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ANTI-PERIODIC SOLUTIONS FOR HIGH-ORDER CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS

ZUDA HUANG, LEQUN PENG, MIN XU

Abstract. In this article, we study anti-periodic solutions for high-order cellular neural networks with time-varying delays. Sufficient conditions for the existence and exponential stability of anti-periodic solutions are presented.

1. Introduction

In recent years, high-order cellular neural networks (HCNNs) have attracted attention due to their wide range of applications in fields such as signal and image processing, pattern recognition, optimization, and many other subjects. There have been many results on the problem of global stability of equilibrium points and periodic solutions of HCNNs in the literature (see [3, 4, 7, 14, 10, 11, 12, 15]). However, there are only a few references on the problem of existence and stability of anti-periodic solutions. However, the existence of anti-periodic solutions is important in nonlinear differential equation (see [1, 2, 6, 8, 9]). Thus, it is worth while to investigate the existence and stability of anti-periodic solutions for HCNNs.

In this article, we study the anti-periodic solution of the high-order cellular neural network medelled by

$$\begin{aligned} x_i'(t) = -c_i(t)x_i(t) &+ \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\widetilde{\tau}_j(t))) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t) g_j(x_j(t-\tau_j(t))) g_k(x_k(t-\tau_k(t))) + u_i(t), \end{aligned} \tag{1.1}$$

where $i = 1, 2, \dots, n$; $c_i, a_{ij}, b_{ijk}, f_j, g_j, u_i$ are continuous functions on $\mathbb{R}$; $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is the state vector; c_i is a positive parameter; a_{ij} and b_{ijk} are the first and second order connection weights of the neural networks, respectively; f_j and g_j are the activation functions; u_i is an external input to the ith

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neuron; $\widetilde{\tau}_j(t)$ and $\tau_j(t)$ are the time-varying delay that satisfy $0 \le \widetilde{\tau}_j(t) \le \tau$ and $0 \le \tau_j(t) \le \tau$ (τ is a constant).

The initial conditions are

$$x_i(t) = \varphi_i(t), \quad t \in [-\tau, 0],\ i = 1, 2, \dots, n, \tag{1.2}$$

where $\varphi(\cdot) = [\varphi_1(\cdot), \varphi_2(\cdot), \dots, \varphi_n(\cdot)] \in C([-\tau, 0], \mathbb{R}^n)$ and $C([-\tau, 0], \mathbb{R}^n)$ denotes the set of continuous functions.

The rest of this article is organized as follows. In Section 2, we give some notations and preliminary knowledge. In Section 3, we present our main results. In Section 4, we present an example to illustrate the effectiveness of our results. Finally, we give the conclusions in Section 5.

2. Preliminary Results

A continuous function $h : \mathbb{R} \to \mathbb{R}$ is said to be T-anti-periodic on $\mathbb{R}$, if

$$h(t + T) = -h(t) \quad \text{for all } t \in \mathbb{R}.$$

We consider (1.1) under the following assumptions: For $i, j = 1, 2, \dots, n$, it will be assumed that

$$\begin{gathered} c_i(t+T) = c_i(t), \quad \tau_i(t+T) = \tau_i(t), \quad a_{ij}(t+T) f_j(v) = -a_{ij}(t) f_j(-v), \\ \widetilde{\tau}_j(t+T) = \widetilde{\tau}_j(t), \quad b_{ijk}(t+T) = -b_{ijk}(t), \quad u_i(t+T) = -u_i(t), \quad \forall t, v \in \mathbb{R}. \end{gathered} \tag{2.1}$$

and

$$\overline{u} = \max_{1 \le i \le n} \sup_{t \in \mathbb{R}} |u_i(t)|. \tag{2.2}$$

Also we will use the assumptions.

