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INSTABILITY OF ELLIPTIC EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS WITH NON-NEGATIVE RICCI CURVATURE

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Abstract. We prove the nonexistence of nonconstant local minimizers for a class of functionals, which typically appear in scalar two-phase field models, over smooth N-dimensional Riemannian manifolds without boundary and non-negative Ricci curvature. Conversely, for a class of surfaces possessing a simple closed geodesic along which the Gauss curvature is negative, we prove the existence of nonconstant local minimizers for the same class of functionals.

1. Introduction

Let $\mathcal{M}$ be a smooth N-dimensional compact Riemannian manifold without boundary. Consider the functional $\mathcal{E} : H^1(\mathcal{M}) \to \mathbb{R}$ given by

$$\mathcal{E}(u) = \int_{\mathcal{M}} \{\frac{|\nabla u|^2}{2} - F(u)\}\, d\mu, \tag{1.1}$$

where F is a C^2 real function and $H^1(\mathcal{M})$ the usual Sobolev space.

In this work, we are interested in knowing how locally minimizing functions of $\mathcal{E}$ are related to the geometry of $\mathcal{M}$.

We will say that $u_0 \in C^\infty(\mathcal{M})$ is a local minimizer of $\mathcal{E}$ if there exists $\delta > 0$ such that

$$\mathcal{E}(u_0) \le \mathcal{E}(u) \quad \text{whenever} \quad \|u - u_0\|_{H^1(\mathcal{M})} \le \delta.$$

In case the first inequality is strict, i.e., $\mathcal{E}(u_0) < \mathcal{E}(u)$, u_0 is said to be a local isolated minimizer. Our main results are stated in the following theorems.

Theorem 1.1. *Suppose that the Ricci curvature of $\mathcal{M}$ is non-negative. Then any local minimizer of $\mathcal{E}$ is a constant function.*

An interesting condition that shows up in the computations of Theorem 1.1 provides some insight on the structure of $\mathcal{M}$. For $u \in H^1(\mathcal{M})$ we denote by $\mathcal{E}''(u)$ the second variation of $\mathcal{E}$ at u. If u is a non-constant critical point of $\mathcal{E}$, the vector field ∇u spans a real line bundle $\mathcal{I}$ in some open non-empty subset of $\mathcal{M}$. Clearly, it is not true, in general, that $\mathcal{I}$ can be extended to the whole $\mathcal{M}$ in a unique way.

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Nevertheless this is precisely the case when a simple analytical condition, given by the next statement, holds for $\mathcal{E}''(u)$ and $|\nabla u|$.

Theorem 1.2. *Keep the hypothesis of Theorem 1.1 and the previous notation for u, $\mathcal{E}$ and $\mathcal{I}$. Set $v = |\nabla u|$. Then if*

$$(\mathcal{E}''(u)v, v) = 0, \tag{1.2}$$

the bundle $\mathcal{I}$ can be extended to $\mathcal{M}$ and is geodesic. There exists a complete Riemannian submanifold $\mathcal{N} \subset \mathcal{M}$ so that $\mathcal{I}|_{\mathcal{N}}$ is the normal bundle of $\mathcal{N}$ and is orientable. The geodesic flow $\varphi : \mathbb{R} \times \mathcal{N} \to \mathcal{M}$ in the direction of $\mathcal{I}|_{\mathcal{N}}$ is an isometric regular covering map. Denoting by K the group of covering transformations of φ, then K is made of isometries, so that $\mathcal{M}$ is isometric to the quotient $(\mathbb{R} \times \mathcal{N})/K$. If $\mathcal{I}$ is orientable, then K is generated by a nontrivial (affine) translation of $\mathbb{R}$ with some isometry of $\mathcal{N}$. Otherwise K is generated by two involutions of $\mathbb{R} \times \mathcal{N}$.

Regarding Theroem 1.1, we show how to construct non-constant local minimizers on some non-convex surfaces. For that purpose, we introduce a small positive parameter ε in the functional; thus writing by considering the functional

$$\mathcal{E}_\varepsilon(u) = \int_{\mathcal{M}} \{\varepsilon \frac{|\nabla u|^2}{2} - \varepsilon^{-1} F(u)\}\, d\mu, \tag{1.3}$$

and take as F a suitable nonnegative double-well potential which vanishes only at α and β ($\alpha < \beta$). As usual χ_A stands for the characteristic function of a set A.

Theorem 1.3. *Let $\mathcal{M}$ be a surface diffeomorphic to S^2. Assume that there exists a simple closed geodesic $\gamma_0 \subset \mathcal{M}$ so that the Gauss curvature K of $\mathcal{M}$ is negative along γ_0. Then for ε small enough there is a non-constant family $\{u_\varepsilon\}_{\varepsilon>0}$ of local minimizers of $\mathcal{E}_\varepsilon$. Moreover $u_\varepsilon \to 0$, as $\varepsilon \to 0$, in $L^1(\mathcal{M})$ where $u_0 = \alpha\chi_{\mathcal{M}_\alpha} + \beta\chi_{\mathcal{M}_\beta}$ and $\mathcal{M} = \mathcal{M}_\alpha \cup \gamma_0 \cup \mathcal{M}_\beta$ is the partition of $\mathcal{M}$ determined by γ_0 .*

Remark 1.4. It will be shown, under the assumed hypotheses, that condition (1.2) can only happen if $\mathrm{Ric}(\nabla u, \nabla u) \equiv 0$. This fits naturally in the vast field of Ricci-flat compact manifolds. The conclusions of Theorem 1.2 may be an extra analytical tool in the study of such manifolds when combined with several deep results already achieved in this field. Compare, for instance, with Theorem 4.1 of [6], where the authors prove a factorization of a covering space of a Ricci-flat manifold M^n into a product $T^k \times M^{n-k}$, with T^k a flat torus and M^{n-k} a lower dimension Ricci-flat manifold, and k being the first Betti number of M^n. Also, after Yau's results on the existence of non-flat Ricci-flat manifolds it is evident that condition (1.2) cannot happen in a $K3$-surface with its Ricci-flat metric.

Associating local minimizers of $\mathcal{E}$ with the geometry of the domain goes back to 1978 when the authors in [4] and [19] considered the evolution problem

$$\begin{gathered} u_t = \Delta u + f(u) \quad \text{in } \mathbb{R}^+ \times \Omega \\ \partial_\nu u = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f \in C^2(\Omega)$ and ∂_ν stands for the exterior normal derivative.

They showed that if Ω is convex then any non-constant solution to (1.4) is unstable in the Lyapunov sense. In this case it amounts to saying that any local minimizer of the corresponding energy functional is a constant function.

Still for bounded convex domains with homogeneous zero Neumann boundary condition, the same kind of result was obtained for systems of reaction-diffusion equations [11] and [18], Ginzburg-Landau equation [12], reaction-diffusion systems with skew-gradient structure [23], geometric parabolic equation [13] and in the context of permanent currents for the full bi-dimensional Ginzburg-Landau functional in [12], among others. In all of these works the proofs make use in a strong way of the homogeneous Neumann boundary condition on a convex domain.

When $\mathcal{M}$ is a general Riemannian manifold without boundary, the Euler-Lagrange equation for $\mathcal{E}$ yields stationary solutions of the reaction-diffusion equation

$$u_t = \Delta u + f(u) \quad \text{in } \mathbb{R} \times \mathcal{M}\,. \tag{1.5}$$

The only result of this type regarding (1.5) over surfaces was considered in [21] where it was shown that if $\mathcal{M} \subset \mathbb{R}^3$ is a convex surface of revolution then the only stable solutions are the constant ones. Actually the proof consists of showing that (1.1), with $F' = f$, has no nonconstant local minimizer. In this particular case writing the planar curve that generates the surface in appropriate coordinates reduces the domain to an interval thus making the underlying analysis much easier than the general case considered here.

The present work generalizes the latter not only in the dimension of the manifold, but also by showing that only the sign of the (Ricci) curvature is what matters.

We should mention that after this work had been completed it was brought to our attention that the conclusion of Theorem 1.1 had appeared in [10] but with only sketched or incomplete proofs.

In case $\mathcal{M}$ is a bounded domain in $\mathbb{R}^N$ typically $\mathcal{E}_\varepsilon$ models the phase separation phenomenon in the context of van der Waals-Cahn-Hilliard theory whereby u represents the density of a two-phase fluid and is also associated to the motion of phase boundaries (interfaces) by mean curvature (see [9], for instance).

Equation (1.5) has been studied in the context of pattern formation; i.e., existence of nonconstant stable (in the sense of Lyapunov) stationary solution. It may model bio-chemical processes over cell surfaces or propagation of calcium waves over the surface of a fertilized egg, for instance.

In particular Theorem 1.1 implies that (1.5) has no pattern as long as $\mathcal{M}$ has non-negative Ricci curvature. On the other hand Theorem 1.3 gives an example of $\mathcal{M}$ for which (1.5), after a suitable scaling, develops patterns.

