*

$$\frac{|A_j(re^{i\theta})||z|^M}{|A_s(re^{i\theta})|} \to 0,$$

where M is a constant.

3. Proof of main results

Proof of Theorem 1.4. Without loss of generality, we can assume that each $A_j \not\equiv 0$, $j \in \{0, 1, \dots, k-1\}$.

Claim: Each transcendental solution f of equation (1.1) is infinite order.

Suppose to the contrary, there exists a transcendental solution $f(z)$ which has order $\sigma(f) = \sigma < \infty$. By Lemma 2.1, for any given $\epsilon_0 (0 < \epsilon_0 < 1)$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then

$$\frac{|f^{(j)}(z)|}{|f^{(i)}(z)|} \leq |z|^{k\sigma}, \quad i = 0, 1, \dots, k-1; \; j = i+1, \dots, k \tag{3.1}$$

as $z \to \infty$ along $\arg z = \psi_0$. Denote $E_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0, 0 \leq j \leq k\} \cup \{\theta \in [0, 2\pi) : \delta(P_j - P_i, \theta) = 0, 0 \leq j \leq k, 0 \leq i \leq k\}$, so E_2 is a finite set. Suppose that $H_j \subset [0, 2\pi)$ is the exceptional set applying Lemma 2.2 to $A_j (j = 0, 1, \dots, k-1)$. Then $E_3 = \bigcup_{j=0}^{k-1} H_j$ has linear measure zero. Take $\arg z = \psi_0 \in [0, 2\pi) - (E_1 \cup E_2 \cup E_3)$ and write $\delta_j = \delta(P_j, \psi_0)$. We need to treat two cases:

Case (i): Not all δ_0, δ_1,..., δ_{k-1} are negative. By Lemma 2.4, there exists some $t \in \{0, 1, 2, \dots, k-1\}$ such that for $j \neq t$, $M > 0$,

$$|\frac{A_j(re^{i\psi_0})}{A_t(re^{i\psi_0})}||z^M| \to 0, \tag{3.2}$$

as $r \to \infty$. Let $v = \deg(P_t)$, $\delta = \delta(P_t, \psi_0)$. From the observation, it is obvious $\delta > 0$. Now we prove $|f^{(t)}(z)|$ is bounded on the ray $\arg z = \psi_0$. Suppose that it is not. Let

$$M(r, f^{(t)}, \psi_0) = \max\{|f^{(t)}(z)| : 0 \le |z| \le r, \arg z = \psi_0\}.$$

There exists an infinite sequence of points $z_n = r_n e^{i\psi_0}$ such that

$$M(r_n, f^{(t)}, \psi_0) = |f^{(t)}(r_n e^{i\psi_0})|, r_n \to \infty.$$

Take a curve $C_n : z = re^{i\psi_0}$, $0 \le r \le |z_n|$, for each n, we have

$$f^{(t-1)}(z_n) = f^{(t-1)}(0) + \int_C f^{(t)}(u)du.$$

And hence

$$|f^{(t-1)}(z_n)| \le |f^{(t-1)}(0)| + |z_n| \cdot |f^{(t)}(z_n)|$$

holds, which leads to

$$\frac{|f^{(t-1)}(z_n)|}{|f^{(t)}(z_n)|} \le (1 + \circ(1))|z_n|, \quad z_n \to \infty.$$

Furthermore,

$$\frac{|f^{(t-j)}(z_n)|}{|f^{(t)}(z_n)|} \le (1 + \circ(1))|z_n|^j, \quad j = 1, 2, \dots, t. \tag{3.3}$$

as $z_n \to \infty$. Since $f^{(t)} \not\equiv 0$, then by (1.1),

$$\begin{aligned} |A_t(z_n)| \le \frac{|f^{(k)}(z_n)|}{|f^{(t)}(z_n)|} + \dots + |A_{t+1}(z_n)| \cdot \frac{|f^{(t+1)}(z_n)|}{|f^{(t)}(z_n)|} \\ + |A_{t-1}(z_n)| \cdot \frac{|f^{(t-1)}(z_n)|}{|f^{(t)}(z_n)|} + \dots + |A_0(z_n)| \cdot \frac{|f(z_n)|}{|f^{(t)}(z_n)|} \end{aligned} \tag{3.4}$$