- (H1) For $j = 1, 2, \dots, n$, there exist $\overline{g_j} > 0$ such that $|g_j(u)| \le \overline{g_j}$ for all $u \in \mathbb{R}$;
- (H2) for $j = 1, 2, \dots, n$, there exist $L_j > 0$ and $M_j > 0$ such that
$$\begin{gathered} |f_j(u) - f_j(v)| \le L_j |u - v|, \quad |g_j(u) - g_j(v)| \le M_j |u - v|, \\ f_j(0) = 0, \quad g_j(0) = 0, \quad \forall u, v \in \mathbb{R}. \end{gathered}$$
- (H3) There exist constants $\eta > 0, \lambda > 0$ and $\xi_i > 0$, $i = 1, 2, \dots, n$, such that for all $t > 0$,

$$[\lambda - c_i(t)]\xi_i + \Big[\sum_{j=1}^{n} |a_{ij}(t)| L_j \xi_j + \sum_{j=1}^{n} \sum_{k=1}^{n} |b_{ijk}(t)| (\overline{g_k} M_j \xi_j + \overline{g_j} M_k \xi_k)\Big] e^{\lambda \tau} < -\eta < 0.$$

Definition 2.1. Let $x^*(t) = \big(x_1^*(t), x_2^*(t), \dots, x_n^*(t)\big)^T$ be an anti-periodic solution of (1.1) with initial value $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T$. If there exist constants $\lambda > 0$ and $M_\varphi > 1$ such that for every solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of (1.1) with an initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$, with

$$|x_i(t) - x_i^*(t)| \le M_\varphi \|\varphi - \varphi^*\| e^{-\lambda t}, \quad \forall t > 0,\ i = 1, 2, \dots, n,$$

where

$$\|\varphi - \varphi^*\| = \sup_{-\tau \le s \le 0} \max_{1 \le i \le n} |\varphi_i(s) - \varphi_i^*(s)|.$$

Then $x^*(t)$ is said to be globally exponentially stable.

Next, we present two important lemmas, to be used for proving our main results in Section 3.

Lemma 2.2. *Let* (H1)–(H3) *hold. Suppose that* $\widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), \dots, \widetilde{x}_n(t))^T$ *is a solution of* (1.1) *with initial conditions*

$$\widetilde{x}_i(s) = \widetilde{\varphi}_i(s), \quad |\widetilde{\varphi}_i(s)| < \xi_i \frac{\overline{u}+1}{\eta}, \quad s \in [-\tau, 0],\ i = 1, 2, \dots, n. \tag{2.3}$$

Then

$$|\widetilde{\varphi}_i(t)| < \xi_i \frac{\overline{u}+1}{\eta}, \quad \text{for all } t \geq 0,\ i = 1, 2, \dots, n. \tag{2.4}$$

Proof. Assume, by way of contradiction, assume that (2.4) does not hold. Then there must exist $i \in \{1, 2, \dots, n\}$ and $\sigma > 0$ such that

$$|\widetilde{x}_i(\sigma)| = \xi_i \frac{\overline{u}+1}{\eta}, \quad \text{and} \quad |\widetilde{x}_j(t)| < \xi_j \frac{\overline{u}+1}{\eta} \quad \text{for all } t \in (-\tau, \sigma),\ j = 1, 2, \dots, n. \tag{2.5}$$

By directly computing the upper left derivative of $|\widetilde{x}_i(t)|$. under assumptions (H1)–(H3), and (2.5), we deduce that

$$\begin{aligned}
0 &\leq D^+(|\widetilde{x}_i(\sigma)|) \\
&\leq -c_i(\sigma)|\widetilde{x}_i(\sigma)| + \sum_{j=1}^{n} |a_{ij}(\sigma)||f_j(\widetilde{x}_j(\sigma - \widetilde{\tau}_j(\sigma)))| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}(\sigma)||g_j(\widetilde{x}_j(\sigma - \tau_j(\sigma)))|\overline{g_k} + |u_i(\sigma)| \\
&\leq -c_i(\sigma)\xi_i \frac{\overline{u}+1}{\eta} + \sum_{j=1}^{n} |a_{ij}(\sigma)|L_j|\widetilde{x}_j(\sigma - \widetilde{\tau}_j(\sigma))| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}(\sigma)|M_j|\widetilde{x}_j(\sigma - \tau_j(\sigma))|\overline{g_k} + \overline{u} \\
&\leq -c_i(\sigma)\xi_i \frac{\overline{u}+1}{\eta} + \sum_{j=1}^{n} |a_{ij}(\sigma)|L_j\xi_j \frac{\overline{u}+1}{\eta} + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}(\sigma)|M_j\xi_j \frac{\overline{u}+1}{\eta}\overline{g_k} + \overline{u} \\
&= [-c_i(\sigma)\xi_i + \sum_{j=1}^{n} |a_{ij}(\sigma)|L_j\xi_j + \sum_{j=1}^{n}\sum_{k=1}^{n} |b_{ijk}(\sigma)|M_j\xi_j\overline{g_k}] \frac{\overline{u}+1}{\eta} + \overline{u} < 0.
\end{aligned} \tag{2.6}$$

which is a contradiction and implies that (2.4) holds. This completes the proof. □

