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# SOLUTIONS TO INTEGRO-DIFFERENTIAL PARABOLIC PROBLEMS ARISING IN THE PRICING OF FINANCIAL OPTIONS IN A LEVY MARKET

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Abstract. We study an integro-differential parabolic problem modelling a process with jumps and stochastic volatility in financial mathematics. Under suitable conditions, we prove the existence of solutions in a general domain using the method of upper and lower solutions and a diagonal argument.

## 1. Introduction

In recent years there has been an increasing interest in solving PDE problems arising in Financial Mathematics and in particular on option pricing. The standard approach to this problem leads to the study of equations of parabolic type.

In financial mathematics, usually the Black-Scholes model ([6], [9], [14], [15], [16], [21]) is used for pricing derivatives, by means of a backward parabolic partial differential equation. A probability approach of the fundamental theorem of asset pricing is given in [8]. In this model, an important quantity is the volatility which is a measure of the fluctuation (risk) in the asset prices, and corresponds to the diffusion coefficient in the Black-Scholes equation.

In the standard Black-Scholes model, a basic assumption is that the volatility is constant. Several models proposed in recent years, however, allowed the volatility to be non constant or a stochastic variable. For instance, in [13] a model with stochastic volatility is proposed. In this model the underlying security S follows, as in the Black-Scholes model, a stochastic process

$$dS_t = \mu S_t \, dt + \sigma_t S_t \, dZ_t,$$

where $Z$ is a standard Brownian motion.

Unlike the classical model, the variance $v(t) = \sigma^2(t)$ also follows a stochastic process given by

$$dv_t = \kappa(\theta - v(t)) \, dt + \gamma \sqrt{v_t} \, dW_t,$$

where $W$ is another standard Brownian motion. The correlation coefficient between $W$ and $Z$ is denoted by $\rho$:

$$\mathrm{Cov}(dZ_t, dW_t) = \rho \, dt$$

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This leads to a generalized Black-Scholes equation:

$$\begin{aligned}&\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\gamma vS\frac{\partial^2 U}{\partial v\partial S}+\frac{1}{2}v\gamma^2\frac{\partial^2 U}{\partial v^2}+rS\frac{\partial U}{\partial S}\\&+[\kappa(\theta-v)-\lambda v]\frac{\partial U}{\partial v}-rU+\frac{\partial U}{\partial t}=0.\end{aligned}$$

A similar model has been considered in [1], [3]. More general models with stochastic volatility have been considered for example in [5], where the following problem is derived using the Feynman-Kac lemma:

$$\begin{gathered}u_t=\frac{1}{2}Tr\left(M(x,\tau)D^2u\right)+q(x,\tau)\cdot Du,\\u(x,0)=u_0(x)\end{gathered}$$

for some diffusion matrix $M$ and a payoff function $u_0$. In the above $D$ denotes the gradient vector and $D^2$ the Hessian matrix, when $x\in\mathbb{R}^n$.

The Black-Scholes models with jumps arise from the fact that the driving Brownian motion is a continuous process, thus it has difficulties fitting the financial data presenting large fluctuations. The necessity of taking into account the large market movements, and a great amount of information arriving suddenly (i.e. a jump) has led to the study of partial integro-differential equations (PIDE) in which the integral term is modelling the jump.

In [2], [21] the following PIDE in the variables $t$ and $S$ is obtained:

$$\frac{1}{2}\sigma^2S^2F_{SS}+(r-\lambda k)SF_S+F_t-rF+\lambda E\{F(SY,t)-F(S,t)\}=0. \tag{1.1}$$

Here $r$ denotes the riskless rate, $\lambda$ the jump intensity, and $k=E(Y-1)$, where $E$ is the expectation operator and the random variable $Y-1$ measures the percentage change in the stock price if the jump – modelled by a Poisson process – occurs (for details see [2], [21]).

