Electronic Journal of Differential Equations, Vol. 2010(2010), No. 56, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO ELLIPTIC PROBLEMS WITH DISCONTINUITIES AND FREE BOUNDARY CONDITIONS

SABRI BENSID, SIDI MOHAMMED BOUGUIMA

Abstract. We study the nonlinear elliptic problem with discontinuous nonlinearity

$$\begin{gathered} -\Delta u = f(u)H(u-\mu) \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega, \end{gathered}$$

where H is the Heaviside unit function, f, h are given functions and μ is a positive real parameter. The domain Ω is the unit ball in $\mathbb{R}^n$ with $n \geq 3$. We show the existence of a positive solution u and a hypersurface separating the region where $-\Delta u = 0$ from the region where $-\Delta u = f(u)$. Our method relies on the implicit function theorem and bifurcation analysis.

1. Introduction

This article concerns the existence and multiplicity of solutions to the problem

$$\begin{gathered} -\Delta u = f(u)H(u-\mu) \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is the unit ball of $\mathbb{R}^n$ with $n \geq 3$; f, h are given functions; μ is a positive real parameter; and H is the Heaviside function

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0\,. \end{cases}$$

Problem (1.1) can be reformulated as an equivalent free boundary problem: Find $u \in C^2(\Omega \setminus \partial w) \cap C^1(\overline{\Omega})$ such that

$$\begin{gathered} -\Delta u = f(u) \quad \text{in } \ w, \\ -\Delta u = 0 \quad \text{in } \Omega \setminus w, \\ u = h \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $w = \{x \in \Omega : u(x) > \mu\}$ and ∂w is the free boundary to be determined. On each side of the free boundary $\partial w = \{x \in \Omega, u(x) = \mu\}$ the equation $-\Delta u = f(u)$ (the side $u > \mu$) or $\Delta u = 0$ (the side $u < \mu$) is satisfied in the classical sense.

2000 *Mathematics Subject Classification.* 34R35, 35J25.
Key words and phrases. Green function; maximum principle; bifurcation; free boundary problem.

Submitted February 22, 2010. Published April 19, 2010.

If χ_w denotes the characteristic function of the set w, we can write (1.1) as

$$\begin{gathered} -\Delta u = f(u)\chi_w \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

This problem has received less attention and a partial result is obtained by the authors in [2].

The aim of this paper is to improve and complement the result obtained in [2]. We will prove the existence and multiplicity results for (1.1) together with some properties of their free boundaries in the case where the nonlinearity satisfies only the condition

$$\frac{f(\mu)}{\mu} > M_n, \quad M_n = \frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}, \quad \text{for} \quad n \geq 3.$$

Another question left open is whether the free boundary of (1.1) is analytic. In this paper we use the techniques presented in [10] to derive affirmative answer to this problem. For

$$\frac{f(\mu)}{\mu} = M_n,$$

we will show that there exists an exceptional position of the free boundary corresponding to (1.1) with $h = 0$ for which bifurcation can occur. The problem (1.1) is often stated in variational context, but our approach hinges in considering the parametrization of the free boundary as the unknown of the problem. This method allows us to understand the effect of the boundary perturbation on the shape of the free boundary. It is clearly not appropriate to review here the rather extensive literature on discontinuous elliptic problems, and we restrict ourselves to outline, referring the reader who requires more information to the paper [2] for extensive further references. Other methods have been developed for problem (1.1), for example Kolibal [11] recently used numerical schemes to compute solutions for a particular case of (1.1) with $h = 0$. The reader can consult [6, 14] for similar problems with Neumann boundary conditions.

We start by giving more precise definitions and hypotheses on quantities used in this paper. Let Γ be the set $\{x \in \Omega : u(x) = \mu\}$. This set is called the free boundary. Because the nonlinearity in (1.1) is discontinuous, we shall specify precisely the meaning of a solution.

Definition 1.1. By a solution, we mean a function $u \in C^2(\Omega \backslash \Gamma) \cap C^1(\overline{\Omega})$ satisfying problem (1.1).

The free boundary determined by the solution itself separates the region where $u < \mu$ and $-\Delta u = 0$ in the classical sense from the region where $u > \mu$ and $-\Delta u = f(u)$. The following assumptions will be needed throughout the paper. Let λ_1 be the first eigenvalue of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions.