holds as $z_n \to \infty$. So we obtain

$$\begin{aligned} 1 \le \frac{1}{|A_t(z_n)|}\Big(\frac{|f^{(k)}(z_n)|}{|f^{(t)}(z_n)|} + \dots + |A_{t+1}(z_n)| \cdot \frac{|f^{(t+1)}(z_n)|}{|f^{(t)}(z_n)|} \\ + |A_{t-1}(z_n)| \cdot \frac{|f^{(t-1)}(z_n)|}{|f^{(t)}(z_n)|} + \dots + |A_0(z_n)| \cdot \frac{|f(z_n)|}{|f^{(t)}(z_n)|}\Big). \end{aligned} \tag{3.5}$$

Since $\delta > 0$, by Lemma 2.2 and (3.2), it is easy to deduce $\frac{|f^{(k)}(z_n)|}{|f^{(t)}(z_n)||A_t(z_n)|} \to 0$. Then from (3.2), (3.3) and (3.4), the right hand of (3.5) tends to 0 as $z_n \to \infty$, a contradiction. Thus, $|f^{(t)}|$ is bounded on $\arg z = \psi_0 \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$. Assume that $|f^{(t)}(re^{i\psi_0})| \le M_1 (M_1 > 0$ is a constant). Take a curve $C' = \{z : \arg z = \psi_0, 0 \le |z| \le r\}$. Since

$$f^{(t-1)}(z) = f^{(t-1)}(0) + \int_{C'} f^{(t)}(u)du,$$

for large $z = re^{i\psi_0}$, we have $|f^{(t-1)}(z)| \le M_2|z|$ ($M_2 > 0$ is a constant). By induction, we obtain

$$|f(z)| \le M_3|z|^t \le M_4|z|^k. \tag{3.6}$$

(ii) Assume that for any $j : 0 \le j \le k-1$, $\delta(P_j, \psi_0) < 0$. By Lemma 2.4, there exists some $s \in \{0, 1, 2, \dots, k-1\}$, for $j \ne s$, we have

$$\Big|\frac{A_j(re^{i\psi_0})}{A_s(re^{i\psi_0})}\Big| \to 0$$

as $r \to \infty$. Let $v = \deg(P_s)$, $\delta = \delta(P_s, \psi_0)$, then $\delta < 0$. From Lemma 2.2, for any given $\epsilon (0 < \epsilon < 1/2)$,

$$|A_j(re^{i\psi_0})| \le |A_s(re^{i\psi_0})| \le \exp\left((1-\epsilon)\delta r^v\right). \tag{3.7}$$

Suppose that $|f^{(k)}(z)|$ is unbounded on the ray $\arg z = \psi_0$. Let

$$M(r, f^{(k)}, \psi_0) = \max\{|f^{(k)}(z)| : 0 \le |z| \le r, \arg z = \psi_0\}.$$

There exists a infinite sequence of points $z_n = r_n e^{i\psi_0}$ such that

$$M(r_n, f^{(k)}, \psi_0) = |f^{(k)}(r_n e^{i\psi_0})|$$

holds as $r_n \to \infty$. Take a curve $C_n : z = re^{i\psi_0}$, $0 \le r \le |z_n|$. Since $f^{(k-1)}(z_n) = f^{(k-1)}(0) + \int_{C_n} f^{(k)}(u)du$, and on C_n, $|f^{(k)}(z)| \le |f^{(k)}(z_n)|$, we have

$$|f^{(k-1)}(z_n)| \le |f^{(k-1)}(0)| + |z_n| \cdot |f^{(k)}(z_n)|.$$

It follows that

$$\frac{|f^{(k-1)}(z_n)|}{|f^{(k)}(z_n)|} \le (1 + \circ(1))|z_n|.$$

So we have

$$\frac{|f^{(k-j)}(z_n)|}{|f^{(k)}(z_n)|} \le (1 + \circ(1))|z_n|^j, \quad j = 1, 2, \dots, k. \tag{3.8}$$