The following problem is a generalization of (1.1) for $N$ assets with prices $S_1,\dots,S_N$:

$$\begin{aligned}&\sum_{i=1}^N\frac{1}{2}\sigma_i^2S_i^2\frac{\partial^2F}{\partial S_i^2}+\sum_{i\neq j}\frac{1}{2}\rho_{ij}\sigma_i\sigma_jS_iS_j\frac{\partial^2F}{\partial S_i\partial S_j}+\sum_{i=1}^N(r-\lambda k_i)S_i\frac{\partial F}{\partial S_i}+\frac{\partial F}{\partial t}-rF\\&+\lambda\int[F(S_1Y_1,\dots,S_dY_d,t)-F(S_1,\dots,S_d,t)]g(Y_1,\dots,Y_d)dY_1\dots dY_d=0\end{aligned}$$

with

$$\rho_{ij}dt=E\{dz_i,dz_j\}$$

the correlation coefficients and $g$ the probability density function of the random variable $(Y_1,\dots,Y_n)$ modelling the jump sizes.

We recall that the case in which $F$ is increasing and all jumps are negative corresponds to the evolution of a call option near a crash, see [7] and the references therein.

In applications when modelling high frequency data by a Levy –like stochastic process appears to be the best fit (see [20] and its references). Jump-diffusion models are a particular case of Levy processes and indeed stock evolution was soon modelled using various classes of Levy processes (see [4], [19]). When using these more general models, option prices are once again found by solving the resulting partial integro-differential equations. For example, integro-differential equations appear in exponential Levy models, when the market price of an asset is represented as the exponential of a Levy stochastic process. The exponential Levy models have been discussed by several authors (see for example [7, 12]).

When the volatility is stochastic we may consider the process

$$\begin{aligned} dS &= S\sigma dZ + S\mu dt, \\ d\sigma &= \beta\sigma dW + \alpha\sigma dt \end{aligned}$$

where $Z$ and $W$ are two standard Brownian motion with correlation coefficient $\rho$. If $F(S,\sigma,t)$ is the price of an option depending on the price of the asset $S$, then by Ito's lemma [15], it holds:

$$dF(S,\sigma,t) = F_S dS + F_\sigma d\sigma + \mathcal{L}F dt$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2}{\partial \sigma^2} + \rho\sigma^2 S\beta \frac{\partial^2}{\partial S \partial \sigma}$$

Under an appropriate choice of the portfolio the stochastic term of the equation vanishes (for details, see [3]).

A generalized tree process has been developed in [10, 11] that *approximates any Stochastic Volatility model*. Unlike the non-random volatility case the tree construction is stochastic every time it is created, since that is the only way we are able to deal with the huge complexity involved.

If in this model we add a jump component modelled by a compound Poisson process to the process $S$, and we follow Merton [21] we obtain the following PIDE, which is a generalization of the previous equation for stochastic volatility:

$$\begin{aligned} &\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2}\sigma^2\beta^2 \frac{\partial^2 F}{\partial \sigma^2} + \rho\sigma^2\beta S \frac{\partial^2 F}{\partial S \partial \sigma} + (r - \lambda k) S \frac{\partial F}{\partial S} \\ &- \frac{1}{2}\rho\sigma^2\beta \frac{\partial F}{\partial \sigma} + \lambda \int_{\mathbb{R}} [F(SY,\sigma,t) - F(S,\sigma,t)] g(Y) dY - rF = 0 \end{aligned} \tag{1.2}$$

Here, $r$ denotes the riskless rate, $\lambda$ the jump intensity, and $k = E(Y-1)$, where $E$ is the expectation operator, $g$ is the density of the $Y$ random variable and the quantity $Y-1$ measures the percentage change in the stock price if the jump occurs. The jump times are modelled by a Poisson process, for details see [21, 9.2] .

