*Electronic Journal of Differential Equations*, Vol. 2010(2010), No. 63, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# FUNCTIONAL EXPANSION - COMPRESSION FIXED POINT THEOREM OF LEGGETT-WILLIAMS TYPE

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Abstract. This paper presents a fixed point theorem of compression and expansion of functional type in the spirit of the original fixed point work of Leggett-Williams. Neither the entire lower nor the entire upper boundary is required to be mapped inward or outward.

## 1. Introduction

The spirit of the original Leggett-Williams fixed point theorem [10] is to take a subset of the elements in the cone in which $\alpha(x) = a$ and map these outward in the sense that $\alpha(Tx) \geq a$, where $\alpha$ is a concave positive functional defined on the cone. The subset that Leggett-Williams considered can be thought of as the set of all elements of the cone in which $\|x\| \leq b$ and $\alpha(x) = a$. There were no outward conditions on the operator $T$ in the Leggett-Williams fixed point theorem concerning those elements with $\|x\| > b$ and $\alpha(x) = a$, and hence they avoided any invariance-like conditions with respect to one boundary. The entire upper boundary was mapped inward (Leggett-Williams had invariance-like conditions with respect to only the outer boundary). That is, all of the elements in the cone for which $\|x\| = c$ were mapped inward in the sense that $\|Tx\| \leq c$. Leggett-Williams created only a compression result; Leggett-Williams did not create an expansion result.

In this paper we use techniques similar to those of Leggett-Williams that will require only subsets of both boundaries to be mapped inward and outward, respectively. We thus provide more general results than those obtained by using the Krasnosel'skii fixed point theorem [8], prior functional compression-expansion results which mapped at least one boundary inward or outward [1, 3, 5, 6, 10, 11], or the topological generalizations of fixed point theorems introduced by Kwong [9] which require both boundaries to be mapped inward or outward (invariance-like conditions). Moreover, conditions involving the norm in the original Leggett-Williams fixed point theorem are replaced by more general conditions on a convex functional.

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2000 *Mathematics Subject Classification.* 47H10.
*Key words and phrases.* Fixed-point theorems; Leggett-Williams; expansion; compression; positive solutions.

Submitted November 19, 2009. Published May 5, 2010.

## 2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

**Definition 2.1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 2.3.** A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let $\psi$ and $\delta$ be nonnegative continuous functionals on $P$; then, for positive real numbers $a$ and $b$, we define the sets:

$$P(\psi, b) = \{x \in P : \psi(x) \leq b\}, \tag{2.1}$$

$$P(\psi, \delta, a, b) = \{x \in P : a \leq \psi(x) \text{ and } \delta(x) \leq b\}. \tag{2.2}$$

**Definition 2.4.** Let $D$ be a subset of a real Banach space $E$. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then $D$ is a *retract* of $E$, and the map $r$ is a *retraction*. The *convex hull* of a subset $D$ of a real Banach space $X$ is given by

$$\operatorname{conv}(D) = \Big\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1,\ \text{and } n \in \mathbb{N} \Big\}.$$

The next theorem is due to Dugundji and its proof can be found in [4, p. 44].

**Theorem 2.5.** *For Banach spaces $X$ and $Y$, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then $F$ has a continuous extension $\tilde{F} : X \to Y$ such that*

$$\tilde{F}(X) \subset \overline{\operatorname{conv}(F(D))}.$$

**Corollary 2.6.** *Every closed convex set of a Banach space is a retract of the Banach space.*

## 3. Fixed point index

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [7, pp. 82-86]; an elementary proof can be found in [4, pp. 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

**Theorem 3.1.** *Let $X$ be a retract of a real Banach space $E$. Then, for every bounded relatively open subset $U$ of $X$ and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on $\partial U$ (relative to $X$), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

- (G1) *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*
- (G2) *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever $U_1$ and $U_2$ are disjoint open subsets of $U$ such that $A$ has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*
- (G3) *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$;*
- (G4) *Permanence: $i(A, U, X) = i(A, U \cap Y, Y)$ if $Y$ is a retract of $X$ and $A(\overline{U}) \subset Y$;*
- (G5) *Excision: $i(A, U, X) = i(A, U_0, X)$ whenever $U_0$ is an open subset of $U$ such that $A$ has no fixed points in $\overline{U} - U_0$;*
- (G6) *Solution: If $i(A, U, X) \neq 0$, then $A$ has at least one fixed point in $U$.*