Since $f^{(k)} \not\equiv 0$, by (1.1), (3.7) and (3.8), for sufficiently large n, we have

$$1 \le |A_{k-1}(z_n)| \cdot \frac{|f^{(k-1)}(z_n)|}{|f^{(k)}(z_n)|} + \dots + |A_0(z_n)| \cdot \frac{|f(z_n)|}{|f^{(k)}(z_n)|} \le \exp\{(1-\epsilon)\delta |z_n|^v\} \cdot |z_n|^{M_5}, \tag{3.9}$$

where M_5 is a positive constant. This is impossible since $\delta < 0$. Then $f^{(k)}(z)$ is bounded on $\arg z = \psi_0$. Assume that $|f^{(k)}(re^{i\psi_0})| \le M_6 (M_6 > 0)$. We take a curve $C' = \{z : \arg z = \psi_0, 0 \le |z| \le r\}$. Since

$$f^{(k-1)}(z) = f^{(k-1)}(0) + \int_{C'} f^{(k)}(u)du,$$

for sufficiently large $z = re^{i\psi_0}$, by induction, we have

$$|f(z)| \le M_7 |z|^k \qquad (M_7 > 0). \tag{3.10}$$

Combine case (i) and case (ii), for $\arg z = \psi_0 \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$ and $|z| = r \ge r_0(\psi_0) > 0$, we obtain

$$|f(z)| \le M(\psi_0)|z|^k, \tag{3.11}$$

where $M(\psi_0) > 0$ is a constant dependent only on ψ_0.

On the other hand, we can choose $\theta_j \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$ $(j = 1, 2, \dots, n, n+1)$ such that

$$0 \le \theta_1 < \theta_2 \dots < \theta_n < 2\pi, \theta_{n+1} = \theta_1 + 2\pi$$

and

$$\max\{\theta_{j+1} - \theta_j | 1 \le j \le n\} < \frac{\pi}{\sigma + 1}.$$

For any given positive number ϵ, we have

$$\frac{|f(z)|}{|z^k|} \le |f(z)| \le \exp\{\epsilon r^{\sigma+1}\}$$

for sufficiently large $r = |z|$. From (3.10) and Lemma 2.3, $\frac{|f(z)|}{|z^k|} \le M' (M'$ is a positive constant) holds in the sectors $\{z : \theta_j \le argz \le \theta_{j+1}, |z| \ge r\}$ $(j =$

$1, 2, \dots, n$) for sufficiently large r. Therefore, $\frac{|f(z)|}{|z^k|} \leq M''$ holds in the whole plane, where M'' is a positive constant. Thus $f(z)$ is a polynomial. It is a contradiction, and hence $\sigma(f) = \infty$.

Proof of Theorem 1.5. Assume that $f(z)$ ia a transcendental solution of (1.2) with $\sigma(f) = \sigma < +\infty$. Set $\omega = \max\{\sigma(B_j), \sigma(Q_j), 0 \leq j \leq k-1\}$.

(1) If there exist $t, s \in \{0, 1, \dots, k-1\}$, such that $\frac{a_{t,n}}{a_{s,n}} < 0$. By the similar discussion to Theorem 1.4, we take $\arg z = \psi_0 \in [0, 2\pi) - (E_1 \cup E_2 \cup E_3)$. So either $\delta(P_t, \psi_0) > 0$ or $\delta(P_s, \psi_0) > 0$. Therefore, not all δ_0, $\delta_1, \dots, \delta_{k-1}$ are negative. By Lemma 2.5, we can obtain (3.2). Following the proof of (i) of Theorem 1.4, we can get (3.6) and (3.10). Then $\sigma(f) = \infty$.

(2) By Lemma 2.1, for any given ϵ_0 with $0 < \epsilon_0 < \min\{\frac{1}{2}, \frac{n-\omega}{2}\}$, there exists a set $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, we have

$$\frac{|f^{(j)}(z)|}{|f^{(i)}(z)|} \leq |z|^{k\sigma}, \quad i = 0, 1, \dots, k-1;\ j = i+1, \dots, k \tag{3.12}$$

as $z \to \infty$ along $\arg z = \theta$. For $B_j e^{P_j}$, suppose that $H_j' \subset [0, 2\pi)$ is the exceptional set applying Lemma 2.2 to $B_j e^{P_j} (j = 0, 1, \dots, k-1)$. Then $E_5 = \bigcup_{j=0}^{k-1} H_j$ has linear measure zero. Since $\arg a_{0,n} \neq \arg a_{1,n}$, it is obvious that there exists a ray $\arg z = \phi_0 \in [0, 2\pi) \setminus (E_4 \cup E_5)$ such that $\delta(P_0, \phi_0) > 0$ and $\delta(P_1, \phi_0) < 0$. By Lemma 2.2, for sufficiently large r, we have

