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# SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION RELATED TO THE OPLUS OPERATOR

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Abstract. In this article, we consider the equation

$$\oplus^k u(x) = \sum_{r=0}^{m} c_r \oplus^r \delta$$

where $\oplus^k$ is the operator iterated $k$ times and defined by

$$\oplus^k = \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^4 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^4\Big)^k,$$

where $p+q=n$, $x=(x_1,x_2,\dots,x_n)$ is in the $n$-dimensional Euclidian space $\mathbb{R}^n$, $c_r$ is a constant, $\delta$ is the Dirac-delta distribution, $\oplus^0\delta = \delta$, and $k = 0,1,2,3,\dots$. It is shown that, depending on the relationship between $k$ and $m$, the solution to this equation can be ordinary functions, tempered distributions, or singular distributions.

## 1. Introduction

The diamond operator, iterated $k$ times, was studied by Kananthai [2], and is defined by

$$\Diamond^k = \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k, \quad p+q=n, \tag{1.1}$$

where $n$ is the dimension of the space $\mathbb{R}^n$, $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, and $k$ is a nonnegative integer. This operator can be expressed as

$$\Diamond^k = \Delta^k \Box^k = \Box^k \Delta^k \tag{1.2}$$

where $\Delta^k$ is the Laplacian operator iterated $k$ times, defined by

$$\Delta^k = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}\Big)^k, \tag{1.3}$$

and $\Box^k$ is the Ultra-hyperbolic operator iterated $k$ times, defined by

$$\Box^k = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}\Big)^k. \tag{1.4}$$

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Kananthai [2] showed that the convolution

$$u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$$

is a unique elementary solution of the operator $\Diamond^k$, where $R^e_{2k}(x)$ and $R^H_{2k}(x)$ are defined by (2.5) and (2.2) with $\alpha = 2k$ respectively; that is,

$$\Diamond^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta\,. \tag{1.5}$$

Satsanit [7] introduced the $\circledcirc^k$ operator, defined by

$$\circledcirc^k = \Big(\big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k.$$

From (1.3) and (1.4), we obtain

$$\begin{aligned}
\circledcirc^k &= \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k \\
&= \Big(\Big(\frac{\Delta + \Box}{2}\Big)^2 + \Big(\frac{\Delta - \Box}{2}\Big)^2\Big)^k \\
&= \Big(\frac{\Delta^2 + \Box^2}{2}\Big)^k.
\end{aligned} \tag{1.6}$$

The $\oplus^k$ operator has been studied by Kananthai, Suantai and Longani [4], and can be expressed in the form

$$\oplus^k = \Big[\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big]^k \cdot \Big[\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big]^k \tag{1.7}$$

Thus, (1.7) can be written as

$$\oplus^k = \Diamond^k \circledcirc^k, \tag{1.8}$$

where $\Diamond^k$ and $\circledcirc^k$ are defined by (1.1), (1.6) respectively.

The purpose of this article, is finding the solution to the equation

$$\oplus^k u(x) = \sum_{r=0}^{m} c_r \oplus^r \delta \tag{1.9}$$

by using convolutions of the generalized function. It is also shown that the type of solution to (1.9) depends on the relationship between $k$ and $m$, according to the following cases:

(1) If $m < k$ and $m = 0$, then (1.9) has the solution

$$u(x) = c_0 \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right)$$

which is an elementary solution of the $\oplus^k$ operator in Theorem 3.1, is an ordinary function when $6k \geq n$, and is a tempered distribution when $6k < n$.

(2) If $0 < m < k$ then the solution of (1.9) is

$$u(x) = \sum_{r=1}^{m} c_r \left(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*(k-r)}(x))^{*-1}\right)$$

which is an ordinary function when $6k-6r \geq n$ and is tempered distribution when $6k - 6r < n$.

(3) If $m \geq k$ and $k \leq m \leq M$, then (1.9) has the solution

$$u(x) = \sum_{r=k}^{M} c_r \oplus^{r-k} \delta$$

which is only a singular distribution.