*Moreover, $i(A, U, X)$ is uniquely defined.*

## 4. Main Result

**Theorem 4.1.** *Suppose $P$ is a cone in a real Banach space $E$, $\alpha$ is a nonnegative continuous concave functional on $P$, $\beta$ is a nonnegative continuous convex functional on $P$ and $T : P \to P$ is a completely continuous operator. If there exists nonnegative numbers $a$, $b$, $c$ and $d$ such that*

- (A1) *$\{x \in P : a < \alpha(x)$ and $\beta(x) < b\} \neq \emptyset$;*
- (A2) *if $x \in P$ with $\beta(x) = b$ and $\alpha(x) \geq a$, then $\beta(Tx) < b$;*
- (A3) *if $x \in P$ with $\beta(x) = b$ and $\alpha(Tx) < a$ ,then $\beta(Tx) < b$;*
- (A4) *$\{x \in P : c < \alpha(x)$ and $\beta(x) < d\} \neq \emptyset$;*
- (A5) *if $x \in P$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx) > c$;*
- (A6) *if $x \in P$ with $\alpha(x) = c$ and $\beta(Tx) > d$, then $\alpha(Tx) > c$;*

*and if*

- (H1) *$a < c$, $b < d$, $\{x \in P : b < \beta(x)$ and $\alpha(x) < c\} \neq \emptyset$, $P(\beta, b) \subset P(\alpha, c)$, and $P(\alpha, c)$ is bounded then $T$ has a fixed point $x^*$ in $P(\beta, \alpha, b, c)$;*
- (H2) *$c < a$, $d < b$, $\{x \in P : a < \alpha(x)$ and $\beta(x) < d\} \neq \emptyset$, $P(\alpha, a) \subset P(\beta, d)$, and $P(\beta, d)$ is bounded then $T$ has a fixed point $x^*$ in $P(\alpha, \beta, a, d)$.*

*Proof.* We will prove the expansion result (H1). The proof of the compression result (H2) is nearly identical; moreover, a topological proof can be found in [2] for the compression result. If we let

$$U = \{x \in P : \beta(x) < b\},$$
$$V = \{x \in P : \alpha(x) < c\},$$

then the interior of $V - U$ is given by $W = (V - U)^\circ = \{x \in V : b < \beta(x) \text{ and } \alpha(x) < c\}$. Thus $U$, $V$ and $W$ are bounded (they are subsets of $V$ which is bounded by condition (H1)), non-empty (by conditions (A1), (A4) and (H1)) and open subsets of $P$. To prove the existence of a fixed point for our operator $T$ in $P(\beta, \alpha, b, c)$, it is enough for us to show that $i(T, W, P) \neq 0$ since $W$ is the interior of $P(\beta, \alpha, b, c)$. By Corollary 2.6, $P$ is a retract of the Banach space $E$ since it is convex and closed.
**Claim 1:** $Tx \neq x$ for all $x \in \partial U$.

Let $z_0 \in \partial U$, then $\beta(z_0) = b$. We want to show that $z_0$ is not a fixed point of $T$; so suppose to the contrary that $T(z_0) = z_0$. If $\alpha(Tz_0) < a$ then $\beta(Tz_0) < b$ by condition $(A3)$, and if $\alpha(z_0) = \alpha(Tz_0) \geq a$ then $\beta(Tz_0) < b$ by condition (A2). Hence in either case we have that $Tz_0 \neq z_0$, thus $T$ does not have any fixed points on $\partial U$.

**Claim 2:** $Tx \neq x$ for all $x \in \partial V$.

Let $z_1 \in \partial V$, then $\alpha(z_1) = c$. We want to show that $z_1$ is not a fixed point of $T$; so suppose to the contrary that $T(z_1) = z_1$. If $\beta(Tz_1) > d$ then $\alpha(Tz_1) > c$ by condition $(A6)$, and if $\beta(z_1) = \beta(Tz_1) \leq d$ then $\alpha(Tz_1) > c$ by condition $(A5)$. Hence in either case we have that $Tz_1 \neq z_1$, thus $T$ does not have any fixed points on $\partial V$.

Let $w_1 \in \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\}$ (see condition $(A1)$) and let $H_1 : [0,1] \times \overline{U} \to P$ be defined by

$$H_1(t,x) = (1-t)Tx + tw_1.$$

Clearly, $H_1$ is continuous and $H_1([0,1] \times \overline{U})$ is relatively compact.