$$|B_0(re^{i\phi_0}) e^{P_0(re^{i\phi_0})} + Q_0(re^{i\phi_0})| \geq \exp\{(1-\epsilon_0)\delta(P_0, \phi_0) r^n\} \tag{3.13}$$

and

$$\begin{aligned} &|B_1(re^{i\phi_0}) e^{P_1(re^{i\phi_0})} + Q_1(re^{i\phi_0})| \\ &\leq \exp\{(1-\epsilon_0)\delta(P_1, \phi_0) r^n\} \exp\{r^{\omega+\epsilon_0}\} + exp\{r^{\omega+\epsilon_0}\}. \end{aligned} \tag{3.14}$$

So for $j = 2, 3, \dots, k-1$, we obtain

$$\begin{aligned} &|B_j(re^{i\phi_0}) e^{P_j(re^{i\phi_0})} + Q_j(re^{i\phi_0})| \\ &\leq \exp\{(1-\epsilon_0) c_j \delta(P_1, \phi_0) r^n\} \exp\{r^{\omega+\epsilon_0}\} + exp\{r^{\omega+\epsilon_0}\}. \end{aligned} \tag{3.15}$$

From (1.2), we have

$$\begin{aligned} &|A_0(re^{i\phi_0})| \\ &\leq \frac{|f^{(k)}(re^{i\phi_0})|}{|f(re^{i\phi_0})|} + |A_{k-1}(re^{i\phi_0})| \cdot \frac{|f^{(k-1)}(re^{i\phi_0})|}{|f(re^{i\phi_0})|} + \dots + |A_1(re^{i\phi_0})| \frac{|f'(re^{i\phi_0})|}{|f(re^{i\phi_0})|}. \end{aligned} \tag{3.16}$$

Combine (3.12)–(3.16), we have

$$\begin{aligned} &\exp\{(1-\epsilon_0)\delta(P_0, \phi_0) r^n\} \\ &\leq r^{k\sigma} + r^{k\sigma} [(\exp\{(1-\epsilon_0)\delta(P_1, \phi_0) r^n\} \exp\{r^{\omega+\epsilon_0}\} + \exp\{r^{\omega+\epsilon_0}\}) \\ &\quad + \Sigma_{j=2}^{k-1} (\exp\{(1-\epsilon_0) c_j \delta(P_1, \phi_0) r^n\} \exp\{r^{\omega+\epsilon_0}\} + \exp\{r^{\omega+\epsilon_0}\})]. \end{aligned}$$

This is impossible, since $\omega + \epsilon_0 < n$.

4. Proof of Theorem 1.6

Lemma 4.1 ([2]). *Let $f(z)$ be an entire function with $\sigma(f) = \infty$ and $\sigma_2(f) = \alpha < +\infty$, let a set $E \subset [1, \infty)$ has finite logarithmic measure. Then there exists a sequence $\{z_k = r_k e^{i\theta_k}\}$ satisfying $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, and for any given $\epsilon_1 > 0$, as $r_k \to \infty$, we have the following properties:*

(i) If $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then
$$\exp\{r_k^{\alpha-\epsilon_1}\} < v(r_k) < exp\{r_k^{\alpha+\epsilon_1}\},$$
where $v(f)$ is the central index of f.

(ii) If $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given constant $M(> 0)$,
$$r_k^M < v(r_k) < exp\{r_k^{\epsilon_1}\}.$$

Lemma 4.2 ([2]). *Let $A_j (0 \le j \le k-1)$ be an entire function with $\sigma(A_j) \le \sigma < \infty$. Then every non-trivial solution f of (1.2) satisfies $\sigma_2(f) \le \sigma$.*

Proof of Theorem 1.6. Assume that $f(z)$ is a solution of (1.2). Clearly f is entire. Since $P_0 + Q_0 \not\equiv 0$, f can not be a constant function. Compare with two sides of (1.2), f can not be a polynomial whose degree is equal or greater than 1.