Before going that point, the following definitions and some concepts are needed.

## 2. Preliminaries

**Definition 2.1.** Let $x = (x_1, x_2, \dots, x_n)$ be a point in $\mathbb{R}^n$. Define

$$\upsilon = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ be the interior of a forward cone and let $\overline{\Gamma}_+$ denote its closure. For any complex number $\alpha$, define the function

$$R_\alpha^H(\upsilon) = \begin{cases} \frac{\upsilon^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}. \tag{2.3}$$

The function $R_\alpha^H(\upsilon)$ was introduced by Nozaki [5, p. 72] and is called the Ultra-hyperbolic kernel of Marcel Riesz.

It is well known that $R_\alpha^H(\upsilon)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is a distribution of $\alpha$ if $\mathrm{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(\upsilon)$ denote the support of $R_\alpha^H(\upsilon)$ and suppose $\operatorname{supp} R_\alpha^H(\upsilon) \subset \bar{\Gamma}_+$, that is $\operatorname{supp} R_\alpha^H(\upsilon)$ is compact.

From Trione [9, p. 11], $R_{2k}^H(\upsilon)$ is an elementary solution of the operator $\Box^k$; that is,

$$\Box^k R_{2k}^H(\upsilon) = \delta(x). \tag{2.4}$$

**Definition 2.2.** Let $x = (x_1, x_2, \dots, x_n)$ and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$. The elliptic kernel of Marcel Riesz and is defined as

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{2.5}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.6}$$

$\alpha$ is a complex parameter, and $n$ is the dimension of $\mathbb{R}^n$.

It can be shown that $R_{-2k}^e(x) = (-1)^k \Delta^k \delta(x)$ where $\Delta^k$ is defined by (1.3). It follows that $R_0^e(x) = \delta(x)$, [2, p. 118]. Moreover, $(-1)^k R_{2k}^e(x)$ is an elementary solution of the operator $\Delta^k$ [2, Lemma 2.4]; that is,

$$\Delta^k((-1)^k R_{2k}^e(x) = \delta(x). \tag{2.7}$$

**Lemma 2.3.** *The functions $R_{2k}^H(\upsilon)$ and $(-1)^k R_{2k}^e(x)$ are the elementary solutions of the operators $\Box^k$ and $\Delta^k$, defined by (1.4) and (1.3) respectively. The function $R_{2k}^H(\upsilon)$ is defined by (2.2) with $\alpha = 2k$, and $R_{2k}^e(x)$ is defined by (2.5) with $\alpha = 2k$.*

*Proof.* We need to show that $\Box^k R^H_{2k}(\upsilon) = \delta(x)$ which is done in [9, Lemma 2.4]. Also we need to show that $\Delta^k((-1)^k R^e_{2k}(x) = \delta(x)$. which is done in [2, p. 31]. $\square$

**Lemma 2.4.** *The convolution $R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\diamondsuit^k$ iterated $k$ as defined by* (1.1)*.*

For the proof of the above lemma see [2, p. 33].

**Lemma 2.5.** *The functions $R^H_\alpha(x)$ and $R^e_\alpha(x)$ defined by* (2.2) *and* (2.5) *respectively, for $Re(\alpha)$, are homogeneous distributions of order $\alpha - n$ and also a tempered distributions.*

*Proof.* Since $R^H_\alpha(x)$ and $R^e_\alpha(x)$ satisfy the Euler equation,

$$\begin{aligned}(\alpha - n)R^H_\alpha(x) &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R^H_\alpha(x),\\ (\alpha - n)R^e_\alpha(x) &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R^e_\alpha(x),\end{aligned}$$

we have that $R^H_\alpha(x)$ and $R^e_\alpha(x)$ are homogeneous distributions of order $\alpha - n$. Donoghue [1, pp. 154-155] proved that the every homogeneous distribution is a tempered distribution. This completes the proof. $\square$