The previous discussion motivates to consider more general integro-differential parabolic problems. This work is devoted to the study of the solutions to the following general partial integro-differential equation in an unbounded smooth domain:

$$\begin{aligned} Lu - u_t &= \mathcal{G}(t,u) \quad \text{in } \Omega \times (0,T) \\ u(x,0) &= u_0(x) \quad \text{on } \Omega \times \{0\} \\ u(x,t) &= h(x,t) \quad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{1.3}$$

We shall assume that $\Omega \subset \mathbb{R}^d$ is an unbounded smooth domain, $L$ is a second order elliptic operator in non divergence form, namely

$$Lu := \sum_{i,j=1}^{d} a^{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^{d} b^i(x,t) u_{x_i} + c(x,t) u,$$

where the coefficients of $L$ belong to the Hölder Space $C^{\delta,\delta/2}(\overline{\Omega}\times[0,T])$ and satisfy the following two conditions

$$\Lambda|v|^2 \geq \sum_{i,j=1}^{d} a^{ij}(x,t)v_i v_j \geq \lambda|v|^2 \quad (\Lambda \geq \lambda > 0),$$
$$|b^i(x,t)| \leq C, \quad c(x,t) \leq 0.$$

The operator $\mathcal{G}$ is a completely continuous integral operator as the ones defined in (1.1) and (1.2), modelling the jump. More precisely, we shall assume that $\mathcal{G}(t,u) = \int_\Omega g(x,t,u)dx$, where $g$ is a continuous function. In this general model, the case in which $g$ is increasing with respect to $u$ and all jumps are positive corresponds to the evolution of a call option near a crash.

Furthermore, we shall assume that $u_0 \in C^{2+\delta}(\overline{\Omega})$, $h \in C^{2+\delta,1+\delta/2}(\overline{\Omega}\times[0,T])$ and satisfy the compatibility condition

$$h(x,0) = u_0(x) \quad \forall\, x \in \partial\Omega \tag{1.4}$$

We shall prove the existence of solutions of (1.3), using the method of upper and lower solutions. We recall that a smooth function $u$ is called an upper (lower) solution of problem (1.3) if

$$\begin{gathered} Lu - u_t \leq\ (\geq)\mathcal{G}(t,u) \quad \text{in } \Omega\times(0,T) \\ u(x,0) \geq\ (\leq)u_0(x) \quad \text{on } \Omega\times\{0\} \\ u(x,t) \geq\ (\leq)h(x,t) \quad \text{on } \partial\Omega\times(0,T) \end{gathered}$$

Our main result reads as follows:

**Theorem 1.1.** *Let $L$ and $\mathcal{G}$ be the operators defined above. Assume that either:*

- *$\mathcal{G}$ is nonincreasing with respect to $u$, or*
- *there exist some continuous, and increasing one dimensional function $f$ such that $\mathcal{G}(t,u) - f(u)$ is nonincreasing with respect to $u$*

*Furthermore, assume there exist $\alpha$ and $\beta$ a lower and an upper solution of the problem with $\alpha \leq \beta$ in $\Omega\times(0,T)$. Then (1.3) admits a solution $u$ such that $\alpha \leq u \leq \beta$ in $\Omega\times(0,T)$.*

**Remark 1.2.** The existence result above is applicable for any sub-linear $\mathcal{G}$, or more generally, for any $\mathcal{G}$ that is bounded by a polynomial.

## 2. The method of upper and lower solutions

In this section we give a proof of Theorem 1.1. First we solve an analogous problem in a bounded domain; with this aim, we extend the boundary data to the interior of $\Omega\times(0,T)$:

**Lemma 2.1.** *Let $U$ be a smooth and bounded subset of $\Omega$. Then there exists a unique function $\varphi_U \in C^{2+\delta,1+\delta/2}(\overline{U}\times[0,T])$ such that*

$$\begin{gathered} L\varphi_U - (\varphi_U)_t = 0, \\ \varphi_U(x,0) = u_0(x) \quad x \in U \\ \varphi_U(x,t) = h(x,t) \quad (x,t) \in \partial U\times[0,T] \end{gathered}$$