Step 1: We prove that $\sigma(f) = \infty$. If it is not true. Assume $\sigma(f) = \sigma < +\infty$. By Lemma 2.1, for any given $\epsilon_0 (0 < \epsilon_0 < 1)$, there exists a subset $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, there is a constant $R_0 > 1$, such that for $\arg z = \psi_0$ and $|z| > R_0$, we have
$$\frac{|f^{(j)}(z)|}{|f^{(i)}(z)|} \le |z|^{k\sigma}, \quad i = 0, 1, \dots, k-1;\ j = i+1, \dots, k. \tag{4.1}$$
Take a ray $\arg z = \psi_0 \in [0, 2\pi) \setminus E_1$, we consider the following two cases:

Case A1: $\delta(R, \psi_0) > 0$. We claim that $|f^{(d)}(z)|$ is bounded on the ray $\arg z = \psi_0$. Suppose that it is not. Following the proof of Theorem 1.4, we have
$$\frac{|f^{(d-j)}(z_n)|}{|f^{(d)}(z_n)|} \le (1 + \circ(1))|z_n|^j, \quad j = 1, 2, \dots, d, \tag{4.2}$$
as $z_n \to \infty$. Since $f^{(d)} \not\equiv 0$, from (1.2),
$$\begin{aligned} A_d(z) = (-1)(&\frac{f^{(k)}(z)}{f^{(d)}(z)} + \dots + A_{d+1}(z) \cdot \frac{f^{(d+1)}(z)}{f^{(d)}(z)} + A_{d-1}(z) \cdot \frac{f^{(d-1)}(z)}{f^{(d)}(z)} \\ &+ \dots + A_0(z) \cdot \frac{f(z)}{f^{(d)}(z)}) \end{aligned}$$
holds, as $z \to \infty$. By (4.1) and (4.2), as $z_n \to \infty$, we obtain
$$|P_d(e^{R(z_n)}) + Q_d(e^{-R(z_n)})| \le r^M \cdot \Sigma_{j\ne d} |P_j(e^{R(z_n)}) + Q_j(e^{-R(z_n)})|, \tag{4.3}$$
where M is a constant. By (3), we obtain
$$|P_d(e^{R(z_n)}) + Q_d(e^{-R(z_n)})| = |a_{dm_d}| e^{m_d r^s \delta(R,\theta)} (1 + o(1)) \tag{4.4}$$
and
$$|P_j(e^{R(z_n)}) + Q_j(^{-R(z_n)})| \le |a_{jm_j}| e^{m_j r^s \delta(R,\theta)} (1 + o(1)) + M_1, j \ne d, \tag{4.5}$$
where M_1 is a positive constant. Substituting (4.4) and (4.5) into (4.3), we obtain a contradiction since $m_d > m_j (j \ne d) \ge 0$. Hence, $|f^{(d)}(re^{\psi_0})|$ is bounded on the ray $\arg z = \psi_0$. By the similar discussion to Theorem 1.4, we can obtain (3.6).

Case A2: $\delta(R,\psi_0)<0$. By a similar discussion to subcase A1 and noting that (4.4) and (4.5) can be substituted by

$$|P_d(e^{R(z)})+Q_d(e^{-R(z)})| = |b_{dt_d}|e^{t_d r^s \delta(R,\psi_0)}(1+o(1)), \tag{4.6}$$

and

$$|P_j(e^{R(z)})+Q_j(e^{-R(z)})| \le |b_{jt_j}|e^{t_j r^s \delta(R,\psi_0)}(1+o(1))+M_2. \tag{4.7}$$

Thus, we can deduce (3.10).

Combine Case A1 and Case A2, we have (3.11). Following the proof of Theorem 1.4, we can also obtain a contradiction.

Step 2: In this step, we prove $\sigma_2(f)=s$. By Lemma 4.2, we have

$$\sigma_2(f)\le s. \tag{4.8}$$

Now we assume that $\sigma_2(f)=\alpha<s$, we will get a contradiction.

Recall the Wiman-Valiron theory [9], there exists a subset $E_3\subset(1,\infty)$ that has finite logarithmic measure, such that for $|z|=r\notin E_3\cup[0,1]$ and $|f(z)|=M(r,f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = (\frac{v(r)}{z})^j(1+o(1))(j=1,2,\dots,k), \tag{4.9}$$

where $v(r)$ is central index of $f(z)$.