**Lemma 2.6.** *The convolution $R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution.*

*Proof.* Choose $\operatorname{supp} R^H_\alpha(x) = K \subset \Gamma_+$ where $K$ is a compact set. Then $R^H_\alpha(x)$ is a tempered distribution with compact support. By Donoghue [1, pp. 156-159], $R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution. $\square$

**Lemma 2.7** (Convolution of $R^e_\alpha(x)$ and $R^H_\alpha(x)$)**.** *Let $R^e_\alpha(x)$ and $R^H_\alpha(x)$ defined by* (2.5) *and* (2.2) *respectively, then we obtain the following:*

(1) $R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x)$ *when $\alpha$ and $\beta$ are complex parameters;*
(2) $R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$ *when $\alpha$ and $\beta$ are integers, except when both $\alpha$ and $\beta$ are odd.*

*Proof.* For the first formula, see [1, p. 158]. For the second formula, when $\alpha$ and $\beta$ are both even integers; see [3]. For the case $\alpha$ is odd and $\beta$ is even or $\alpha$ is even and $\beta$ is odd, by Trione [8], we have

$$\Box^k R^H_\alpha(x) = R^H_{\alpha-2k}(x) \tag{2.8}$$

and

$$\Box^k R^H_{2k}(x) = \delta(x), \quad k = 0, 1, 2, 3, \dots \tag{2.9}$$

where $\Box^k$ is the Ultra-hyperbolic operator iterated $k$-times defined by

$$\Box^k = \Big(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^k.$$

Now let $m$ be an odd integer. We have $\Box^k R^H_m(x) = R^H_{m-2k}(x)$ and

$$R^H_{2k}(x) * \Box^k R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x)$$

or

$$\big(\Box^k R^H_{2k}(x)\big) * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x),$$
$$\delta * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x).$$

Since $m$ is odd, hence $m-2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k$, $\beta = m-2k$, we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$$

when $\alpha$ is nonnegative even and $\beta$ is odd.

For the case when $\alpha$ is negative even and $\beta$ is odd, by (2.8) we have

$$\Box^k R^H_0(x) = R^H_{-2k}(x)$$

or $\Box^k\delta = R^H_{-2k}(x)$, where $R^H_0(x) = \delta$. Now when $m$ is odd,

$$R^H_{-2k}(x) * \Box^k R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x)$$

or

$$\big(\Box^k\delta\big) * \Box^k R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x),$$
$$\delta * \Box^{2k} R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_{m-2(2k)}(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m-2k$, now $\alpha$ is negative even and $\beta$ is odd. Then we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x).$$

That completes the proof. $\square$

## 3. Main Results

**Theorem 3.1.** *Given the equation*

$$\oplus^k G(x) = \delta(x), \tag{3.1}$$

*where $\oplus^k$ is the oplus operator iterated $k$ times defined by* (1.8), *$\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}^n$, and $k$ is a nonnegative integer. Then*

$$G(x) = \big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} \tag{3.2}$$

*is a Green's function or an elementary solution for the operator $\oplus^k$, where*

$$C(x) = \frac{1}{2}R^H_4(x) + \frac{1}{2}(-1)^2 R^e_4(x), \tag{3.3}$$

*where $C^{*k}(x)$ denotes the convolution of $C$ with itself $k$ times, $\big(C^{*k}(x)\big)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $G(x)$ is a tempered distribution.*

For a proof of the above theorem, see [6].

**Theorem 3.2.** *For $0 < r < k$,*

$$\begin{aligned} &\oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) \\ &= \Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*(k-r)}(x))^{*-1}\Big) \end{aligned} \tag{3.4}$$

*and for $k \le m$,*

$$\oplus^m \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) = \oplus^{m-k}\delta. \tag{3.5}$$

*Proof.* For $0 < r < k$, from (3.1),

$$\oplus^k \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) = \delta.$$

Thus,

$$\oplus^{k-r} \oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) = \delta$$

or

$$\oplus^{k-r}\delta * \oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) = \delta.$$

Convolving both sides by $\Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*k}(x))^{*-1}\Big)$, we obtain

$$\begin{aligned} &\oplus^{k-r} \Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*k}(x))^{*-1}\Big) \\ &* \oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) \\ &= \Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*(k-r)}(x))^{*-1}\Big) * \delta. \end{aligned}$$