*Moreover, if $\alpha$ and $\beta$ are a lower and an upper solution of this reduced problem with $\alpha \leq \beta$ in $\Omega \times (0,T)$, then*

$$\alpha(x,t) \leq \varphi_U(x,t) \leq \beta(x,t)$$

*for $(x,t) \in \overline{U} \times [0,T]$.*

*Proof.* Existence and uniqueness follow immediately from [17, Thm. 10.4.1] and the compatibility condition (1.4). By the maximum principle, it is clear that if $\alpha \leq \beta$ are a lower and an upper solution, then

$$\alpha(x,t) \leq \varphi_U(x,t) \leq \beta(x,t)$$

□

**Lemma 2.2.** *Let $U \subset \mathbb{R}^d$ a bounded smooth domain, let $\tilde{T} < T$ let $\varphi_U$ be defined as in Lemma 2.1. Furthermore assume that $\alpha$, $\beta$ are lower respectively upper solutions of the initial problem* (1.3). *Then the problem*

$$\begin{aligned} Lu - u_t &= \mathcal{G}(t,u) \quad \text{in } U \times (0,\tilde{T}) \\ u(x,0) &= u_0(x) \quad \text{in } U \times \{0\} \\ u(x,t) &= \varphi_U(x,t) \quad \text{in } \partial U \times (0,\tilde{T}) \end{aligned} \tag{2.1}$$

*admits at least one solution $u$ with $\alpha \leq u(x,t) \leq \beta$ for $x \in U, 0 \leq t \leq \tilde{T}$.*

*Proof.* Suppose first that $\mathcal{G}$ is nonincreasing with respect to $u$. Set $u^0 = \alpha$ and $V = U \times (0,\tilde{T})$, observe that for simplicity, it is possible to assume that $\alpha = 0$. By the previous lemma, and using the fact that the problem

$$\begin{aligned} Lv - v_t &= \mathcal{G}(t,u^n) \quad \text{in } U \times (0,\tilde{T}) \\ v(x,0) &= u_0(x) \quad \text{in } U \times \{0\} \\ v(x,t) &= \varphi_U(x,t) \quad \text{in } \partial U \times (0,\tilde{T}) \end{aligned} \tag{2.2}$$

where $u_n$ is a given function, has a unique solution $v \in W_p^{2,1}(V)$ (see [18] for details); we may define $u^{n+1} \in W_p^{2,1}(V)$ where

$$W_p^{2,1}(V) = \{v \in L^p : v_{x_i}, v_{x_i x_j} \in L^p, v_t \in L^p\}$$

(see [17, 18]), as the unique solution of the problem

$$\begin{aligned} Lu^{n+1} - u_t^{n+1} &= \mathcal{G}(t,u^n) \quad \text{in } U \times (0,\tilde{T}) \\ u^{n+1}(x,0) &= u_0(x) \quad \text{in } U \times \{0\} \\ u^{n+1}(x,t) &= \varphi_U(x,t) \quad \text{in } \partial U \times (0,\tilde{T}) \end{aligned} \tag{2.3}$$

We claim that

$$\alpha \leq u^n(x,t) \leq u^{n+1}(x,t) \leq \beta \quad \forall (x,t) \in \overline{U} \times [0,\tilde{T}],\ \forall n \in \mathbb{N}$$

Indeed, by the maximum principle it follows that $u^1 \geq \alpha$. For example assume that there exists $(x_0,t_0) \in \overline{U} \times [0,\tilde{T}]$ such that $u^1(x_0,t_0) < \alpha(x_0,t_0)$. As $u^1|_{\partial\overline{U} \times [0,\tilde{T}]} \geq \alpha|_{\partial\overline{U} \times [0,\tilde{T}]}$, we deduce that $(x_0,t_0) \in U \times (0,\tilde{T})$, and we may assume that $(x_0,t_0)$ is a maximum of $\alpha - u^1$. Therefore, $\nabla(\alpha - u^1)(x_0,t_0) = 0$ (then $(\alpha - u^1)_t(x_0,t_0) = 0$) and $\Delta(\alpha - u^1)(x_0,t_0) < 0$. Since $L$ is strictly elliptic, $L(\alpha - u^1)(x_0,t_0) < 0$, but we have that $L(u^1) - u_t^1 = \mathcal{G}(t,\alpha) \leq L(\alpha) - \alpha_t$, contradiction.