If $\sigma_2(f)=\alpha$ $(0<\alpha<s)$, from Lemma 4.1, we can take a sequence of points $\{z_n=r_ne^{i\theta_n}\}$ satisfying $|f(z_n)|=M(r_n,f)$, $\theta_n\in[0,2\pi)$, $lim_{n\to\infty}\theta_n=\theta_0\in[0,2\pi)$, for any given $\epsilon_1(0<\epsilon_1<min\{\alpha,s-\alpha\})$ and $r_n\notin E_2\cup E_3\cup[0,1]$, we obtain

$$\exp\{r_n^{\alpha-\epsilon_1}\}<v(r_k)<\exp\{r_n^{\alpha+\epsilon_1}\}, \tag{4.10}$$

as $r_n\to\infty$. If $\sigma_2(f)=\alpha=0$, then for any positive constant M, we have

$$r_n^M<v(r_n)<exp\{r_n^{\epsilon_1}\}, \tag{4.11}$$

as $r_n\to\infty$.

In the following, we consider three cases:

Case B1: $\delta(R,\theta_0)>0$. From (1.2), we have

$$\begin{aligned} A_d(z)(\frac{f^{(d)}(z)}{f(z)}) = (-1)\Big\{&\frac{f^{(k)}(z)}{f(z)}+\dots+A_{d+1}(z)\cdot\frac{f^{(d+1)}(z)}{f(z)}\\ &+A_{d-1}(z)\cdot\frac{f^{(d-1)}(z)}{f(z)}+\dots+A_0(z)\Big\}. \end{aligned} \tag{4.12}$$

For sufficiently large n, $\delta(R,\theta_n)>0$ since $\theta_n\to\theta_0$. For the point range $\{z_n=r_ne^{i\theta_n}\}$, combine (2.1), (4.9) and (4.12), we obtain

$$\begin{aligned} &|a_{dm_d}|e^{m_d r_n^s\delta(R,\theta_n)}|1+o(1)|(\frac{v(r_n)}{r_n})^d\\ &\le (\frac{v(r_n)}{r_n})^k+\dots+(\frac{v(r_n)}{r_n})^{d+1}(|a_{d+1m_{d+1}}|e^{m_{d+1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\ &\quad+(\frac{v(r_n)}{r_n})^{d-1}(|a_{d-1m_{d-1}}|e^{m_{d-1}r_n^s\delta(R,\theta_n)}|1+o(1)|\\ &\quad+\dots+|a_{0m_0}|e^{m_0 r_n^s\delta(R,\theta_n)}|1+o(1)|. \end{aligned}$$

By (4.10) or (4.11), we obtain

$$\begin{aligned}
&|a_{dm_d}|e^{m_d r_n^s \delta(R,\theta_n)}|1+o(1)|(\frac{\exp(dr_n^{\alpha-\epsilon_1})}{r_n^d})\\
&\leq (\frac{\exp(kr_n^{\alpha+\epsilon_1})}{r_n^k}) + \dots + (\frac{\exp((d+1)r_n^{\alpha+\epsilon_1})}{r_n^{d+1}})(|a_{d+1m_{d+1}}|e^{m_{d+1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + (\frac{\exp((d-1)r_n^{\alpha+\epsilon_1})}{r_n^{d-1}})(|a_{d-1m_{d-1}}|e^{m_{d-1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + \dots + |a_{0m_0}|e^{m_0 r_n^s\delta(R,\theta_n)}|1+o(1)|
\end{aligned}$$

or

$$\begin{aligned}
&|a_{dm_d}|e^{m_d r_n^s \delta(R,\theta_n)}|1+o(1)|(\frac{r_n^M}{r_n^d})\\
&\leq (\frac{\exp(kr_n^{\epsilon_1})}{r_n^k}) + \dots + (\frac{\exp((d+1)r_n^{\epsilon_1})}{r_n^{d+1}})(|a_{d+1m_{d+1}}|e^{m_{d+1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + (\frac{\exp((d-1)r_n^{\epsilon_1})}{r_n^{d-1}})(|a_{d-1m_{d-1}}|e^{m_{d-1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + \dots + |a_{0m_0}|e^{m_0 r_n^s\delta(R,\theta_n)}|1+o(1)|.
\end{aligned}$$

Since $m_d > m_j (j \neq d)$ and $\alpha + \epsilon_1 < s$, the above two inequalities are impossible. This shows case B1 can not occur.