Remark 2.3. It follows that the bounded solution $\widetilde{x}(t)$ can be defined on $[0, \infty)$ according to the theory of functional differential equations in [5].

Lemma 2.4. *Suppose that* (H1)–(H3) *hold. Let* $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ *be the solution of* (1.1) *with initial value* $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T$*, and* $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ *be the solution of* (1.1) *with initial value* $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$*. Then there exist constants* $M_\varphi > 1$ *such that*

$$|x_i(t) - x_i^*(t)| \leq M_\varphi \|\varphi - \varphi^*\| e^{-\lambda t}, \quad \forall t > 0,\ i = 1, 2, \dots, n.$$

Proof. Let $y(t) = \{y_j(t)\} = \{x_j(t) - x_j^*(t)\} = x(t) - x^*(t)$. Then

$$
\begin{aligned}
y_i'(t) = -c_i(t)[x_i(t) - x_i^*(t)] &+ \sum_{j=1}^n a_{ij}(t)\big[f_j(x_j(t-\widetilde{\tau}_j(t))) - f_j(x_j^*(t-\widetilde{\tau}_j(t)))\big] \\
&+ \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(t)\big[g_j(x_j(t-\tau_j(t)))g_k(x_k(t-\tau_k(t))) \\
&- g_j(x_j^*(t-\tau_j(t)))g_k(x_k^*(t-\tau_k(t)))\big]
\end{aligned}
$$

where $i = 1, 2, \dots, n$. Next, define a Lyapunov functional as

$$
V_i(t) = |y_i(t)|e^{\lambda t}, \quad i = 1, 2, \dots, n. \tag{2.7}
$$

By (2.6) and (2.7) it follows that

$$
\begin{aligned}
D^+(V_i(t)) &\le D^+(|y_i(t)|)e^{\lambda t} + \lambda|y_i(t)|e^{\lambda t} \\
&\le (\lambda - c_i(t))|y_i(t)|e^{\lambda t} + \{\sum_{j=1}^n |a_{ij}|(t)L_j|y_j(t-\widetilde{\tau}_j(t))| \\
&\quad + \sum_{j=1}^n \sum_{k=1}^n |b_{ijk}(t)|[\overline{g_k}M_j|y_j(t-\tau_j(t))| + \overline{g_j}M_k|y_k(t-\tau_k(t))|]\}e^{\lambda t}
\end{aligned} \tag{2.8}
$$

where $i = 1, 2, \dots, n$. Let $m^* > 1$ denote a real number such that

$$
m^*\xi_i > \|\varphi - \varphi^*\| = \sup_{-\tau \le s \le 0} \max_{1 \le j \le n} |\varphi_j(s) - \varphi_j^*(s)| > 0, \quad i = 1, 2, \dots, n.
$$

Then by (2.7), we have

$$
V_i(t) = |y_i(t)|e^{\lambda t} < m^*\xi_i, \quad \text{for all } t \in [-\tau, 0],\ i = 1, 2, \dots, n.
$$

Thus we can claim that

$$
V_i(t) = |y_i(t)|e^{\lambda t} < m^*\xi_i, \quad \text{for all } t > 0,\ i = 1, 2, \dots, n. \tag{2.9}
$$

Otherwise, there must exist $i \in \{1, 2, \dots, n\}$ and $t_i > 0$ such that

$$
V_i(t_i) = m^*\xi_i, \quad V_j(t) < m^*\xi_j, \quad \forall t \in [-\tau, t_i),\ j = 1, 2, \dots, n. \tag{2.10}
$$