**Claim 3:** $H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$.

Suppose not; that is, there exists $(t_1, x_1) \in [0,1] \times \partial U$ such that $H(t_1, x_1) = x_1$. Since $x_1 \in \partial U$ we have that $\beta(x_1) = b$. Either $\alpha(Tx_1) < a$ or $\alpha(Tx_1) \geq a$.

Case 1: $\alpha(Tx_1) < a$. By condition $(A3)$ we have $\beta(Tx_1) < b$, which is a contradiction since

$$\begin{aligned} b = \beta(x_1) &= \beta((1-t_1)Tx_1 + t_1 w_1) \\ &\leq (1-t_1)\beta(Tx_1) + t_1\beta(w_1) < b. \end{aligned}$$

Case 2: $\alpha(Tx_1) \geq a$. We have that $\alpha(x_1) \geq a$ since

$$\begin{aligned} \alpha(x_1) &= \alpha((1-t_1)Tx_1 + t_1 w_1) \\ &\geq (1-t_1)\alpha(Tx_1) + t_1\alpha(w_1) \geq a, \end{aligned}$$

and thus by condition (A2) we have $\beta(Tx_1) < b$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$, and thus by the homotopy invariance property $(G3)$ of the fixed point index

$$i(T,U,P) = i(w_1,U,P),$$

and by the normality property (G1) of the fixed point index

$$i(T,U,P) = i(w_1,U,P) = 1.$$

Let $w_2 \in \{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\}$ (see condition (A4)) and let

$$H_2 : [0,1] \times \overline{V} \to P$$

be defined by

$$H_2(t,x) = (1-t)Tx + tw_2.$$

Clearly, $H_2$ is continuous and $H_2([0,1] \times \overline{V})$ is relatively compact.

**Claim 4:** $H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V$.

Suppose not; that is, there exists $(t_2, x_2) \in [0,1] \times \partial V$ such that $H(t_2, x_2) = x_2$. Since $x_2 \in \partial V$ we have that $\alpha(x_2) = c$. Either $\beta(Tx_2) \leq d$ or $\beta(Tx_2) > d$.

Case 1: $\beta(Tx_2) > d$. By condition $(A6)$ we have $\alpha(Tx_2) > c$, which is a contradiction since

$$\begin{aligned} c = \alpha(x_2) &= \alpha((1-t_2)Tx_2 + t_2 w_2) \\ &\geq (1-t_2)\alpha(Tx_2) + t_2\alpha(w_2) > c. \end{aligned}$$

Case 2 : $\beta(Tx_2) \leq d$. We have that $\beta(x_2) \leq d$ since

$$\begin{aligned} \beta(x_2) &= \beta((1-t_2)Tx_2 + t_2 w_2) \\ &\leq (1-t_2)\beta(Tx_2) + t_2\beta(w_2) \leq d, \end{aligned}$$

and thus by condition (A5) we have $\alpha(Tx_2) > c$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V$ and thus by the homotopy invariance property $(G3)$ of the fixed point index

$$i(T,V,P) = i(w_2,V,P),$$

and by the solution property (G6) of the fixed point index (since $w_2 \notin V$ the index cannot be nonzero) we have

$$i(T,V,P) = i(w_2,V,P) = 0.$$

Since $U$ and $W$ are disjoint open subsets of $V$ and $T$ has no fixed points in $\overline{V} - (U \cup W)$ (by claims 1 and 2), by the additivity property (G2) of the fixed point index

$$i(T,V,P) = i(T,U,P) + i(T,W,P).$$

Consequently, we have

$$i(T,W,P) = -1,$$

and thus by the solution property (G6) of the fixed point index the operator $T$ has a fixed point $x^* \in W \subset P(\beta,\alpha,b,c)$. $\square$

## 5. Application

In this section we will illustrate the key techniques for verifying the existence of a positive solution for a boundary value problem using our main result. In particular, under the expansion condition (H1) we apply the properties of a Green's function, bound the nonlinearity by constants over some intervals, and use concavity to deal with a singularity. To proceed, consider the second-order nonlinear focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1), \tag{5.1}$$

$$x(0) = 0 = x'(1), \tag{5.2}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous. If $x$ is a fixed point of the operator $T$ defined by

$$Tx(t) := \int_0^1 G(t,s) f(x(s)) ds,$$

where

$$G(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1] \times [0,1]$$

is the Green's function for the operator $L$ defined by $Lx(t) := -x''$ with right-focal boundary conditions $x(0) = 0 = x'(1)$, then it is well known that $x$ is a solution of the boundary value problem (5.1), (5.2). Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in $t$.