Case B2: $\delta(R,\theta_0) < 0$. For sufficiently large n, $\delta(R,\theta_n) < 0$ since $\theta_n \to \theta_0$. Following the discussion of Subcase B1, we have

$$\begin{aligned}
&|b_{dt_d}|e^{-t_d r_n^s \delta(R,\theta_n)}|1+o(1)|(\frac{\exp(dr_n^{\alpha-\epsilon_1})}{r_n^d})\\
&\leq (\frac{\exp(kr_n^{\alpha+\epsilon_1})}{r_n^k}) + \dots + (\frac{\exp((d+1)r_n^{\alpha+\epsilon_1})}{r_n^{d+1}})(|b_{d+1t_{d+1}}|e^{-t_{d+1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + (\frac{\exp((d-1)r_n^{\alpha+\epsilon_1})}{r_n^{d-1}})(|b_{d-1t_{d-1}}|e^{-t_{d-1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + \dots + |b_{0t_0}|e^{-t_0 r_n^s\delta(R,\theta_n)}|1+o(1)|
\end{aligned}$$

or

$$\begin{aligned}
&|b_{dt_d}|e^{-t_d r_n^s \delta(R,\theta_n)}|1+o(1))|(\frac{r_n^M}{r_n^d})\\
&\leq (\frac{\exp(kr_n^{\epsilon_1})}{r_n^k}) + \dots + (\frac{\exp((d+1)r_n^{\epsilon_1})}{r_n^{d+1}})(|b_{d+1t_{d+1}}|e^{-t_{d+1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + (\frac{\exp((d-1)r_n^{\epsilon_1})}{r_n^{d-1}})(|b_{d-1t_{d-1}}|e^{-t_{d-1}r_n^s\delta(R,\theta_n)})|1+o(1)|\\
&\quad + \dots + |b_{0t_0}|e^{-t_0 r_n^s\delta(R,\theta_n)}|1+o(1)|.
\end{aligned}$$

Since $t_d > t_j (j \neq d)$ and $\alpha + \epsilon_1 < s$, we also obtain a contradiction.

Case B3: $\delta(R,\theta_0) = 0$. If there exists a subsequence of $\{\theta_n\}$ such that $\delta(R,\theta_n) > 0$ or $\delta(R,\theta_n) < 0$. Then by case B1 and case B2, we can get a contradiction.

Now, suppose that for sufficiently large n, $\delta(R,\theta_n) = 0$. Then we consider three subcases: $\delta_{s-1}(R,\theta) < 0$;$\delta_{s-1}(R,\theta) > 0$; $\delta_{s-1}(R,\theta) = 0$. If $\delta_{s-1}(R,\theta) < 0$ or $\delta_{s-1}(R,\theta) > 0$. Then replace $\delta(R,\theta)$ by $\delta_{s-1}(R,\theta)$ in the case B1 and B2, we can obtain a contradiction. If $\delta_{s-1}(R,\theta)=0$, from the previous discussion in case B3, the remain case is $\delta_{s-1}(R,\theta)=0$ and $\delta_{s-1}(R,\theta_n)=0$ for sufficiently large n. Then we can consider $\delta_{s-2}(R,\theta)$, and we can also obtain a contradiction. On the analogy by this, the remain case is that $\delta_j(R,\theta_n)=0$ for $j\in\{1,2,\dots,s\}$ and for sufficiently large n.

Rewriting (1.2), we have

$$\begin{aligned}(-\frac{v(r_n)}{z_n})^k(1+o(1)) &= A_{k-1}(z_n)(\frac{v(r_n)}{z_n})^{k-1}(1+o(1))+\dots \\ &\quad + A_d(z_n)(\frac{v(r_n)}{z_n})^d(1+o(1))+\dots+A_0(z_n).\end{aligned}\tag{4.13}$$

For $z_n=r_ne^{\theta_n}$, since $\delta_j(R,\theta_n)=0$ for $j\in\{1,2,\dots,s\}$, it leads to

$$|A_j(z_n)|=|P_j(e^{R(z_n)})+Q_j(e^{-R(z_n)})|\le M, j\in\{1,2,\dots,k\},\tag{4.14}$$

where M is a constant. From (4.14), we obtain

$$v(r_n)\le Br_n^k,\tag{4.15}$$

where B is a constant. However, this contradicts (4.10) and (4.11). Therefore, case B3 can not occur.

Combining case B1, B2 and B3, we have $\sigma_2(f)=s$. $\square$

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