Setting $f = F'$ then clearly critical points of $\mathcal{E}$ satisfy the semi-linear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{on } \mathcal{M}. \tag{1.6}$$

A smooth solution u of the above equation is said to be weakly stable if the quadratic form satisfies

$$E(\varphi) = \int_{\mathcal{M}} \{\frac{|\nabla \varphi|^2}{2} - f'(u)\varphi^2\}\, d\mu \geq 0, \tag{1.7}$$

in $H^1(\mathcal{M})$. Otherwise u is called weakly unstable. Then it follows immediately from the proof of Theorem 1.1 that any nonconstant solution to the above equation is weakly unstable as long as $\mathcal{M}$ has non-negative Ricci curvature.

This article is organized as follows. In Section 2 in addition to recalling some notation of Riemmanian Geometry we prove some preliminary results, Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2 and Section 4 to the proof of Theorem 1.3.

2. Geometric background and notation

Let $\mathcal{M}$ be an N-dimension ($N \geq 2$) Riemannian manifold without boundary, and $T\mathcal{M}$, $T^*\mathcal{M}$ its tangent and cotangent bundles, respectively. Let $T_s^r(\mathcal{M}) = (T\mathcal{M})^{\otimes r} \otimes (T^*\mathcal{M})^{\otimes s}$, for non-negative integers r and s. For an integer $k \geq 0$ let $\mathcal{A}^k T^*\mathcal{M}$ be the alternate k-bundle of $T^*\mathcal{M}$. Given any real vector bundle $\mathcal{F}$ over $\mathcal{M}$ we denote by $G(\mathcal{F})$ the set of its smooth sections and by $G^k(\mathcal{F}) = G(\mathcal{A}^k T^*\mathcal{M} \otimes \mathcal{F})$ the smooth sections of *k-forms on* $\mathcal{M}$ *with coefficients in* $\mathcal{F}$.

The contraction is a natural coupling $c : T_1^1(\mathcal{M}) \to T_0^0(\mathcal{M})$ given by $c(v \otimes \omega) = \omega(v)$, where $v \otimes \omega$ is a decomposable tensor of $T\mathcal{M} \otimes T^*\mathcal{M}$. The contraction extends to $c : T_s^r(\mathcal{M}) \to T_{s-1}^{r-1}(\mathcal{M})$ for any $r, s \geq 1$, by putting $c(v_1 \otimes \cdots \otimes v_r \otimes \omega_s \otimes \cdots \otimes \omega_1) = \omega_1(v_1)\ v_2 \otimes \cdots \otimes v_r \otimes \omega_s \otimes \cdots \otimes \omega_2$. Indeed, when $r = s = 1$ the contraction is just the trace operator on linear homomorphisms $T\mathcal{M} \to T\mathcal{M}$.

Let $\nabla : G(T_0^1(\mathcal{M})) \longrightarrow G^1(T_0^1(\mathcal{M}))$ be the Levi-Civita connection on $\mathcal{M}$. It is well known that ∇ can be extended in a unique way to an operator $\overline{\nabla} : G(T_s^r(\mathcal{M})) \longrightarrow G^1(T_s^r(\mathcal{M}))$ such that Leibnitz rule is preserved and commutes with the contraction [15]. We abuse notation and write $\overline{\nabla} = \nabla$ whenever r, s are not both zero. When $f \in G(T_0^0(\mathcal{M}))$ is just a smooth function we preserve the usual notation $\nabla f = (df)^* \in G(T_0^1(\mathcal{M}))$. It then follows that

$$\nabla(T \otimes W) = \nabla T \otimes W + T \otimes \nabla W, \quad \forall T \in G(T_s^r(\mathcal{M})) \text{ and } \forall W \in G(T_q^p(\mathcal{M}))\,, \tag{2.1}$$

$$\nabla c(T) = c(\nabla T)\,, \quad \text{for any contraction } c : T_s^r(\mathcal{M}) \to T_{s-1}^{r-1}(\mathcal{M})\,. \tag{2.2}$$

Notice that we identify

$$\begin{aligned} &(T\mathcal{M})^{\otimes r} \otimes (T^*\mathcal{M})^{\otimes s} \otimes (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q} \\ &\cong (T\mathcal{M})^{\otimes r} \otimes (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes s} \otimes (T^*\mathcal{M})^{\otimes q}\,, \end{aligned} \tag{2.3}$$

and similarly, by sticking the 1-form component of a section of $\mathcal{A}^1 T^*\mathcal{M} \otimes (T_s^r(\mathcal{M}))$ on the left of the covariant part we have $\mathcal{A}^1 T^*\mathcal{M} \otimes T_s^r(\mathcal{M}) \cong T_{s+1}^r(\mathcal{M})$. These identifications are necessary for (2.1) and (2.2) to make sense. They also allow us to define the composition $\nabla(\nabla T)$ for any $T \in G(T_s^r(\mathcal{M}))$.

Some combinations of $\otimes$ and $c(\cdot)$ deserve special notation. For tensors $T \in G(T_s^1(\mathcal{M}))$ and $W \in G(T_q^1(\mathcal{M}))$ we write $TW = c(W \otimes T)$. When $s = 1$ and $q = 1$, TW is the composition of the endomorphisms T with W, and if $q = 0$ TW is the image of the vector W under T. In particular, if $s \geq 2$ and W_1, W_2 are vector fields we set $T(W_1, W_2) = [TW_2]W_1$.

Let $F \in T_3^1(\mathcal{M})$ be the Riemann tensor of $\mathcal{M}$. Then for any vector fields X, Y, Z and W locally defined we have

$$\begin{aligned} F(X, Y, Z, W) :=& \langle [FZ](Y, X), W\rangle \\ =& \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle\,. \end{aligned} \tag{2.4}$$

The proof of the next lemma is straightforward and is omitted.

Lemma 2.1. *Let $V \in G(T_0^1(\mathcal{M}))$. Then the skew-symmetric component with respect to the cotangent factors of $\nabla(\nabla V)$ is FV. This is equivalent to*

$$[\nabla(\nabla V)](X, Y) - [\nabla(\nabla V)](Y, X) = [FV](Y, X)$$

for any vectors X, Y.

We define the Ricci tensor of $\mathcal{M}$ as $\mathrm{Ric}(V,W) = -c([FW]V)$, for any V, W vector fields. Observe that if $\{s_i : i = 1, \dots, n\}$ is any local orthonormal basis of $T\mathcal{M}$ then $\mathrm{Ric}(V,W) = \sum_{i=1}^n F(s_i, V, W, s_i)$.

By a non-negative Ricci manifold, $\mathcal{M}$, is a manifold that satisfies $\mathrm{Ric}(V,V) \geq 0$ for any $V \in T\mathcal{M}$.

The following lemma will be useful in our approach.

Lemma 2.2. *Let V and W be vector fields over $U \subset \mathcal{M}$ open. Then*

$$\mathrm{c}([\nabla(\nabla V)]W - \nabla_W(\nabla V)) = \mathrm{Ric}(W,V)\,. \tag{2.5}$$

Proof. We choose an orthonormal basis $\{s_1, s_2, \dots, s_n\}$ locally defined and compute

$$\begin{aligned}
&\mathrm{c}([\nabla(\nabla V)]W - \nabla_W(\nabla V)) \\
&= \sum_i \langle [\nabla_{s_i}(\nabla V)]W - [\nabla_W(\nabla V)]s_i, s_i\rangle \\
&= \sum_i \langle \nabla_{s_i}[(\nabla V)W] - (\nabla V)\nabla_{s_i}W - \nabla_W[(\nabla V)s_i] + (\nabla V)\nabla_W s_i, s_i\rangle \\
&= \sum_i \langle \nabla_{s_i}\nabla_W V - \nabla_W\nabla_{s_i}V - \nabla_{[s_i,W]}V, s_i\rangle \\
&= \sum_i F(s_i, W, V, s_i) = \mathrm{Ric}(V,W)\,.
\end{aligned}$$

□

Let $\mathcal{M}$ be a Riemann surface and let $\gamma_0 \subset \mathcal{M}$ be a simple closed arcwise parametrized geodesic. Assume there exists a smooth unitary orthogonal vector field η defined along γ_0, or equivalently, a neighborhood of γ_0 is oriented. Extend η to a geodesic vector field on a vicinity $\mathcal{V}$ of γ_0. If $\varphi_t(p) = \varphi(t,p)$ is the flow of η one can restrict $\mathcal{V}$, if necessary, and choose $\delta > 0$ so that the map $\varphi : [-\delta, \delta] \times \gamma_0 \to \mathcal{V}$ is a diffeomorphism.

Let t and x be the coordinate functions of the inverse map $\varphi^{-1} : \mathcal{V} \to [-\delta, \delta] \times \gamma_0$, $\varphi^{-1}(p) = (t(p), x(p))$. For any $\sigma : [0,1] \to \mathcal{V}$ a smooth curve we denote by $\overline{\sigma}$ its projection over γ_0,

$$\overline{\sigma}(s) = x \circ \sigma(s)\,, \quad 0 \leq s \leq 1\,. \tag{2.6}$$

Notice that we abuse language and denote by σ either a curve or its trace, according to the context. Similarly, $|\sigma|$ denotes the length of the curve, but for a two dimensional region $U \subset \mathcal{M}$, $|U|$ denotes its area.

The next lemma is well known to geometers, and can be found in the literature.