By theorem 3.1,

$$\begin{aligned} &\delta * \oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) \\ &= \Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*(k-r)}(x))^{*-1}\Big) * \delta. \end{aligned}$$

It follows that

$$\begin{aligned} &\oplus^r \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) \\ &= \Big(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{6(k-r)}(x)) * (C^{*(k-r)}(x))^{*-1}\Big) \end{aligned}$$

as required. For $k \le m$

$$\begin{aligned} &\oplus^m \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) \\ &= \oplus^{m-k} \oplus^k \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right). \end{aligned}$$

It follows that

$$\oplus^m \left(((-1)^{3k} R^e_{6k}(x) * R^H_{6k}(x)) * (C^{*k}(x))^{*-1}\right) = \oplus^{m-k}\delta$$

by Theorem 3.1. This completes the proof. □

**Theorem 3.3.** *Consider the linear differential equation*

$$\oplus^k u(x) = \sum_{r=0}^{m} c_r \oplus^r \delta, \tag{3.6}$$

*where*

$$\oplus^k = \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^4 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^4\Big)^k,$$

*$p+q=n$, $n$ is odd with $p$ odd and $q$ even, or $n$ is even with $p$ odd and $q$ odd, $x\in\mathbb{R}^n$, $c_r$ is a constant, $\delta$ is the Dirac-delta distribution, and $\oplus^0\delta=\delta$. Then the type of solution to (3.6) depends on the relationship between $k$ and $m$, according to the following cases:*

(1) *If $m<k$ and $m=0$, then (3.6) has solution*
$$u(x)=c_0\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{*-1}\right)$$
*which is an elementary solution of the $\oplus^k$ operator in Theorem 3.1, when $6k\geq n$, and is a tempered distribution when $6k<n$.*

(2) *If $0<m<k$, then the solution of (3.6) is*
$$u(x)=\sum_{r=1}^{m}c_r\Big(((-1)^{3(k-r)}R^e_{6(k-r)}(x)*R^H_{6(k-r)}(x))*(C^{*(k-r)}(x))^{*-1}\Big)$$
*which is an ordinary function when $6k-6r\geq n$, and is a tempered distribution when $6k-6r<n$.*

(3) *If $m\geq k$ and $k\leq m\leq M$, then (3.6) has solution*
$$u(x)=\sum_{r=k}^{M}c_r\oplus^{r-k}\delta$$
*which is only a singular distribution.*

*Proof.* (1) For $m=0$, we have $\oplus^k u(x)=c_0\delta$, and by Theorem 3.1 we obtain
$$u(x)=\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{*-1}\right)$$
Now, $(-1)^{3k}R^e_{6k}(x)$ and $R^H_{6k}(x)$ are the analytic function for $6k\geq n$ and also $(-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)*(C^{*k}(x))^{-1}$ exists and is an analytic function by (3.2). It follows that $(-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)*(S^{*k}(x))^{-1}$ is an ordinary function for $6k\geq n$. By Lemma 2.3 with $\alpha=6k$, $(-1)^{3k}R^e_{6k}(x)$ and with $\alpha=6k$, $R^H_{6k}(x)$ are tempered distribution with $6k<n$, we obtain $(-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)*(C^{*k}(x))^{-1}$ exists and is a tempered distribution.