Let $\tau \in (0,1)$ and define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E : x \text{ is nonnegative, nondecreasing, concave and } x(\tau) \geq \tau x(1)\};$$

for $x \in P$, define the concave functional $\alpha$ on $P$ by

$$\alpha(x) := \min_{t\in[\tau,1]} x(t) = x(\tau)$$

and the convex functional $\beta$ on $P$ by

$$\beta(x) := \max_{t\in[0,1]} x(t) = x(1).$$

In the following theorem, we demonstrate how to apply the expansive condition of Theorem 4.1 to prove the existence of at least one positive solution to (5.1), (5.2).

**Theorem 5.1.** *If $\tau \in (0,1)$ is fixed, $b$ and $c$ are positive real numbers with $3b \leq c$, and $f : [0,\infty) \to [0,\infty)$ is a continuous function such that*

(a) *$f(w) > \frac{c}{\tau(1-\tau)}$ for $w \in \left[c, \frac{c}{\tau}\right]$,*
(b) *$f(w)$ is decreasing for $w \in [0, b\tau]$ with $f(b\tau) \geq f(w)$ for $w \in [b\tau, b]$, and*
(c) *$\int_0^\tau sf(bs)\,ds < \frac{2b - f(b\tau)(1-\tau^2)}{2}$,*

*then the focal problem (5.1), (5.2) has at least one positive solution $x^* \in P(\beta,\alpha,b,c)$.*

*Proof.* If we let $a = b\tau$ and $d = c/\tau$, then we have that $a < c$ and $b < d$ since $3b \leq c$. For $x \in P(\beta,\alpha,b,c)$, if $t \in (0,1)$, then by the properties of the Green's function $(Tx)''(t) = -f(x(t))$ and $Tx(0) = 0 = (Tx)'(1)$. For any $y, w \in [0,1]$ with $y \leq w$ we have the following important property of the Green's function,

$$\min_{s\in[0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w}; \tag{5.3}$$

thus for any $x \in P$ we have that

$$\begin{aligned}\alpha(Tx) = Tx(\tau) &= \int_0^1 G(\tau,s)\ f(x(s))\,ds \\ &\geq \int_0^1 \tau G(1,s) f(x(s))\,ds = \tau Tx(1) = \tau\beta(Tx).\end{aligned}$$

Therefore we have that $T : P \to P$. By the Arzela-Ascoli Theorem it is a standard exercise to show that $T$ is a completely continuous operator using the properties of $G$ and $f$. We also point out that $P(\alpha,c)$ is a bounded subset of the cone $P$, since if $x \in P(\alpha,c)$, then

$$\tau\beta(x) \leq \alpha(x) \leq c,$$

and so

$$\|x\| = \beta(x) \leq \frac{\alpha(x)}{\tau} \leq \frac{c}{\tau}.$$

Also, if $x \in P(\beta,b)$, then

$$\alpha(x) \leq \beta(x) \leq b < c,$$

and hence $P(\beta,b) \subset P(\alpha,c)$.

For any $M \in (2b,c)$ the function $x_M$ defined by

$$x_M(t) \equiv \int_0^1 MG(t,s)ds = \frac{Mt(2-t)}{2} \in P(\beta,\alpha,b,c),$$

since

$$\alpha(x_M) = x_M(\tau) = \frac{M\tau(2-\tau)}{2} < \frac{c\tau(2-\tau)}{2} \le c$$

and

$$\beta(x_M) = x_M(1) = \frac{M}{2} > b.$$

Consequently we have that $\{x \in P : b < \beta(x) \text{ and } \alpha(x) < c\} \neq \emptyset$.