Lemma 2.3. *Suppose that the gaussian curvature K is strictly negative on $\mathcal{V}$. We have:*

- (a) *Let $p_0, p_1 \in \mathcal{V}$ and σ be any smooth simple curve joining p_0 and p_1. Then*
  - (a1) *$|\sigma| \geq |t(p_1) - t(p_0)|$ and equality holds if and only if σ reparametrizes the geodesic segment $t \mapsto \varphi_t(p)$ between p_0 and p_1.*
  - (a2) *$|\sigma| \geq |\overline{\sigma}|$ and equality holds if and only if $\sigma = \overline{\sigma} \subset \gamma_0$.*
- (b) *Let $J \subset \gamma_0$ be an interval or $J = \gamma_0$. Let $0 < \delta_0 \leq \delta$ and U be any of the sets $\varphi([0, \delta_0] \times J)$ or $\varphi([-\delta_0, 0] \times J)$. Then $|U| > \delta_0\,|J|$.*

3. Nonexistence of nonconstant minimizers

This section is devoted to the proofs of Theorems 1.1 and 1.2, which in turn will be applications of the identities established in the next two lemmas.

Recall that the Riemannian metric of $\mathcal{M}$ induces metrics in any tensor product $T^r_s(\mathcal{M})$, as well as in their spaces of sections. If $T, W \in T^1_1(\mathcal{M})$ then their inner-product (fiberwise) is computed as $\langle T, W\rangle = c(c(T \otimes W^*))$, being W^* the (metric) transpose of the endomorphism $W : T\mathcal{M} \to T\mathcal{M}$.

If V is a C^1 vector field on $\mathcal{M}$ we set $\operatorname{div}(V) = c(\nabla V)$. The hessian of a C^2 function u on $\mathcal{M}$ is $H_u = \nabla(\nabla u)$. The Laplacean of u is then $\Delta u = c(H_u) = \operatorname{div}(\nabla u)$.

The Riemannian measure on $\mathcal{M}$ will be denoted by $d\mu$. By a *component* of a topological space we always mean a *connected component*.

Lemma 3.1. *Let V be a C^2 vector field on $\mathcal{M}$ and u a C^3 function on $\mathcal{M}$. Then*

$$\Delta(Vu) - V(\Delta u) = \operatorname{div}(\nabla V^*\nabla u) + \langle H_u, \nabla V\rangle + \operatorname{Ric}(\nabla u, V)\,. \tag{3.1}$$

Proof. We first notice that

$$\nabla(Vu) = [\mathrm{d}(Vu)]^* = \nabla V^*\nabla u + H_u\, V\,. \tag{3.2}$$

Then

$$\begin{aligned}\Delta(Vu) - V(\Delta u) &= \mathrm{c}(\nabla[\nabla V^*\nabla u + H_u\, V]) - \mathrm{c}(\nabla_V H_u)\\ &= \mathrm{c}(\nabla[\nabla V^*\nabla u]) + \mathrm{c}([\nabla H_u]V + H_u\nabla V - \nabla_V H_u)\\ &= \operatorname{div}(\nabla V^*\nabla u) + \mathrm{c}([\nabla H_u]V - \nabla_V H_u) + \mathrm{c}(H_u\nabla V)\,.\end{aligned} \tag{3.3}$$

Applying Lemma 2.2 to the second summand of term (3.3) and observing that $\mathrm{c}(H_u\nabla V) = \langle H_u, \nabla V\rangle$ we arrive at

$$\Delta(Vu) - V(\Delta u) = \operatorname{div}(\nabla V^*\nabla u) + \operatorname{Ric}(\nabla u, V) + \langle H_u, \nabla V\rangle\,,$$

and the proof is complete. □

Remark 3.2. Lemma 3.1 is central in the next constructions of this section. Indeed, it somehow appears in [21], where its full geometric significance is shadowed by the high symmetry of that case. The main idea there, which holds in general, is a commutation relation between the Laplacian operator and a particular directional derivative, namely, the normalized gradient of u.

Let u be a non-constant critical point of $\mathcal{E}$ with $F' = f$. Then

$$\frac{d}{dt}\mathcal{E}(u+tv)|_{t=0} = -\int_{\mathcal{M}} (\Delta u + f(u))\, v\, d\mu = 0, \quad \forall v \in H^1(\mathcal{M})\,. \tag{3.4}$$

The linearization of the operator $\Delta + f(\cdot)$ at u yields an operator $\mathcal{L} : H^1(\mathcal{M}) \to H^{-1}(\mathcal{M})$ defined by

$$\mathcal{L}(u)v = \Delta v + i(f'(u)\, v)\,, \tag{3.5}$$

where $i : H^1(\mathcal{M}) \to H^{-1}(\mathcal{M})$ is the Sobolev inclusion $H^1 \subset H^{-1}$. Let $(\cdot,\cdot) : H^{-1} \times H^1 \to \mathbb{R}$ be the canonical pairing of a vector space and its dual. Then

$$\frac{d^2}{dt^2}\mathcal{E}(u+tv)|_{t=0} = (\mathcal{E}''(u)v, v) = -(\mathcal{L}(u)v, v)\,.$$

For the next lemma we temporarily drop any hypothesis about Ricci curvature. It will be immediate that for Ricci non-negative manifolds the quadratic form associated to $\mathcal{L}$ is not sign definite. Define

$$U := \{\nabla u \neq 0\} \subset \mathcal{M} .$$

Let V be the unitary vector field $V = \frac{\nabla u}{|\nabla u|}$ over U.

Lemma 3.3. *Let $v = |\nabla u|$. Then*

$$(\mathcal{L}(u)v, v) = \int_{\mathcal{M}} |\nabla u|^2 \big(|\nabla V|^2 + \mathrm{Ric}(V, V)\big)\, d\mu . \tag{3.6}$$

Proof. The function u is of class C^3, hence V is C^2. In the open set U we have $V(\Delta u + f(u)) = 0$, thus

$$\Delta(Vu) + f'(u)(Vu) = \Delta(Vu) - V(\Delta u) . \tag{3.7}$$

Applying Lemma 3.1 directly to the righthand side of (3.3) we get

$$\Delta(Vu) + f'(u)(Vu) = \mathrm{div}(\nabla V^* \nabla u) + \langle H_u, \nabla V\rangle + \mathrm{Ric}(\nabla u, V) . \tag{3.8}$$

The covariant derivative of V is

$$\nabla V = \frac{1}{|\nabla u|} H_u - \nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3} . \tag{3.9}$$

A computation shows that ∇V is orthogonal to the tensor $\nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3}$. Recalling that $v = |\nabla u| = Vu$ we obtain

$$\langle H_u, \nabla V\rangle = |\nabla u| \big\langle \frac{1}{|\nabla u|} H_u - \nabla u \otimes \frac{(H_u \nabla u)^*}{|\nabla u|^3}, \nabla V \big\rangle = v\, |\nabla V|^2 . \tag{3.10}$$

Let W be any vector in the tangent space over a point of U. Since V is unitary we have

$$\langle \nabla V^* V, W\rangle = \langle V, \nabla_W V\rangle = \frac{1}{2} W|V|^2 = 0 . \tag{3.11}$$

Thus $\mathrm{div}(\nabla V^* \nabla u) = \mathrm{div}(v \nabla V^* V)$ vanishes identically. With the help of (3.10) equation (3.8) turns into

$$\Delta v + f'(u)v = v\, |\nabla V|^2 + v\ \mathrm{Ric}(V, V) . \tag{3.12}$$

Notice that v vanishes in $\mathcal{M} - U$. Looking at the left-hand side of the above identity as a distribution it becomes clear that its support is contained in $\overline{U}$. Therefore, applying it on $v \in H^1(\mathcal{M})$ one obtains

$$(\mathcal{L}(u)v, v) = \int_{\mathcal{M}} |\nabla u|^2 \big(|\nabla V|^2 + \mathrm{Ric}(V, V)\big)\, d\mu , \tag{3.13}$$

which completes the proof. □

Remark 3.4. Let $p \in M$ be a non-critical point of u. The level set $S = \{x : u(x) = u(p)\}$ is a regular hypersurface near p. It can be seen that $\nabla V = A + (\nabla_V V) \otimes V^*$, where $A : TS \to TS$ is the *shape operator* respect to V of the second fundamental form of the inclusion $S \subset M$. By setting $c = |\nabla_V V|$ the squared norm of ∇V becomes

$$|\nabla V|^2 = |A|^2 + c^2 . \tag{3.14}$$

Therefore, $|\nabla V|^2$ is the sum of the square of the principal curvatures of S plus the square of the curvature of the flow of ∇u.

Remark 3.5. In the unidimensional case $\mathcal{M} = S^1$ a direct proof of instability can be given. Endow S^1 with a metric so that $|S^1| = l$. Functions on S^1 are identified with functions on $[0, l]$ satisfying certain boundary conditions. In this case the Euler-Lagrange equation for $\mathcal{E}$ is

$$\begin{gathered} u''(t) + f(u(t)) = 0, \quad 0 < t < l \\ u(0) = u(l), \quad u'(0) = u'(l)\,. \end{gathered} \tag{3.15}$$

Its linearization becomes $\mathcal{L}(u)v = v'' + f'(u)v$. Assume by contradiction that u is a non-constant local minimizer of $\mathcal{E}$. Then $(\mathcal{L}(u)v, v) \le 0$, and due to Lemma 3.3 we get $\mathcal{L}(u)v = 0$. Hence $v = |u'|$ is an eigenfunction associated to the zero eigenvalue.