Moreover, as $\mathcal{G}$ is non-increasing, we have that

$$Lu^1 - u^1_t = \mathcal{G}(t,\alpha) \geq \mathcal{G}(t,\beta) \geq L\beta - \beta_t$$

and hence $u^1 \leq \beta$. Inductively, as $u^{n-1} \leq u^n$, it follows that:

$$Lu^{n+1} - u^{n+1}_t = \mathcal{G}(t,u^n) \leq \mathcal{G}(t,u^{n-1}) = Lu^n - u^n_t$$

and thus $u^{n+1} \geq u^n$.

We recall that in order to prove the previous claims we proceed by induction: assume for example that $u^1(x_0,t_0) > \beta(x_0,t_0)$ for some $(x_0,t_0) \in \overline{U} \times [0,\tilde{T}]$. As $u^1|_{\partial\overline{U}\times[0,\tilde{T}]} \leq \beta|_{\partial\overline{U}\times[0,\tilde{T}]}$, we deduce that $(x_0,t_0) \in U \times (0,\tilde{T})$, and we may assume that $(x_0,t_0)$ is a maximum of $u^1 - \beta$. Therefore, $\nabla(u^1-\beta)(x_0,t_0) = 0$ (then $(u^1-\beta)_t(x_0,t_0) = 0$) and $\Delta(u^1-\beta)(x_0,t_0) < 0$, and as $L$ is strictly elliptic, $L(u^1-\beta)(x_0,t_0) < 0$.

We have that $L(u^1-\beta)-(u^1-\beta)_t = \mathcal{G}(t,\alpha)-(L(\beta)-(\beta)_t) \geq \mathcal{G}(t,\alpha)-\mathcal{G}(t,\beta) \geq 0$. Then $L(u^1-\beta)(x_0,t_0) \geq (u^1-\beta)_t(x_0,t_0) = 0$, which is a contradiction. Next, we assume as inductive hypothesis that $u^n \geq u^{n+1}$. As before, if $u^n(x_0,t_0) - u^{n+1}(x_0,t_0) > 0$ is a maximum, we will obtain the same contradiction with $L(u^n - u^{n+1})(x_0,t_0) \geq (u^n - u^{n+1})_t(x_0,t_0) = 0$. In the same way as before, it follows that $u^{n+1} \leq \beta$.

We now define

$$u(x,t) = \lim_{n\to\infty} u^n(x,t).$$

In [18, Chapter 7] the $L^p$-estimates for this type of differential equations are given, specifically that the $W^{2,1}_p$-norm of $u^n - u^m$ can be controlled by its $L^p$-norm and the $L^p$-norm of its image by the operator $L - \partial_t$, what results in the following:

$$\begin{aligned}
&\|D^2(u^n-u^m)\|_{L^p(V)} + \|(u^n-u^m)_t\|_{L^p(V)} \\
&\leq c\left(\|L(u^n-u^m)-(u^n-u^m)_t\|_{L^p(V)} + \|u^n-u^m\|_{L^p(V)}\right).
\end{aligned}$$

By construction,

$$L(u^n-u^m)-(u^n-u^m)_t = \mathcal{G}(\cdot,u^{n-1}) - \mathcal{G}(\cdot,u^{m-1}).$$

As $\mathcal{G}$ is a completely continuous operator, using the fact that $\alpha \leq u^n \leq \beta$ and Lebesgue's dominated convergence theorem it follows that $\{u^n\}$ is a Cauchy sequence in $W^{2,1}_p(V)$. Hence $u^n \to u$ in the $W^{2,1}_p$-norm, and then $u$ is a strong solution of the problem.