Combining (2.8) with (2.10), we obtain

$$
\begin{aligned}
0 &\le D^+(V_i(t_i) - m^*\xi_i)\\
&= D^+(V_i(t_i))\\
&\le (\lambda - c_i(t_i))|y_i(t_i)|e^{\lambda t_i} + \{\sum_{j=1}^n |a_{ij}(t_i)|L_j|y_j(t_i - \widetilde{\tau}_j(t_i))|\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n |b_{ijk}(t_i)|[\overline{g_k}M_j|y_j(t_i-\tau_j(t_i))| + \overline{g_j}M_k|y_k(t_i-\tau_k(t_i))|]\}e^{\lambda t_i}\\
&\le (\lambda - c_i(t_i))m^*\xi_i + \sum_{j=1}^n |a_{ij}|(t_i)L_j m^*\xi_j e^{\lambda\tau}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n |b_{ijk}(t_i)|\overline{g_k}M_j m^*\xi_j e^{\lambda\tau} + \sum_{j=1}^n\sum_{k=1}^n |b_{ijk}(t_i)|\overline{g_j}M_k m^*\xi_k e^{\lambda\tau}\\
&= \{(\lambda - c_i(t_i))\xi_i + \big[\sum_{j=1}^n |a_{ij}(t_i)|L_j\xi_j\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n |b_{ijk}(t_i)|(\overline{g_k}M_j\xi_j + \overline{g_j}M_k\xi_k)\big]e^{\lambda\tau}\}m^*
\end{aligned} \tag{2.11}
$$

It is clear that

$$
(\lambda - c_i)\xi_i + [\sum_{j=1}^n |a_{ij}|(t_i)L_j\xi_j + \sum_{j=1}^n\sum_{k=1}^n |b_{ijk}(t_i)|(\overline{g_k}M_j\xi_j + \overline{g_j}M_k\xi_k)]e^{\lambda\tau} > 0
$$

This contradicts (H3), then (2.9) holds. Letting $M_\varphi > 1$, such that

$$
\max_{1\le i\le n}\{m^*\xi_i\} \le M_\varphi\|\varphi - \varphi^*\|, \quad i = 1, 2, \dots, n. \tag{2.12}
$$

In view of (2.9) and (2.12), we obtain

$$
|x_i(t) - x_i^*(t)| = |y_i(t)| \le \max_{1\le i\le n}\{m\xi_i\}e^{-\lambda t} \le M_\varphi\|\varphi - \varphi^*\|e^{-\lambda t},
$$

where $i = 1, 2, \dots, n$, $t > 0$. This completes the proof. □

Remark 2.5. If $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ is the T-anti-periodic solution of (1.1), it follows from Lemma 2.2 and the Definition 2.1 that $x^*(t)$ is globally exponentially stable.

3. Main Results

In this section,we present our main result that there exists the exponentially stable anti-periodic solution of (1.1).

Theorem 3.1. *Assume that* (H1)–(H3) *are satisfied. Then* (1.1) *has exactly one T-anti-periodic solution $x^*(t)$. Moreover, this solution is globally exponentially stable.*

Proof. Let $v(t) = (v_1(t), v_2(t), \dots,\ v_n(t))^T$ is a solution of (1.1) with initial conditions

$$
v_i(s) = \varphi_i^v(s), \quad |\varphi_i^v(s)| < \xi_i\frac{\overline{u}+1}{\eta}, \quad s \in (-\tau, 0],\ i = 1, 2, \dots, n. \tag{3.1}
$$

Thus according to Lemma 2.1, the solution $v(t)$ is bounded and

$$|v_i(t)| < \xi_i \frac{\overline{u}+1}{\eta}, \quad \text{for all } t \in \mathbb{R},\ i = 1,2,\dots,n. \tag{3.2}$$