A direct computation shows that u' is also an eigenfunction of the zero eigenvalue of $\mathcal{L}(u)$. Then $w = u' + |u'|$ is an eigenfunction and since w vanishes in an open interval the Unique Continuation Theorem gives us $w \equiv 0$. Hence $u' \equiv 0$, what goes against the hypothesis. This shows that the first eigenvalue of $\mathcal{L}(u)$ is positive and there are no non-constant local minimizers of $\mathcal{E}$.

In view of Lemma 3.3 the proof of Theorem 1.1 is now immediate if we strengthen the hypothesis to $\operatorname{Ric} > 0$ on $\mathcal{M}$. Indeed, one can show that $\operatorname{Ric} > 0$ on some open set of $\mathcal{M}$ suffices for the positivity of $(\mathcal{L}(u)v, v)$, by using the Unique Continuation Theorem together with the contradiction assumption that the first eigenvalue of $\mathcal{L}(u)$ is zero.

We will rather give a unified proof for the case $\operatorname{Ric} \ge 0$. This requires a few more lemmas dealing with the more delicate case $\nabla V = 0$ and $\operatorname{Ric} = 0$ on U. It will follow after a series of steps rich on tricky details. The main ingredients are the level sets of u and the behaviour of the geodesics of $\mathcal{M}$ respect to the critical points of u.

The remaining results of this section do not demand that u be bounded or belong to any particular Sobolev Space. We will skip for a while any functional analytic concerns, and assume that $\mathcal{M}$ is an arbitrary complete, not necessarily compact, Riemann manifold, and u is a *classical* solution to equation (1.6). The compactness of $\mathcal{M}$ will be implicitly invoked back only in the proofs of Theorems 1.1 and 1.2.

For the next six lemmas and corollaries, we assume that

$$|\nabla V| = 0 \quad \text{in } U\,,$$

unless otherwise stated. In particular we obtain that V is a parallel vector field over U. From equation (3.9) we also get

$$H_u = V \otimes (H_u\, V)^* = \Delta u\, V \otimes V^*\,. \tag{3.16}$$

For any $p \in \mathcal{M}$ define $\mathcal{N}_p$ as the component of the level set $\{x \in \mathcal{M} : u(x) = u(p)\}$ that contains p.

Lemma 3.6. *If $p \in U$ then $\mathcal{N}_p \subset U$. Further, $\mathcal{N}_p$ is a complete geodesic Riemannian submanifold of codimension 1 of $\mathcal{M}$ and $|\nabla u| > 0$ is constant on $\mathcal{N}_p$.*

Proof. Let U_p be a component of U and C_p a component of $U_p \cap \mathcal{N}_p$ so that $p \in C_p$. Clearly C_p is a codimension 1 submanifold of $\mathcal{M}$. If $X, Y \in T(C_p) \subset TU$ we have

$$\langle \nabla_X Y, V\rangle = X\langle Y, V\rangle - \langle Y, \nabla_X V\rangle = 0 \tag{3.17}$$

for V is parallel and X, Y are orthogonal to V. This shows that C_p is geodesic.

Letting $q \in C_p$ and $X \in T_q(C_p)$, we have $\nabla_X \nabla u = H_u(X) = 0$. Therefore ∇u is parallel and $|\nabla u| \neq 0$ is constant along C_p. If $\overline{q}$ is an adherent point of C_p then

$\nabla u(\overline{q})$ is non-zero so that $\overline{q} \in U_p$. This shows that C_p is closed in $\mathcal{M}$, and since U_p is open, C_p is also open as a topological subspace of $\mathcal{N}_p$. Therefore by the conexity we have $C_p = \mathcal{N}_p \subset U_p$.

The geodesic completeness of $\mathcal{N}_p$ follows from the Theorem of Rinow and Hopf [3] and the fact that $\mathcal{M}$ is complete. □

Lemma 3.7. *Let $\gamma : \mathbb{R} \to \mathcal{M}$ be an arclength parametrized geodesic, and $h(t) = u(\gamma(t))$ for all $t \in \mathbb{R}$. Assume that h is non-constant, and let (a, b) be a component of $\gamma^{-1}(U)$. Then*

(a) *h is strictly monotone in (a, b).*
(b) *Assume $a \in \mathbb{R}$, and let $p = \gamma(a)$. Then p is a critical point of u and $H_u(p) \neq 0$.*
(c) *Under the same hypothesis as (b) let $r = b - a \in \mathbb{R} \cup \{+\infty\}$. Then $(a - r, a)$ is also a component of $\gamma^{-1}(U)$. Further, $h(t)$ is symmetric respect to $t = a$; i.e., $h(a - s) = h(a + s)$ for all $s \in \mathbb{R}$.*
(d) *Under the same hypothesis as (c), assume also $b \in \mathbb{R}$. Then h is periodic of period $2r$.*

Proof. For all $t \in \mathbb{R}$ we have $h'(t) = \langle \nabla u, \gamma'(t) \rangle$. This justifies the existence of the interval (a, b), since h is non-constant. For all $t \in (a, b)$ we can write $h'(t) = |\nabla u| \langle V, \gamma'(t) \rangle$. Both of V and γ' are parallel along γ, hence $\langle V, \gamma'(t) \rangle = k$ is a constant in (a, b). We must have $k \neq 0$, otherwise the geodesic γ would be entirely contained in $\mathcal{N}_{\gamma(t_0)}$, for any $t_0 \in (a, b)$, and h would be constant. Hence k and $|\nabla u|$ are non-zero in (a, b) and part (a) is proved.

We compute the second derivative of h for any $t \in (a, b)$,

$$h''(t) = \frac{d}{dt} \langle \nabla u, \gamma'(t) \rangle = \langle H_u(\gamma'(t)), \gamma'(t) \rangle = \Delta u(\gamma(t)) k^2 \,,$$

in view of (3.16). Then $h(t)$ is a solution to the $2^{\rm nd}$ order equation

$$h'' + k^2 \, f(h) = 0 \tag{3.18}$$

on (a, b). If $a \in \mathbb{R}$, h satisfies the initial condition $h(a) = u(p)$, $h'(a) = 0$. By uniqueness of the initial-value problem the constant function $t \mapsto u(p)$ is not a solution of that problem, and therefore $u(p)$ is not a root of f. Hence, $h''(a) = -k^2 \, f(u(p)) \neq 0$, and $H_u(p)$ does not vanish. This concludes part (b).

Due to $h''(a) \neq 0$, there is a small left open neighborhood of a where $h'(t) \neq 0$, and hence $\gamma(t) \in U$ for $t < 0$ small. Therefore, there is a component of $\gamma^{-1}(U)$ of the form (c, a), for some $c \in (-\infty, a)$. Let $J = (0, \min\{r, a - c\})$.

We define $h_-(s) = h(a - s)$ and $h_+(s) = h(a + s)$ for all $s \in \mathbb{R}$. Then $h_-(0) = h_+(0) = h(a)$, $h_-'(0) = h_+'(0) = 0$. Further, for $s \in J$ there are suitable constants k_-, k_+ that play the role of k on (3.18):

$$\begin{gathered}
h_-'' + k_-^2 \, f(h_-) = 0 \,, \\
h_+'' + k_+^2 \, f(h_+) = 0 \,.
\end{gathered}$$

Again uniqueness for this problem will give us $h_- \equiv h_+$ as long as we show that $k_-^2 = k_+^2$.

Let $V_-(s) = V(\gamma(a - s))$ and $V_+(s) = V(\gamma(a + s))$ for all $s > 0$ small. Both of V_- and V_+ can be continuously extended by parallel transport along γ to vectors $\tilde{V}_-$ and $\tilde{V}_+$, respectively, on $T_p\mathcal{M}$. We claim that the (unitary) vectors $\tilde{V}_-$ and $\tilde{V}_+$ are colinear. The (symmetric polinomials on the) eigenvalues of the continuous symmetric tensor H_u are continuous. The special form of H_u on U, given by equation (3.16), implies that for all small $s > 0$, $H_u(\gamma(a \pm s))$ has a zero eigenvalue of multiplicity at least $N-1$, which is inherited by $H_u(p)$. The remaining eigenvalue of $H_u(p)$, $\Delta u(p)$, has to be non-zero (after part (b)) and simple. This is an open condition, and the eigenspace associated to this eigenvalue varies continuously, close to p. It is generated by V on U, therefore, we have $\tilde{V}_- = \pm \tilde{V}_+$. Since

$$\begin{aligned} k_- &= \lim_{s\to 0^+} \langle V_-(s), \gamma'(a-s)\rangle = \langle \tilde{V}_-, \gamma'(a)\rangle \\ k_+ &= \lim_{s\to 0^+} \langle V_+(s), \gamma'(a+s)\rangle = \langle \tilde{V}_+, \gamma'(a)\rangle , \end{aligned} \tag{3.19}$$

we obtain $|k_-| = |k_+|$, hence $h_-(s) = h_+(s)$ for $s \in J$. Critical points of h_- and h_+ happen together in this range and correspond to intersections of $\gamma(t)$ with the border of U. Therefore $0 < s \mapsto \gamma(a-s)$ cannot leave U before $s = r$, and since the argument is symmetric, we conclude that $a - c = r$ and $\gamma^{-1}(U)$ contains $(a-r, a)$ as a component, which proves part (c).