Suppose now that the only condition on $\mathcal{G}(t,u)$ is that there exist some increasing, continuous function $f$ such that $\mathcal{G}(t,u) - f(u)$ is nonincreasing with respect to $u$. We define $u^{n+1} \in W^{2,1}_p(V)$ as the unique solution of the problem

$$\begin{gathered}
Lu^{n+1} - u^{n+1}_t - f(u^{n+1}) = \mathcal{G}(t,u^n) - f(u^n) \quad \text{in } U \times (0,\tilde{T}) \\
u^{n+1}(x,0) = u_0(x) \quad \text{in } U \times \{0\} \\
u^{n+1}(x,t) = \varphi_U(x,t) \quad \text{in } \partial U \times (0,\tilde{T})
\end{gathered} \tag{2.4}$$

As before, we claim that

$$0 \leq u^n(x,t) \leq u^{n+1}(x,t) \leq \beta(x,t) \quad \forall (x,t) \in \overline{U} \times [0,\tilde{T}],\ \forall n \in \mathbb{N}_0$$

Indeed, by the maximum principle it follows that $u^1 \geq 0$; moreover,

$$Lu^1 - u^1_t - f(u^1) = \mathcal{G}(t,0) - f(0) \geq \mathcal{G}(t,\beta) - f(\beta) \geq L\beta - \beta_t - f(\beta)$$

and hence $u^1 \le \beta$. Inductively,

$$\begin{aligned} Lu^{n+1} - u_t^{n+1} - f(u^{n+1}) &= \mathcal{G}(t, u^n) - f(u^n) \\ &\le \mathcal{G}(t, u^{n-1}) - f(u^{n-1}) = Lu^n - u_t^n - f(u^n) \end{aligned}$$

Thus $u^{n+1} \ge u^n$. In the same way as before it follows that $u^{n+1} \le \beta$. Proving any of these claims is similar with what we did earlier. Indeed, assume that for some $(x_0, t_0) \in \bar{U} \times [0, \tilde{T}]$ we have $u^{n+1}(x_0, t_0) < u^n(x_0, t_0)$. By continuity we may assume that $(x_0, t_0)$ is a maximum of $u^n - u^{n+1}$. As before, $L(u^n - u^{n+1})(x_0, t_0) < 0$, but from the induction hypothesis above:

$$L(u^n - u^{n+1})(x_0, t_0) \ge (u_t^n - u_t^{n+1})(x_0, t_0) + f(u^n(x_0, t_0)) - f(u^{n+1}(x_0, t_0)) \ge 0,$$

contradiction. The expression is non-negative since $(x_0, t_0)$ is a maximum point and $f$ is increasing.

The rest of the proof follows as in the other case. □

**Proof of Theorem 1.1.** We approximate the domain $\Omega$ by a non-decreasing sequence $(\Omega_N)_{N\in\mathbb{N}}$ of bounded smooth sub-domains of $\Omega$, which can be chosen in such a way that $\partial\Omega$ is also the union of the non-decreasing sequence $\Omega_N \cap \Omega$.

Then, using Lemma 2.2 define $u^N$ as a solution of the problem

$$\begin{gathered} Lu - u_t = \mathcal{G}(t, u) \quad \text{in } \Omega_N \times (0, T - \frac{1}{N}) \\ u(x, 0) = u_0(x) \quad \text{in } \Omega_N \times \{0\} \\ u(x, t) = h(x, t) \quad \text{in } \partial\Omega_N \times (0, T - \frac{1}{N}) \end{gathered} \tag{2.5}$$

such that $\alpha = 0 \le u^N \le \beta$ in $\Omega_N \times (0, T - \frac{1}{N})$. Define $V_N = \Omega_N \times (0, T - \frac{1}{N})$ and choose $p > d$. For $M > N$, we have that

$$\begin{aligned} &\|D^2(u^M)\|_{L^p(V_N)} + \|(u^M)_t\|_{L^p(V_N)} \\ &\le c\left(\|Lu^M - (u^M)_t\|_{L^p(V_N)} + \|u^M\|_{L^p(V_N)}\right) \\ &\le c\left(\|\mathcal{G}(t, u^M)\|_{L^p(V_N)} + \|\beta\|_{L^p(V_N)}\right) \le C \end{aligned}$$

for some constant $C$ depending only on $N$. By Morrey imbedding, there exists a subsequence that converges uniformly on $\overline{V}_N$. Using a standard diagonal argument, we may extract a subsequence (still denoted $\{u^M\}$) such that $u^M$ converges uniformly to some function $u$ over compact subsets of $\Omega \times (0, T)$. For $V = U \times (0, \tilde{T})$, $U \subset \Omega$, $\tilde{T} < T$, taking $M, N$ large enough we have that

$$\begin{aligned} &\|D^2(u^N - u^M)\|_{L^p(V)} + \|(u^N - u^M)_t\|_{L^p(V)} \\ &\le c\left(\|L(u^N - u^M) - (u^N - u^M)_t\|_{L^p(V)} + \|u^N - u^M\|_{L^p(V)}\right) \end{aligned}$$