From (2.1), we obtain

$$\begin{aligned}
&((-1)^{m+1} v_i(t+(m+1)T))' \\
&= (-1)^{m+1}\{-c_i(t+(m+1)T)v_i(t+(m+1)T) \\
&\quad + \sum_{j=1}^{n} a_{ij}(t+(m+1)T) f_j(v_j(t+(m+1)T - \widetilde{\tau}_j(t+(m+1)T))) \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t+(m+1)T) g_j(v_j(t+(m+1)T - \tau_j(t+(m+1)T))) \\
&\quad \times g_k(v_k(t+(m+1)T - \tau_k(t+(m+1)T))) + u_i(t+(m+1)T)\} \\
&= -c_i(t)[(-1)^{m+1} v_i(t+(m+1)T)] + \sum_{j=1}^{n} a_{ij}(t) \\
&\times f_j[(-1)^{m+1} v_j((t+(m+1)T) - \widetilde{\tau}_j(t+(m+1)T))] \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ijk}(t) g_j[(-1)^{m+1} v_j((t+(m+1)T) - \tau_j(t+(m+1)T))] \\
&\quad \times g_k[(-1)^{m+1} v_k((t+(m+1)T) - \tau_k(t+(m+1)T))] + u_i(t)
\end{aligned} \tag{3.3}$$

where $i = 1,2,\dots,n$. Thus $(-1)^{m+1} v(t+(m+1)T)$ are the solutions of (1.1) on $\mathbb{R}$ for any natural number m. Then, from Lemma 2.2, there exists a constant $M > 0$ such that

$$\begin{aligned}
&|(-1)^{m+1} v_i(t+(m+1)T) - (-1)^m v_i(t+mT)| \\
&\le M e^{-\lambda(t+mT)} \sup_{-\tau\le s\le 0} \max_{1\le i\le n} |v_i(s+T) + v_i(s)| \\
&\le 2e^{-\lambda(t+mT)} M \max_{1\le i\le n}\{\xi_i \frac{\overline{u}+1}{\eta}\}, \quad \forall t+mT > 0,\ i = 1,2,\dots,n.
\end{aligned} \tag{3.4}$$

Thus, for any natural number m, we have

$$(-1)^{m+1} v_i(t+(m+1)T) = v_i(t) + \sum_{k=0}^{m} [(-1)^{k+1} v_i(t+(k+1)T) - (-1)^k v_i(t+kT)]. \tag{3.5}$$

Hence,

$$|(-1)^{m+1} v_i(t+(m+1)T)| \le |v_i(t)| + \sum_{k=0}^{m} |(-1)^{k+1} v_i(t+(k+1)T) - (-1)^k v_i(t+kT)|, \tag{3.6}$$

where $i = 1,2,\dots,n$. In view of (3.4), we can choose a sufficiently large constant $N > 0$ and a positive constant α such that

$$|(-1)^{m+1} v_i(t+(m+1)T) - (-1)^m v_i(t+mT)| \le \alpha (e^{-\lambda T})^m, \tag{3.7}$$

for all $m > N$, $i = 1,2,\dots,n$, on any compact set of $\mathbb{R}$. Obviously, together with (3.5), (3.6) and (3.7), $\{(-1)^m v(t+mT)\}$ uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ on any compact set of $\mathbb{R}$.

Now we show that $x^*(t)$ is T-anti-periodic solution of (1.1). Firstly, $x^*(t)$ is T-anti-periodic, since

$$\begin{aligned} x^*(t+T) &= \lim_{m\to\infty}(-1)^m v(t+T+mT) \\ &= -\lim_{(m+1)\to\infty}(-1)^{m+1}v(t+(m+1)T) = -x^*(t). \end{aligned}$$

secondly, we prove that $x^*(t)$ is a solution of (1.1). Because of the continuity of the right-hand side of (1.1), (3.3) implies that $\{((-1)^{m+1}v(t+(m+1)T))'\}$ uniformly converges to a continuous function on any compact subset of $\mathbb{R}$. Thus, letting $m\to\infty$, we can easily obtain

$$\begin{aligned} \frac{d}{dt}\{x_i^*(t)\} &= -c_i(t)x_i^*(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j^*(t-\widetilde{\tau}_j(t))) \\ &\quad + \sum_{j=1}^n\sum_{k=1}^n b_{ijk}(t)g_j(x_j^*(t-\tau_j(t)))g_k(x_k^*(t-\tau_k(t))) + u_i(t), \end{aligned} \tag{3.8}$$

where $i=1,2,\dots,n$. Therefore, $x^*(t)$ is a solution of (1.1).