Part (d) is now immediate. Clearly the symmetry of $h(t)$ holds with respect to any critical point of h. If $r = b - a$ is finite then we get $h(a + r + s) = h(a+r-s) = h(a-r+s)$ for any $0 < s < r$. In particular, an inductive argument shows that $\{a + mr : m \in \mathbb{Z}\}$ are all critical points of $h(t)$. The period of h is $2r$ since it intercalates increasing with decreasing intervals between consecutive critical points. $\square$

Remark 3.8. ¿From part (c) of the above lemma, we have $h'(a+s) = -h'(a-s)$. Selecting $s > 0$ small, we obtain

$$h'(a+s) = k_+|\nabla u|_{\gamma(a+s)} = -k_-|\nabla u|_{\gamma(a-s)} = -h'(a-s)\,. \tag{3.20}$$

Therefore, $k_- = -k_+$ and $\tilde{V}_- = -\tilde{V}_+$.

As a consequence of Lemma 3.7 we get $H_u(p) \neq 0$ and $\Delta u(p) \neq 0$ for any critical point p of u, since there is a point $q \in \mathcal{M}$ with $u(q) \neq u(p)$ and a geodesic $\gamma(t)$ joining p to q. Further, the set of critical points of u is $\partial U = M - U$.

Proof of Theorem 1.1. By Lemma 3.3 along with the condition $\mathrm{Ric} \geq 0$ we deduce that $(\mathcal{L}(u)v, v) \geq 0$. We will show that this inequality is strict, so u cannot be a local minimum of $\mathcal{E}$. The case where $\nabla V \neq 0$ is straightforward from the Lemma, so we assume in the sequel that $\nabla V \equiv 0$ on U.

Suppose by contradiction that the first eigenvalue of $\mathcal{L}(u)$ is non-positive. Then $(\mathcal{L}(u)v, v) = 0$ and v must be an eigenfunction of $\mathcal{L}(u)$ associated to the zero eigenvalue. Since $f'(u)v$ is continuous, standard elliptic regularity applied to

$$\Delta v + f'(u)\, v = 0 \quad \text{on } \mathcal{M} \tag{3.21}$$

gives us $v \in C^2(\mathcal{M})$. Computing the gradient of v in U we obtain

$$\nabla v = \nabla|\nabla u| = H_u(V) = \Delta u\, V\,. \tag{3.22}$$

Let p be a critical point of u and $\gamma(t)$ be a geodesic satisfying the hypotheses on Lemma 3.7, so that $\gamma(0) = p$. Following the notation in the proof of the Lemma we have, by part (b), that $\Delta u(p) \neq 0$. On the other hand, Remark 3.8 gives us

$$\lim_{t\to 0^+} V_{\gamma(t)} = -\lim_{t\to 0^-} V_{\gamma(t)} \neq 0\,. \tag{3.23}$$

This shows that ∇v is not even continuous at p, what contradicts the C^2 regularity of v. The only remedy is granting that the first eigenvalue of $\mathcal{L}(u)$ is positive, which completes the proof. □

Note that V defines a line subundle of $T\mathcal{M}|_U$ that can be extended over ∂U by taking the only simple eigenspace of H_u (associated to the non-zero eigenvalue) near critical points. This justifies the next result.

Corollary 3.9. *There exists a geodesic line bundle $\mathcal{I} \subset T\mathcal{M}$ so that $\mathcal{I}|_U$ is spanned by V.*

Choose a point $p_0 \in U$ and let U_0 be its correspondent component of U. Denote $\mathcal{N}_0 = \mathcal{N}_{p_0}$. We would like to extend the field $V|_{U_0}$ to the whole of $\mathcal{M}$ by means of the bundle $\mathcal{I}$. The flow of such extension would, then, be generated by isometries, and routine arguments would give us a covering map $\varphi : \mathbb{R} \times \mathcal{N}_0 \to \mathcal{M}$, from which one would quickly derive the results of Theorem 1.2. This case has already been researched in greater generality, for instance, in [2].

Here is where the orientability of $\mathcal{I}$ comes in. Clearly, such an extension of $V|_{U_0}$ is possible if and only if $\mathcal{I}$ is orientable (as a real vector bundle). Both of orientable and non-orientable cases can happen to $\mathcal{I}$, leading to two different constructions for $\mathcal{M}$. In order to keep generality and to shorten the proofs, we give a definition of φ independent of $\mathcal{I}$.

For any $p \in \mathcal{N}_0$ let $t \in \mathbb{R} \mapsto \varphi_t(p)$ be the geodesic defined by $\varphi_0(p) = p$ and $\varphi_0'(p) = V_p$. Then $\varphi : \mathbb{R} \times \mathcal{N}_0 \to \mathcal{M}$ is smooth.

Lemma 3.10. *There is an open interval (a, b) so that $\varphi : (a, b) \times \mathcal{N}_0 \to U_0$ is an isometry.*

Proof. Let $(a, b) \ni 0$ be the maximal interval for which $\varphi_t(p_0)$ belongs to U_0. If $q \in \mathcal{N}_0$ is any other point we see that $u(\varphi_t(p_0)) = u(\varphi_t(q))$ for $t \in \mathbb{R}$, since both functions satisfy the same differential equation (3.18) with same initial conditions. Due to Lemma 3.7 it follows that (a, b) keeps the maximality property above stated, for any $q \in \mathcal{N}_0$.

Since V is parallel and equals $\varphi_0'(p)$ on p, it holds $\varphi_t'(p) = V_{\varphi_t(p)}$ for all $t \in (a, b)$. Therefore $t \mapsto \varphi_t$ are integral curves of $V|_{U_0}$. Two such curves do not intersect, and because $u(\varphi_t(p))$ is monotone the curve $\varphi_s(q)$ cannot be a reparametrization of $\varphi_t(p)$, for any $(s, q) \in (a, b) \times \mathcal{N}_0$ with $q \neq p$. This concludes injectivity of $\varphi : (a, b) \times \mathcal{N}_0 \to U_0$. Notice that φ is the flow of V restricted to $\mathcal{N}_0$, hence it is an isometry with its image. The set $\varphi((a, b) \times \mathcal{N}_0)$ is open.

Now we show that the image of φ is closed in U_0. Let $q \in U_0$ be an adherent point of $\varphi((a, b) \times \mathcal{N}_0)$, and $\sigma : [0, 1] \to U_0$ be a smooth curve with $\sigma(0) = p_0$, $\sigma(1) = q$. Let $I = \sigma^{-1}(U_0)$, I is open in $[0, 1]$ and non-empty. Using that φ is a local isometric coordinate chart one see that I is closed, hence $I = [0, 1]$ and q belongs to the image of φ. The image of φ is then open and closed in U_0, and by convexity, we have $\varphi((a, b) \times \mathcal{N}_0) = U_0$. □

Following the notation of Lemmas 3.7 and 3.10 we consider the case $b \in \mathbb{R}$. Then $\varphi_b(\mathcal{N}_0) \subset \partial U_0$. Let $\hat{p} = \varphi_b(p_0)$. We have $\varphi_b(\mathcal{N}_0) = \mathcal{N}_{\hat{p}}$, since φ_t preserves level sets of u. Surprisingly, $\mathcal{N}_{\hat{p}}$ may not be isometric to $\mathcal{N}_0$. This question relates to whether the curve $t \mapsto \varphi_t(p)$ does *leave* U_0 when it crosses the border at $t = b$.

Let U_1 be the component of U that contains $\varphi_t(\mathcal{N}_0)$ for all $b < t < 2b - a$.

Lemma 3.11. *$\mathcal{N}_{\hat{p}}$ is a geodesic complete submanifold of $\mathcal{M}$. The map $\varphi_b : \mathcal{N}_0 \to \mathcal{N}_{\hat{p}}$ is a local isometry. It is a bijection if and only if $\mathcal{I}|_{U_0 \cup \mathcal{N}_{\hat{p}}}$ is orientable, and it holds $U_1 \neq U_0$. Otherwise φ_b is a two-fold covering map onto $\mathcal{N}_{\hat{p}}$ and $U_1 = U_0$.*

Proof. Let $p \in \mathcal{N}_0$ and $\mathcal{V} \ni \varphi_b(p)$ be a simply connected open neighborhood of $\varphi_b(p)$. There is a local trivialization of $\mathcal{I}|_{\mathcal{V}}$ by means of a unitary parallel vector field $\tilde{V}$, so that $\tilde{V}_{\varphi_b(p)} = \varphi_b'(p)$. By continuity, $\varphi_b'(q) = \tilde{V}_{\varphi_b(q)}$ for any $q \in \varphi_b^{-1}(\mathcal{V})$. Again, uniqueness of the parallel transport along a curve subject to the same initial conditions gives us $\tilde{V}_{\varphi_{b+s}}(q) = \varphi_{b+s}'(q)$ for all s small enough. Restricting $\mathcal{V}$ if necessary we see that φ is the flow of a unitary killing field defined on the open set $\varphi((a, b+\varepsilon) \times \varphi_b^{-1}(\mathcal{V})) \cup \mathcal{V}$, for some $\varepsilon > 0$ small. Hence φ_b is a local isometry of $\mathcal{N}_0$ onto $\mathcal{N}_{\hat{p}}$. From that it also follows that $\mathcal{N}_{\hat{p}}$ is geodesic and complete.