By construction,

$$L(u^N - u^M) - (u^N - u^M)_t = \mathcal{G}(t, u^{N-1}) - \mathcal{G}(t, u^{M-1})$$

As before, using that $\mathcal{G}$ is continuous, and that $\alpha \le u^N \le \beta$, by dominated convergence it follows that $\{u^N\}$ is a Cauchy sequence in $W_p^{2,1}(V)$. Hence $u^N \to u$ in the $W_p^{2,1}$-norm, and then $u$ is a classical solution in $V$. It follows that $u$ satisfies the equation on $\Omega \times (0, T)$. Furthermore, it is clear that $u(x, 0) = u_0(x)$. For $M > N$ we have that $u_M(x, t) = u_N(x, t)$ for $x \in \partial\Omega \cap \partial\Omega_N$, $t \in (0, T - \frac{1}{N})$. Thus, it follows that $u$ satisfies the boundary condition $u(x, t) = h(x, t)$ on $\partial\Omega \times [0, T)$. □

**Example 2.3.** *Consider the problem*

$$\begin{gathered} Lu - u_t = \int_{\mathbb{R}} (u(x,t) - u(xy,t))\,\nu(y)dy \quad \text{in } \Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{on } \Omega \times \{0\} \\ u(x,t) = h(x,t) \quad \text{on } \partial\Omega \times (0,T) \end{gathered} \tag{2.6}$$

*where* $\nu(y) = \frac{M}{\sqrt{\pi}} e^{-M^2 y^2}$, *a Gaussian kernel, and* $\Omega \subseteq \mathbb{R}^d$.

We shall verify that we can apply Theorem 1.1. To do so we need to show that the operator $\mathcal{G}$ has the properties of the theorem and that the problem admits an upper and a lower solution. Note that we can write $\mathcal{G}$ as:

$$\mathcal{G}(t,x,u) = u(x,t) - E[u(xY,t)],$$

where the expectation is calculated with respect to some variable $Y$ with density $\nu(y)$. Now take $f(u) = u$ which is obviously a continuous and increasing functional and observe that:

$$\mathcal{G}(t,x,u) - f(u) = -E[u(xY,t)],$$

which is clearly non-increasing with respect to $u$ (the expectation is an increasing operator).

Thus, the operator $\mathcal{G}$ satisfies the hypothesis of the theorem and we just need to find upper and lower solutions for the problem. To do so, notice that $\mathcal{G}(t,x,0) = 0$, thus $\alpha \equiv 0$ is a lower solution of the problem (with suitable boundary conditions). Let us define $\beta(x,t) = k(T-t)^{-\frac{d}{2}} e^{\frac{\theta}{T-t}|x|^2}$, with $k > 0$, $\theta > 0$. We will show that $\beta$ is an upper solution to the problem.

A straightforward computation shows that $\beta$ satisfies

$$\begin{aligned} L\beta - \beta_t = \beta\Big\{ \big(\frac{2\theta}{T-t}\big)^2 \sum_{i,j=1}^d a^{ij} x_i x_j + \frac{2\theta}{T-t} \sum_{i=1}^d a^{ii} \\ + \frac{2\theta}{T-t} \sum_{i=1}^d b^i x_i - \big[\frac{d}{2(T-t)} + \frac{\theta}{(T-t)^2}|x|^2\big]\Big\} \end{aligned}$$