(2) For the case $0<m<k$, we have
$$\oplus^k u(x)=c_1\oplus\delta+c_2\oplus^2\delta+\dots+c_m\oplus^m\delta.$$
We convolved both sides of the above equation by $(-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)*(C^{*k}(x))^{-1}$ to obtain
$$\begin{aligned}
&\oplus^k\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{-1}\right)*u(x)\\
&=c_1\oplus\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{-1}\right)\\
&\quad+c_2\oplus^2\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{-1}\right)\\
&\quad+\dots+c_m\oplus^m\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{-1}\right).
\end{aligned}$$
By Theorems 3.1 and 3.2, we obtain
$$\begin{aligned}
u(x)&=c_1\Big(((-1)^{3(k-1)}R^e_{6(k-1)}(x)*R^H_{6(k-1)}(x))*(C^{*(k-1)}(x))^{*-1}\Big)\\
&\quad+c_2\Big(((-1)^{4(k-2)}R^e_{6(k-2)}(x)*R^H_{6(k-2)}(x))*(C^{*(k-2)}(x))^{*-1}\Big)\\
&\quad+\dots+c_m\Big(((-1)^{3(k-m)}R^e_{6(k-m)}(x)*R^H_{6(k-m)}(x))*(C^{*(k-m)}(x))^{*-1}\Big)
\end{aligned}$$

or

$$u(x)=\sum_{r=1}^{m}c_r\left(((-1)^{3(k-r)}R^e_{6(k-r)}(x)*R^H_{6(k-r)}(x))*(C^{*(k-r)}(x))^{*-1}\right).$$

Similarly, as in the case(1), $u(x)$ is an ordinary function for $6k-6r\geq n$ and is a tempered distribution for and $6k-6r<n$.

(3) For the case $m\geq k$ and $k\leq m\leq M$, we have

$$\oplus^k u(x)=c_k\oplus^k\delta+c_{k+1}\oplus^{k+1}\delta+\cdots+c_M\oplus^M\delta.$$

Convolved both sides of the above equation by $(-1)^{3k}R^e_{6k}(x)*R_{6k}(x)*(C^{*k}(x))^{*-1}$ to obtain

$$\begin{aligned}&\oplus^k\left(((-1)^{3k}R^e_{4k}(x)*R^H_{6k}(x))*(S^{*k}(x))^{*-1}\right)*u(x)\\&=c_k\oplus^k\left(((-1)^{2k}R^e_{6k}(x)*R^H_{6k}(x))*(S^{*k}(x))^{-1}\right)\\&\quad+c_{k+1}\oplus^{k+1}\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{*-1}\right)\\&\quad+\cdots+c_M\oplus^M\left(((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x))*(C^{*k}(x))^{*-1}\right).\end{aligned}$$

By Theorems 3.1 and 3.2 again, we obtain

$$u(x)=c_k\delta+c_{k+1}\oplus\delta+c_{k+2}\oplus^2\delta+\cdots+c_M\oplus^{M-k}\delta=\sum_{r=k}^{M}c_r\oplus^{r-k}\delta.$$

Since $\oplus^{r-k}\delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. This completes the proofs. □

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## References

[1] W. F. Donoghue; *Distributions and Fourier Transform*, Academic Press, New York, 1969.

[2] A. Kananthai; *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 1997, pp 27-37.

[3] A. Kananthai; *On the distribution related to the Ultra-hyperbolic equation*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 1997, pp. 101-106.

[4] A. Kananthai, S. Suantai, V. Longani; *On the operator $\oplus^K$ related to the wave equation and Laplacian*, Applied Mathematics and Computational, Elsevier Science Inc.New York, (2002), pp 219-229.

[5] Y. Nozaki; *On Riemann-Liouville integral of Ultra-hyperbolic type*, Kodai Math. Sem. Rep. 6(2)(1964) 69-87.

[6] W. Satsanit; *Green function and Fourier transform Oplus operator*, Electronic Journal of Differential Equations, Vol. 2010(2010), No. pp 1-14.

[7] W. Satsanit; *On the Green function of the operator $(\circledcirc+m^4)^k$ related to the Klein-Gordon operator and Helmholtz operator*, International of Pure and Applied Mathematics, Vol. 57, No 1, 2009, pp. 77-86.

[8] S. E. Trione; *On the Ultra-hyperbolic kernel*, Trabajos de Mathematica, 116 preprint (1987).

[9] S. E. Trione, M. Aguirre Tellez; *The distribution convolution products of Marcel Riesz's Ultra-hyperbolic kernel*, Ravista de la Union Mathematica Argentina 39 (1995) 115-124. preprint(1987).

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