Now assume φ_b is injective. Then $(t, q) \in (a, b] \times \mathcal{N}_0 \mapsto \varphi_t'(q)$ is a well defined trivialization of $\mathcal{I}|_{U_0 \cup \mathcal{N}_{\hat{p}}}$, so it is orientable. If $\varphi_t(p)$ belongs to U_0 for some $t \in (b, 2b-a)$ then there is $s \in (a, b)$ and $q \in \mathcal{N}_0$ with $\varphi_s(q) = \varphi_t(p)$. Both geodesics have velocities on the bundle $\mathcal{I}$, so they must be opposite since $u(\varphi_t(p))$ is decreasing on t. Therefore $\varphi_t(p)$ is a backward reparametrization of $\varphi_s(q)$ and we get $\varphi_b(p) = \varphi_b(q)$, contradicting injectivity. Hence there must be $U_0 \neq U_1$.

On the other hand, if there are distinct points $p, q \in \mathcal{N}_0$ with $\varphi_b(p) = \varphi_b(q)$ one clearly has $\varphi_b'(p) = -\varphi_b'(q)$, since both velocities lie in the same fiber of $\mathcal{I}$ and cannot be equal. Therefore no orientation of $\mathcal{I}|_{U_0}$ can be extended to a larger set on $\mathcal{M}$ containing $\mathcal{N}_{\hat{p}}$; i.e., $\mathcal{I}|_{U_0 \cup \mathcal{N}_{\hat{p}}}$ is non-orientable. In this case it holds $\varphi_{2b}(p) = q$, hence $\varphi_{2b}(\mathcal{N}_0) = \mathcal{N}_0$, what indicates that $U_0 = U_1$. Restricting φ_b to suitable vicinities $\mathcal{V}_p$, $\mathcal{V}_q$ of p and q, respectively, we may write $\varphi_{2b}|_{\mathcal{V}_p} = (\varphi_b|_{\mathcal{V}_q})^{-1} \circ \varphi_b|_{\mathcal{V}_p}$, what shows that φ_{2b} is locally an isometry *without fixed points* and $\varphi_{2b}^2 = Id_{\mathcal{N}_0}$. This proves that $\varphi_b : \mathcal{N}_0 \to \mathcal{N}_{\hat{p}}$ is a two-fold covering map. $\square$

Recall that an involution of a Riemannian manifold is an isometry I such that $I^2 = id$.

Lemma 3.12. *$\varphi : \mathbb{R} \times \mathcal{N}_0 \to \mathcal{M}$ is a regular isometric covering map. Denote by $K = Aut(\mathbb{R} \times \mathcal{N}_0, \varphi)$ the group of covering transformations of φ. Then, if $\mathcal{I}$ is orientable, K is either trivial or cyclic generated by the metric product of a translation of $\mathbb{R}$ with an isometry of $\mathcal{N}_0$. If $\mathcal{I}$ is not orientable K is generated by at most two involutions of $\mathbb{R} \times \mathcal{N}_0$.*

Proof. If u has no critical points then $U_0 = U = \mathcal{M}$ and φ is the (regular) trivial covering map, $\mathcal{I}$ is orientable and $K = \{Id\}$. Otherwise $\partial U_0 \neq \emptyset$ and we assume b on Lemma 3.10 is finite.

Following Lemma 3.11 we let $\mathcal{N}_{\hat{p}} = \varphi_b(\mathcal{N}_0)$ be a component of the border of U_0. If there is another component U_1 of U that cobounds U_0 through $\mathcal{N}_{\hat{p}}$ then we can choose $p_1 \in U_1$ with $u(p_1) = u(p_0)$ and let $\mathcal{N}_1 = \mathcal{N}_{p_1}$. Let $\psi : (a, b) \times \mathcal{N}_1 \to U_1$ be the map analogous to φ. It can be seen from the proof of Lemma 3.11 that $\varphi_t'(p) \in \mathcal{I}_{\varphi_t(p)}$ for all $t \in \mathbb{R}$, $p \in \mathcal{N}_0$. Then $\psi_b(\mathcal{N}_1) = \varphi_b(\mathcal{N}_0) = \mathcal{N}_{\hat{p}}$. It is clear that $\varphi_{2b}(\mathcal{N}_0) = \mathcal{N}_1$ and $\varphi_{b+s}(p) = \psi_{b-s}(\varphi_{2b}(p))$ for all $s \in \mathbb{R}$, $p \in \mathcal{N}_0$. Therefore φ is an isometry from $(a, 2b-a) \times \mathcal{N}_0$ onto $U_0 \cup \mathcal{N}_{\hat{p}} \cup U_1$.

On the other hand, if U_0 *self-bounds* at $\mathcal{N}_{\hat{p}}$ as described by Lemma 3.11, the function ψ above defined equals φ, and $\mathcal{N}_1 = \mathcal{N}_0$. Hence $\varphi : (a, 2b-a) \times \mathcal{N}_0 \to U_0 \cup \mathcal{N}_{\hat{p}}$ is a two-fold isometric covering map.

If $a = -\infty$ we are done. Otherwise there is another component $\mathcal{N}_{\hat{q}}$ of ∂U_1, $\mathcal{N}_{\hat{q}} \neq \mathcal{N}_{\hat{p}}$. The above constructions can be repeated, extending the isometric covering property of φ to the interval $(a, 3b-2a)$. This can also be performed backwards on t, starting on $t=a$. An inductive argument gives us that $\varphi : \mathbb{R} \times \mathcal{N}_0 \to \mathcal{M}$ is a covering map, and a local isometry.

If φ is injective we have again the trivial covering, and $K = \{Id\}$. In this case one clearly has $\mathcal{I}$ orientable. We assume in the remaining of this proof that φ is not injective.

Suppose first that $\mathcal{I}$ is orientable. Let $\varphi_{t_1}(p_1) = \varphi_{t_2}(p_2)$ for different pairs $(t_1, p_1), (t_2, p_2) \in \mathbb{R} \times \mathcal{N}_0$. Then $\varphi'_{t_1}(p_1) = \varphi'_{t_2}(p_2)$, so $\varphi_t(p_1)$ is an orientation preserving reparametrization of $\varphi_s(p_2)$. There is $\tau > 0$ with $\varphi_\tau(\mathcal{N}_0) = \mathcal{N}_0$, and τ can be taken the smallest positive number with such property. Then φ_τ is an isometry of $\mathcal{N}_0$.

Consider the automorphism of the covering space $\mathbb{R} \times \mathcal{N}_0$ given by $g_\tau(t,p) = (t-\tau, \varphi_\tau(p))$. A quick computation shows that the subgroup generated by g_τ acts transitively on the preimage $\varphi^{-1}(q)$ for all $q \in \mathcal{M}$. Since K is completely defined by some subgroup of the permutations group of $\varphi^{-1}(q)$ it becomes $K = \{g_\tau^n : n \in \mathbb{Z}\}$, and the covering map is regular.

Now consider $\mathcal{I}$ not orientable. Reasoning similarly to the previous case we can find $C \neq 0$ so that $\varphi_C : \mathcal{N}_0 \to \mathcal{N}_{\hat{p}}$, $\hat{p} = \varphi_C(p_0)$, is a two-fold covering, and $\varphi_{2C} : \mathcal{N}_0 \to \mathcal{N}_0$ is an involution. We can pick C so that $|C| > 0$ is minimum. Then $g_C(t,p) = (2C - t, \varphi_{2C}(p))$ is an involution of $\mathbb{R} \times \mathcal{N}_0$ and a covering transformation. If φ is a two-fold covering then the orbits of $\{Id, g_C\}$ acting on $\mathbb{R} \times \mathcal{N}_0$ are all the preimages of points of $\mathcal{M}$. Hence φ is regular and $K = \{Id, g_C\}$.

If φ is not a two-fold covering let (t_2, p_2), (t_1, p_1) and $g_C(t_1, p_1)$ be three distinct points in the preimage of a fixed point $q \in \mathcal{M}$. The velocities of the geodesics $s \mapsto \varphi_s(p_1)$ and $s \mapsto \varphi_{2C-s}(\varphi_{2C}(p_1))$ are opposite over q, and we can assume, without loss of generality, that $\varphi'_{t_2}(p_2) = \varphi'_{t_1}(p_1)$. Again there is $\tau > 0$ minimum such that $\varphi_\tau(\mathcal{N}_0) = \mathcal{N}_0$ and $\varphi'_\tau(p) = V_{\varphi_\tau(p)}$ for any $p \in \mathcal{N}_0$. Define g_τ as in the $\mathcal{I}$ orientable case.