Using the fact that $\sum_{i=1}^d a^{ii} \leq \Lambda$, and that $2\sum_{i=1}^d b^i x_i \leq \varepsilon |x|^2 + \frac{1}{\varepsilon}\|b\|_\infty^2$ for some small $\varepsilon$, we deduce that

$$\frac{1}{\beta}(L\beta - \beta_t) \leq (4\theta\Lambda - 1 + \varepsilon(T-t))\,\frac{\theta |x|^2}{(T-t)^2} + \frac{1}{T-t}[2\theta\Lambda - \frac{d}{2} + \frac{1}{\varepsilon}\theta\|b\|_\infty^2].$$

Taking $\varepsilon < 1/T$, and

$$\theta \leq \min\big\{\frac{1-T\varepsilon}{4\Lambda}, \frac{d\varepsilon}{2\|b\|_\infty^2 + 4\Lambda}\big\},$$

it follows that with this particular choice of $\theta$, the $\beta$ defined above has the property that

$$L\beta - \beta_t \leq 0.$$

Finally, if we show that $\mathcal{G}(t,x,\beta) \geq 0$ and if the boundary conditions are chosen so that $0 \leq u_0(x) \leq \beta(x,0)$ and $0 \leq h(x,t) \leq \beta(x,t)$ for $x \in \partial\Omega$, then we have that $\alpha \equiv 0$ and $\beta$ are respectively a lower and an upper solution of the problem.

Let us show that $\mathcal{G}(t,x,\beta) \geq 0$ for our $\beta$. To simplify notation write $\beta = ce^{\alpha x^2}$, with $c = k(T-t)^{-\frac{d}{2}}$, $\alpha = \frac{\theta}{T-t}$ and assume that $x^2 < \frac{M^2}{2\alpha} = \frac{M^2}{2\theta^2(T-t)^2}$, where $M$ is the positive constant from the definition of $\nu$. For convenience, we work in a one dimensional domain and we require that $M > \sqrt[4]{2}$. We note that $\theta$ in the definition of $\beta$ may be chosen as small as we want so the restriction on the domain of $x$ is irrelevant. Then

$$\mathcal{G}(t,x,\beta) = \int_{\mathbb{R}} \big(ce^{\alpha x^2} - ce^{\alpha x^2 y^2}\big)\nu(y)dy = ce^{\alpha x^2} - c\frac{M}{\sqrt{M^2 - x^2\alpha}} = f(x)$$

and

$$\frac{df}{dx}(x) = c\alpha x\Big(2e^{\alpha x^2} - \frac{M}{\sqrt{(M^2 - x^2\alpha)^3}}\Big).$$

We note that $c, \alpha > 0$, and the term within parentheses is always positive for any $x$ within the domain $\{x \in R : x^2 < \frac{M^2}{2\alpha}\}$ if $M > \sqrt[4]{2}$. Indeed, since $\alpha x^2 > 0$ we always have

$$2e^{\alpha x^2} - \frac{M}{\sqrt{(M^2 - x^2\alpha)^3}} \geq 2 - \frac{M}{\sqrt{(M^2 - x^2\alpha)^3}} \geq 0$$

The last inequality holds for any $x$ with property

$$x^2 \leq \frac{1}{\alpha}\Big(M^2 - \frac{M^{2/3}}{2^{2/3}}\Big).$$

However, recall that the domain of $x$ is $x^2 < \frac{M^2}{2\alpha}$ and note that if $M > \sqrt[4]{2}$,

$$\frac{M^2}{2\alpha} < \frac{1}{\alpha}\Big(M^2 - \frac{M^{2/3}}{2^{2/3}}\Big),$$

thus the term within parentheses in $\frac{df}{dx}$ is positive for all $x$ within the domain.

Therefore, the sign of the derivative is the sign of $x$, which means that the function $f(x)$ attains its minimum at $x = 0$. As $f(0) = 0$, we conclude that $f(x) \geq 0$ for all $x$ in the domain and therefore $\beta$ is an upper solution of the problem.

Finally, we note that we can use the procedure outlined within the proof of the Lemma 2.2 to construct an approximate solution of the original problem. As this construction is straightforward we do not insist on details.

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