Similarly, for any $L \in \left(\frac{2b}{2-\tau}, 2b\right)$ the function $x_L$ defined by

$$x_L(t) \equiv \int_0^1 LG(t,s)ds = \frac{Lt(2-t)}{2} \in \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\},$$

since

$$\alpha(x_L) = x_L(\tau) = \frac{L\tau(2-\tau)}{2} > b\tau = a$$

and

$$\beta(x_L) = x_L(1) = \frac{L}{2} < b.$$

Likewise, for any $J \in \left(\frac{2c}{\tau(2-\tau)}, \frac{2c}{\tau}\right)$, the function $x_J$ defined by

$$x_J(t) \equiv \int_0^1 JG(t,s)ds = \frac{Jt(2-t)}{2} \in \{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\},$$

since

$$\alpha(x_J) = x_J(\tau) = \frac{J\tau(2-\tau)}{2} > c$$

and

$$\beta(x_J) = x_J(1) = \frac{J}{2} < \frac{c}{\tau} = d.$$

We have that both

$$\{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\} \neq \emptyset,$$

and

$$\{x \in P : c < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset,$$

and hence conditions (A1) and (A4) of Theorem 4.1 are satisfied.

**Claim 1:** $\beta(Tx) < b$ for all $x \in P$ with $\beta(x) = b$ and $\alpha(x) \ge a$. Let $x \in P$ with $\beta(x) = b$ and $\alpha(x) \ge a$. By the concavity of $x$, for $s \in [0,\tau]$ we have

$$x(s) \ge \left(\frac{x(\tau)}{\tau}\right)s \ge bs,$$

and for all $s \in [\tau, 1]$, we have $b\tau \le x(s) \le b$. Hence by properties (b) and (c), it follows that

$$\begin{aligned}
\beta(Tx) &= \int_0^1 G(1,s)\, f(x(s))\, ds = \int_0^1 sf(x(s))\, ds \\
&= \int_0^\tau sf(x(s))\, ds + \int_\tau^1 sf(x(s))\, ds \\
&\le \int_0^\tau sf(bs)\, ds + f(b\tau)\int_\tau^1 s\, ds \\
&< \frac{2b - f(b\tau)(1-\tau^2)}{2} + \frac{f(b\tau)(1-\tau^2)}{2} = b.
\end{aligned}$$

**Claim 2:** If $x \in P$ and $\alpha(Tx) < a$, then $\beta(Tx) < b$. Let $x \in P$ with $\alpha(Tx) < a$. Thus by the properties of $G(t,s)$ given in (5.3),

$$\begin{aligned}
\beta(Tx) &= \int_0^1 G(1,s)\ f(x(s))\,ds \\
&\le \big(\frac{1}{\tau}\big)\int_0^1 G(\tau,s) f(x(s))\,ds \\
&= \big(\frac{1}{\tau}\big)\alpha(Tx) < \big(\frac{a}{\tau}\big) = b.
\end{aligned}$$

**Claim 3:** $\alpha(Tx) > c$ for all $x \in P$ with $\alpha(x) = c$ and $\beta(x) \le d$. Let $x \in P$ with $\alpha(x) = c$ and $\beta(x) \le d$. Then for $s \in [\tau, 1]$ we have

$$c \le x(s) \le d = \frac{c}{\tau}.$$

Hence by property (a),

$$\begin{aligned}
\alpha(Tx) &= \int_0^1 G(\tau,s) f(x(s))\,ds \ge \int_\tau^1 G(\tau,s)\ f(x(s))\,ds \\
&= \int_\tau^1 \tau f(x(s))\,ds > \int_\tau^1 \frac{c}{1-\tau}\,ds = c.
\end{aligned}$$

**Claim 4:** If $x \in P$ and $\beta(Tx) > d$, then $\alpha(Tx) > c$. Let $x \in P$ with $\beta(Tx) > d$. Again by the properties of $G$ given in (5.3),

$$\begin{aligned}
\alpha(Tx) &= \int_0^1 G(\tau,s) f(x(s))\,ds \\
&\ge \tau \int_0^1 G(1,s) f(x(s))\,ds \\
&= \tau\beta(Tx) > \tau d = c.
\end{aligned}$$

Therefore, the expansion hypotheses of Theorem 4.1 have been satisfied; thus the operator $T$ has at least one fixed point $x^* \in P(\beta, \alpha, b, c)$, which is a desired solution of (5.1), (5.2). □

**Example.** Let $b = 1$ $c = 5$, and $\tau = 1/2$. Then the boundary value problem

$$x'' + \frac{1}{\sqrt{x}} + e^{x-2} = 0,$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least one positive solution $x^*$ which can be verified by the above theorem, with $1 \le x^*(1)$ and $x^*(\tau) \le 5$.

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