Now let (t,p) be any point in $\varphi^{-1}(q) \ni (t_1, p_1)$. If $\varphi'_t(p) = \varphi'_{t_1}(p_1)$ then there is an integer n such that $(t,p) = g_\tau^n(t_1, p_1)$. Otherwise $(t,p) = g_\tau^n \circ g_C(t_1, p_1)$. This shows that the action of K is transitive on the preimages and the covering map is regular. Further K is generated by $\{g_\tau, g_C\}$. A careful check travelling forth and back on the geodesics $t \mapsto \varphi_t(p)$ reveals that $\varphi_\tau \circ \varphi_{2C} \circ \varphi_\tau \circ \varphi_{2C} = Id_{\mathcal{N}_0}$. Defining $D = C - \frac{\tau}{2}$ and $g_D(t,p) = (2D - t, \varphi_{2D}(p))$ we see that $g_D = g_\tau \circ g_C$ is an involution of $\mathbb{R} \times \mathcal{N}_0$ and $\{g_C, g_D\}$ generates K. This completes the proof. □

Proof of Theorem 1.2. Let u be a non-constant critical point of $\mathcal{E}$ with $(\mathcal{L}(u)v, v) = -(\mathcal{E}''(u)v, v) = 0$. Clearly the manifold $\mathcal{N}$ in the theorem stands for $\mathcal{N}_0$.

The proof then follows from the sequence of the lemmas and corollaries numbering from 3.6 through 3.12. The assertion that $\mathcal{M} \simeq (\mathbb{R} \times \mathcal{N})/K$ is a standard fact in Topology [20] and the metric is induced from $\mathbb{R} \times \mathcal{N}$ through the local isometry φ. □

4. Existence of nonconstant minimizers

This section is devoted to showing that if $\mathcal{M}$ fails to have non-negative Ricci curvature then Theorem 1.1 may not hold. This will be accomplished by showing that there are non-convex surfaces for which $\mathcal{E}_\varepsilon$ has non-constant local minimizers, for ε small enough.

The procedure we follow consists of finding the limit of the energies $\mathcal{E}_\varepsilon$ in the sense of $\Gamma-$convergence and then using a result of De Giorgi which roughly states that close (in some specified topology) to an isolated minimizer of the Γ-limit problem there is a minimizer of the original one.

Throughout this section, $\mathcal{M}$ will denote a surface diffeomorphic to S^2. For the reader's convenience we give the definition of $\Gamma-$convergence which is going to be used.

A family $\{\Lambda_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of real-extended functionals defined in $L^1(\mathcal{M})$ is said to Γ-converge in $L^1(\mathcal{M})$, as $\varepsilon \to 0$, to a functional $\Lambda_0 : L^1(\mathcal{M}) \longrightarrow \mathbb{R} \cup \{\infty\}$, if:

- For each $v \in L^1(\mathcal{M})$ and for any family $\{v_\varepsilon\}$ in $L^1(\mathcal{M})$ such that $v_\varepsilon \to v$ in $L^1(\mathcal{M})$, as $\varepsilon \to 0$, it holds that $\Lambda_0(v) \leq \liminf_{\varepsilon\to 0} \Lambda_\varepsilon(v_\varepsilon)$.
- For each $v \in L^1(\mathcal{M})$ there is a family $\{w_\varepsilon\}$ in $L^1(\mathcal{M})$ such that $w_\varepsilon \to v$ in $L^1(\mathcal{M})$, as $\varepsilon \to 0$ and $\Lambda_0(v) \geq \limsup_{\varepsilon\to 0} \Lambda_\varepsilon(w_\varepsilon)$.

Convergence in this sense will be denoted by $\Gamma^- \lim_{\varepsilon\to 0^+} \Lambda_\varepsilon = \Lambda_0$. The definitions and results we need about functions of bounded variation defined on $\mathcal{M}$ are provided below. We set

$$\mathcal{G}(\mathcal{M}) := \{g : g \text{ is a } C^1 \text{ section of } T\mathcal{M},\ |g(x)| \leq 1,\ \forall\ x \in \mathcal{M}\} \tag{4.1}$$

and let $\mathcal{H}^N$ denote the usual N-dimensional Hausdorff measure. Given $u : \mathcal{M} \to \mathbb{R}$ we define

$$|Du|(\mathcal{M}) := \sup_{g\in\mathcal{G}(\mathcal{M})} \int_{\mathcal{M}} u \operatorname{div}(g)\, d\mathcal{H}^2 \,. \tag{4.2}$$

A real function $u \in L^1(\mathcal{M})$ has bounded variation in $\mathcal{M}$ if $|Du|(\mathcal{M}) < \infty$. See [7] when $\mathcal{M}$ is a bounded domain in $\mathbb{R}^N$. The set

$$BV(\mathcal{M}) := \{u : \mathcal{M} \to \mathbb{R};\ u \in L^1(\mathcal{M}) \text{ and } |Du|(\mathcal{M}) < \infty\}$$

is a Banach space with the norm $\|u\|_{BV} = \|u\|_{L^1} + |Du|(\mathcal{M})$.

Letting χ_A denote the characteristic function of a set $A \subset \mathcal{M}$, we have

$$|D\chi_A|(\mathcal{M}) = \sup_{g\in\mathcal{G}(\mathcal{M})} \int_A \operatorname{div}(g)\, d\mathcal{H}^2 \,. \tag{4.3}$$

The perimeter of a set $A \subset \mathcal{M}$ is defined by $\operatorname{Per}_{\mathcal{M}}(A) := |D\chi_A|(\mathcal{M})$. If the border of A in $\mathcal{M}$ is at least C^2 then $|D\chi_A|(\mathcal{M}) = \mathcal{H}^1(\partial A \cap \mathcal{M})$.

Throughout this section we assume that the potential F in (3) satisfies:

- $F : \mathbb{R} \to \mathbb{R}$ is C^2
- $F \geq 0$ and $F(t) = 0$ if and only if $t \in \{\alpha, \beta\},\ \alpha < \beta$.
- $\exists\ t_0 > 0,\ c_1 > 0,\ c_2 > 0,\ k > 2$ such that $c_1 t^k \leq F(t) \leq c_2 t^k$, for $|t| \geq t_0$.

For convenience we denote the space of functions of bounded variation in $\mathcal{M}$ taking only two values, α and β, by $BV(\mathcal{M}, \{\alpha, \beta\})$.

The computation of the $\Gamma-$limit of $\mathcal{E}_\varepsilon$ when $\mathcal{M}$ is a bounded domain in R^N is standard by now. However no such result is available in the literature when $\mathcal{M}$ is a surface. Nevertheless the proof found in [1] can be adapted to our case in a natural manner thus yielding the following result.

Theorem 4.1. *Let $\mathcal{E}_\varepsilon : L^1(\mathcal{M}) \to \mathbb{R}$ be defined by*

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \int_{\mathcal{M}} \left[\varepsilon \frac{|\nabla u|^2}{2} - \varepsilon^{-1} F(u)\right] d\mathcal{H}^2 & \text{if } u \in H^1(\mathcal{M}) \\ \infty & \text{if } u \in L^1(\mathcal{M}) \backslash H^1(\mathcal{M})\,. \end{cases} \tag{4.4}$$

Then $\Gamma^- \lim_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon = \mathcal{E}_0$ *where*

$$\mathcal{E}_0(u) = \begin{cases} \lambda |D\chi_{\{u=\alpha\}}|(\mathcal{M}) & \text{if } u \in BV(\mathcal{M}, \{\alpha,\beta\}) \\ \infty & \text{otherwise} \end{cases} \tag{4.5}$$

and

$$\lambda = \int_0^1 \sqrt{F(s)}\, ds\,. \tag{4.6}$$

We say that v_0 in $L^1(\mathcal{M})$ is an L^1-local minimizer of the functional $\Lambda_0 : L^1(\mathcal{M}) \mapsto \mathbb{R} \cup \{\infty\}$ if there is $r > 0$ such that

$$\Lambda_0(v_0) \le \Lambda_0(v) \quad \text{whenever} \quad 0 < \|v - v_0\|_{L^1(\mathcal{M})} < r\,.$$

Moreover, if $\Lambda_0(v_0) < \Lambda_0(v)$ for $0 < \|v - v_0\|_{L^1(\mathcal{M})} < r$, then v_0 is called an isolated L^1-local minimizer of Λ_0.

The next result, which we use for finding a family of minimizers for (1.3), is due to De Giorgi and can be found in its abstract form in [5]. A proof, with the hypotheses on F given above, can be found in [16], since the replacement of Lebesgue measure with Haussdorf measure does not affect the arguments used.

Theorem 4.2. *Suppose that a sequence of real-extended functionals* $\{\Lambda_\varepsilon\}$ *and* Λ_0 *satisfy*

- (i) $\Gamma^- \lim_{\varepsilon\to 0^+} \Lambda_\varepsilon = \Lambda_0$
- (ii) *Any sequence* $\{v_\varepsilon\}_{\varepsilon>0}$ *such that* $\Lambda_\varepsilon(v_\varepsilon) \le C < \infty$ *for all* $\varepsilon > 0$*, is compact in* $L^1(\mathcal{M})$.
- (iii) *There exists an isolated* L^1*-local minimizer* v_0 *of* Λ_0.

Then there exist $\varepsilon_0 > 0$ *and a family* $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ *such that*

- v_ε *is an* L^1*-local minimizer of* Λ_ε*, and*
- $\|v_\varepsilon - v_0\|_{L^1(\mathcal{M})} \to 0$*, as* $\varepsilon \to 0$.