Finally, by applying Lemma 2.2, it is easy to check that $x^*(t)$ is globally exponentially stable. This completes the proof. $\square$

4. An Example

In this section, a simple example is provided to illustrate our results. Consider the high-order cellular neural network with delays

$$\begin{aligned} x_1'(t) &= -x_1(t) + \frac{1}{4}|\sin t|f_1(x_1(t-1)) + \frac{1}{36}|\cos t|f_2(x_2(t-2)) \\ &\quad + \frac{1}{72}\sin t g_1^2(x_1(t-1)) + \frac{1}{36}\cos t g_1(x_1(t-1))g_2(x_2(t-2)) \\ &\quad + \frac{1}{72}\cos t g_2^2(x_2(t-2)) + \frac{1}{9}\sin t, \\ x_2'(t) &= -x_2(t) + \frac{1}{36}|\cos t|f_1(x_1(t-1)) + \frac{1}{4}|\sin t|f_2(x_2(t-2)) \\ &\quad + \frac{1}{72}\cos t g_1^2(x_1(t-1)) + \frac{1}{36}\cos t g_1(x_1(t-1))g_2(x_2(t-2)) \\ &\quad + \frac{1}{72}\sin t g_2^2(x_2(t-2)) + \frac{1}{9}\sin t\,, \end{aligned} \tag{4.1}$$

where $f_1(x)=f_2(x)=x$, $g_1(x)=g_2(x)=\arctan x$, $c_1(t)=c_2(t)=1, u_1(t)=\frac{1}{9}\sin t$, $u_2(t)=\frac{1}{9}\sin t$, $a_{11}(t)=a_{22}(t)=\frac{1}{4}|\sin t|$, $a_{12}(t)=a_{21}(t)=\frac{1}{36}|\cos t|$, $b_{111}(t)=b_{222}(t)=\frac{1}{72}\sin t$, $b_{112}(t)=b_{121}(t)=b_{122}(t)=b_{211}(t)=b_{212}(t)=b_{221}(t)=\frac{1}{72}\cos t$. Noting that

$$L_1=L_2=M_1=M_2=1, \quad \overline{g_1}=\overline{g_2}=\frac{\pi}{2}.$$

Therefore, there exist constants $\eta=\frac{1}{2}$, $\lambda=\frac{1}{1800}$ and $\xi_1=\xi_2=1$, such that for all $t>0$, $i=1,2$, there holds

$$[\lambda-c_i(t)]\xi_i + [\sum_{j=1}^2 a_{ij}(t)L_j\xi_j + \sum_{j=1}^2\sum_{k=1}^2 b_{ijk}(t)(\overline{g_k}M_j\xi_j+\overline{g_j}M_k\xi_k)]e^{\lambda\tau} < -\eta,$$

which implies that system (4.1) satisfy all the conditions in Theorem 3.1. Hence, (4.1) has exactly one π-anti-periodic solution. Moreover, this solution is globally exponentially stable.

This fact is verified in the numerical simulation in Figure 1.

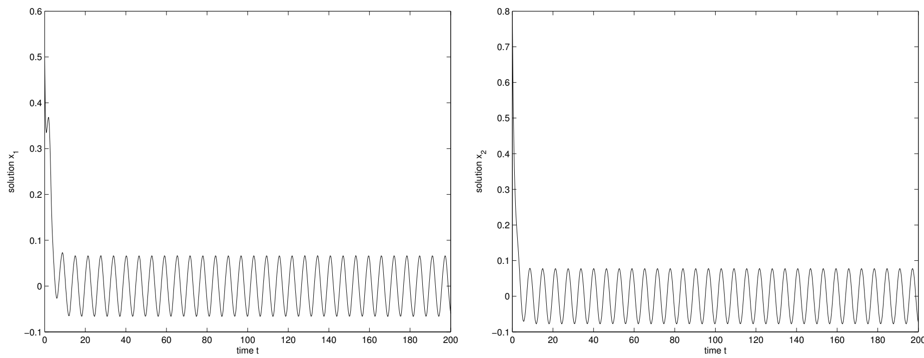

FIGURE 1. Numerical solution $(x_1(t), x_2(t))$ of system (4.1) for $(\varphi_1(s), \varphi_2(s)) = (0.5, 0.8)$.

We remark that (4.1) is a very simple form of high-order cellular neural networks with delays. However, the results in the references can not be applicable for obtaining existence and exponential stability of the anti-periodic solutions. This makes our results new.

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