The growth condition on F is required in order to have the hypothesis on compactness (ii) satisfied. We also take, without loss of generality, $\lambda = 1$ on equation (4.6).

For any $u \in BV(\mathcal{M}, \{\alpha,\beta\})$ we denote by γ its boundary curve, i.e., $\gamma = \partial\{p \in \mathcal{M} : u(p) = \alpha\}$. Similarly, for any such γ there are exactly two distinct functions in $BV(\mathcal{M}, \{\alpha,\beta\})$ with γ as boundary curve. It holds $\mathcal{E}_0(u) = |\gamma|$. Given $r > 0$ there exists $\tilde{u} \in BV(\mathcal{M}, \{\alpha,\beta\})$ so that $\tilde{\gamma}$ is the disjoint union of a finite number of smooth closed curves satisfying

- $\|u - \tilde{u}\|_{BV} < r$;
- $|\gamma| \ge |\tilde{\gamma}|$.

We set

$$\begin{aligned} &BV_s(\mathcal{M}, \{\alpha,\beta\}) \\ &= \{u \in BV(\mathcal{M}, \{\alpha,\beta\}) : \gamma \subset \mathcal{M} \text{ is a smooth 1-dimensional submanifold}\}\,. \end{aligned} \tag{4.7}$$

Now we assume that a simple closed geodesic γ_0 is separable, i.e., $\mathcal{M} - \{\gamma_0\}$ has two components. Let $u_0 \in BV_s(\mathcal{M}, \{\alpha,\beta\})$ be the function associated to γ_0 so that $u_0 = \alpha\chi_{M_\alpha} + \beta\chi_{\mathcal{M}_\beta}$ with $M_i = \{p \in \mathcal{M} : u_0(p) = i\}$ $(i = \alpha,\beta)$.

Theorem 4.3. *Under the hypotheses and notation of Theorem 1.3 it holds that* u_0 *is an* $L^1(\mathcal{M})$*-local isolated minimizer of* $\mathcal{E}_0$.

Proof. Let $\mathcal{V}$ be the neighborhood constructed in preparation for Lemma 2.3. We choose $0 < \delta_0 < \delta$ and define $\mathcal{V}_0 = \varphi([-\delta_0, \delta_0] \times \gamma_0)$. We claim that any $r > 0$ with

$$r < |\beta - \alpha|\, \delta_0 \, \min\big\{\delta - \delta_0, \frac{|\gamma_0|}{2}\big\} \tag{4.8}$$

will verify $\mathcal{E}_0(u) > \mathcal{E}_0(u_0)$ whenever $u \in BV(\mathcal{M}, \{\alpha, \beta\})$ and $0 < \|u - u_0\|_{L^1} < r$.

The discussion prior to the theorem allows us to restrict our attention to competing functions $u \in BV_s(\mathcal{M}, \{\alpha, \beta\})$. Let γ be the boundary curve of a given u. A differential topology argument (see [8]) allows us to consider γ in generic position with $\partial\mathcal{V}_0$ and $\partial\mathcal{V}$, or equivalently, γ is transversal to the boundaries of $\mathcal{V}_0$ and $\mathcal{V}$. In particular, each connected component of $\gamma \cap \mathcal{V}_0$ is diffeomorphic to either $S^1 \subset \operatorname{int} \mathcal{V}_0$ or $[0, 1] \subset \mathcal{V}_0$ and endpoints contained in $\partial\mathcal{V}_0$. We define

$$D = \{\sigma : \sigma \text{ is a component of } \gamma \cap \mathcal{V}_0\}\,,$$
$$I = \cup_{\sigma \in D} \overline{\sigma} \subset \gamma_0\,.$$

Lemma 4.4. *Let $u \in BV_s(\mathcal{M}, \{\alpha, \beta\})$ with $\|u - u_0\| < r$. Then*

$$|I| > \max\big\{|\gamma_0| - (\delta - \delta_0), \frac{|\gamma_0|}{2}\big\}.$$

Proof. For each $\sigma \in D$, $\overline{\sigma}$ is a closed segment of γ_0. Hence,

$$J := \gamma_0 - I = \cup_{i=1}^m J_i\,,$$

where each J_i is an open interval of γ_0, and the J_i's are pairwise disjoint. The construction leading to J clearly yields

$$\gamma \cap \varphi([-\delta_0, \delta_0] \times J_i) = \emptyset \quad \text{for } 1 \le i \le m\,.$$

Therefore, u is constant in $\varphi([-\delta_0, \delta_0] \times J_i)$. Since u_0 switches its value over J_i we conclude that $|u - u_0| = |\beta - \alpha|$ in one of the regions $\varphi([-\delta_0, 0] \times J_i)$ or $\varphi([0, \delta_0] \times J_i)$. Applying Lemma 2.3 part (b) we derive

$$\|u - u_0\|_{L^1(\varphi([-\delta_0, \delta_0] \times J_i))} > |\beta - \alpha|\delta_0\, |J_i|\,.$$

Thus

$$\begin{aligned} r > \|u - u_0\|_{L^1} > \sum_{i=1}^m |\beta - \alpha|\delta_0\, |J_i| = |\beta - \alpha|\delta_0(|\gamma_0| - |I|) \\ \Rightarrow |I| > |\gamma_0| - \frac{r}{|\beta - \alpha|\delta_0}\,. \end{aligned}$$

Together with (4.8) the above inequality readily implies the Lemma. □

We set a little more notation: for any $\sigma \in D$ let $\rho = \rho(\sigma)$ be the component of γ that contains σ as an arc. We are led to three cases:

(i) If there is some $\rho(\sigma) \not\subset \mathcal{V}$ then there is an arc $\tilde{\sigma} \subset \rho$ joining a point of $\partial\mathcal{V}_0$ to a point of $\partial\mathcal{V}$. Lemma 2.3 (part (a)) gives us $|\tilde{\sigma}| \ge \delta - \delta_0$ and then

$$|\gamma| \ge |\tilde{\sigma}| + \sum_{\sigma \in D} |\sigma| \ge \delta - \delta_0 + |I| > |\gamma_0|\,,$$

in view of Lemma 4.4.

(ii) If there is some $\rho(\sigma) \subset \mathcal{V}$ that is freely homotopic to γ_0 within $\mathcal{V}$ then the intersection number of ρ with any geodesic ray $t \mapsto \varphi_t(x)$ is ± 1. Denoting by $\bar{\rho}$ the projection of ρ over γ_0 we get $\bar{\rho} = \gamma_0$. Hence, Lemma 2.3 part (a2) gives us

$|\gamma| \geq |\rho| \geq |\gamma_0|$. The strictness $|\gamma| > |\gamma_0|$ comes from $\|u - u_0\|_{L^1} > 0$, since there must be another component $\rho' \neq \rho$ of γ or ρ is not equal to γ_0.

(iii) Assume that neither (i) nor (ii) occurs. If for some $\sigma \in D$ we have $\bar{\rho} = \gamma_0$ we conclude similarly to case (ii) above, hence $|\gamma| > |\gamma_0|$. Otherwise, let p and q be points of ρ so that their projections over γ_0 are the end points of the segment $\bar{\rho} \subset \gamma_0$. Let σ_1 and σ_2 be the two distinct arcs of ρ joining p and q ($\sigma_i \subset \mathcal{V}$, $i = 1, 2$), with projections respectively $\bar{\sigma_1}$ and $\bar{\sigma_2}$. Since the intersection number of ρ with the ray $t \mapsto \varphi_t(x)$ is 0 we have $\bar{\sigma_1} = \bar{\sigma_2} = \bar{\rho}$. Hence $|\rho| = |\sigma_1| + |\sigma_2| > 2|\bar{\sigma_1}|$. Fixing ρ we see that any $\sigma \in D$ that is an arc of ρ satisfies $\overline{\sigma} \subset \bar{\sigma_1}$. Then

$$|\bigcup_{\sigma \in D\,, \sigma \subset \rho} \overline{\sigma}| \leq |\bar{\sigma_1}| < \frac{1}{2}|\rho|\,,$$

from which we derive

$$|\gamma| = \sum_{\rho \text{ a component of } \gamma} |\rho| > 2|I| > |\gamma_0|\,. \tag{4.9}$$

Therefore, $\mathcal{E}_0(u) = |\gamma| > |\gamma_0| = \mathcal{E}_0(u_0)$ if $0 < \|u - u_0\|_{L^1} < r$ and the theorem is proved. □

Proof of Theorem 1.3. As mentioned before, Theorem 1.3 is just an application of Theorem 4.2 for $\Lambda_\varepsilon = \mathcal{E}_\varepsilon$, whose hypotheses we now verify. Indeed (i) is nothing but Theorem 4.1 and (ii) may be found in [22], for instance. Although the proof of (ii) in [22] is rendered for $\mathcal{M}$ a bounded domain in $\mathbb{R}^N$ the proof holds equally well in our case.

As for (iii) it has been verified in Theorem 4.3 above. □

The following result seems to be known, though we have not been able to find it in the literature. It is a consequence of the procedure used in this section along with Theorem 1.1.

Lemma 4.5. *Let $\mathcal{M}$ be a compact Riemann surface with no boundary and having nonnegative Gaussian curvature. Then $\mathcal{M}$ has no closed nonintersecting isolated minimizing geodesic.*

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