- (F1) The function f is k-lipstchitzian, non-decreasing, positive and there exist two strictly positive constants $k, \beta > 0$ such that $f(s) \leq ks + \beta$ with $k < \min\{\lambda_1, 1\}$.
- (F2) The function f is differentiable and constant on the interval of the form $[0, c]$ where $c > \frac{\beta}{2n-k}$.

(F3) There exists $\mu^* > 0$ such that

$$\frac{f(\mu^*)}{\mu^*} = M_n, \quad M_n = \frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}.$$

The main result of this paper is reads as follows.

Theorem 1.2. *(a) Assume that* (F1), (F2) *are satisfied and suppose that there exists* $\mu > 0$ *such that*

$$\frac{f(\mu)}{\mu} > M_n, \quad M_n = \frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}, \quad \text{for } n \geq 3.$$

Let $\|h\|_\infty = \max_{x\in\partial\Omega} |h(x)|$. *If* h *is small enough,* $\|h\|_\infty < \mu$, *then* (1.1) *has at least two positive solutions and the free boundaries are analytic hypersurfaces.*

(b) Assume that (F1), (F2), (F3) *are satisfied. There is an exceptional value* $r_0 \in (0, 1)$ *at which the reduced problem* (1.1) *with* $h = 0$ *has a bifurcation: there is a solution of* (1.1) *with* $h = 0$ *having free boundary in polar coordinates of the form*

$$\{(r, \theta) \in (0, 1) \times S,\ r = r_0 + s\phi_{00} + o(s)\}$$

for all s *in a neighborhood of zero, where* ϕ_{00} *is a given constant and* S *is the unit sphere in* $\mathbb{R}^n$.

The proof of this theorem is given in several steps. The hypothesis (F2) is rather technical and allows us to avoid some tedious computations.

This paper is organized as follows. In Section 2, we give existence results for the reduced problem (1.1) with $h = 0$. Section 3 is devoted to the statements of the main results. Finally, Section 4 is devoted to regularity of the free boundary.

2. The reduced problem

This section deals with the existence of solutions for the reduced problem (1.1) with $h = 0$.

Proposition 2.1. *(a) Assume that* (F1), (F2), (F3) *are satisfied. Then* (1.1) *with* $h = 0$ *has a solution* $u > 0$ *such that the free boundary* $\{(r, \theta) \in \Omega; u(r, \theta) = \mu^*\}$ *is the sphere of radius* $r_0 = (\frac{2}{n})^{\frac{1}{n-2}}$.

(b) Assume that (F1), (F2) *are satisfied. If* $\frac{f(\mu)}{\mu} > M_n$, *then* (1.1) *with* $h = 0$ *has two positive and radial solutions and their free boundaries are respectively spheres with radii* r_1 *and* r_2 *different from* $(\frac{2}{n})^{\frac{1}{n-2}}$.

The approach we shall adopt in our analysis is to find radial solutions of (1.1) with $h = 0$. For this purpose, we start by establishing useful estimates.

Lemma 2.2 (a priori estimates)**.** *Assume* (F1). *If* u *is a positive solution of* (1.1) *with* $h = 0$, *then* $0 < u \leq \frac{\beta}{2n-k}$ *in* Ω.

Proof. Let $\overline{u}$ be a supersolution of the reduced problem satisfying

$$\begin{aligned} -\Delta\overline{u} &= k\overline{u} + \beta \quad \text{in } \Omega, \\ \overline{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Since the function $\overline{u} \to k\overline{u} + \beta$ is Lipschitzian, then from the well-known result by [8], it follows that $\overline{u}$ is radial and satisfies

$$\begin{gathered} -r^{1-n}\partial/\partial r(r^{n-1}\partial\overline{u}/\partial r) = k\overline{u} + \beta \quad \text{for } 0 < r < 1 \\ \overline{u}'(0) = 0, \quad \overline{u}(1) = 0. \end{gathered} \tag{2.2}$$

Therefore,

$$\frac{\partial\overline{u}}{\partial\rho}(\rho) = -\rho^{1-n}\int_0^\rho s^{n-1}(k\overline{u}(s) + \beta)ds = -\frac{\beta\rho}{n} - k\rho^{1-n}\int_0^\rho s^{n-1}\overline{u}(s)ds.$$

Integrating on $[0, r]$ gives

$$\overline{u}(r) - \overline{u}(0) = -\frac{\beta r^2}{2n} - k\int_0^r t^{1-n}\int_0^t \tau^{n-1}\overline{u}(\tau)\,d\tau\,dt.$$

Let $r = 1$, it is clear that

$$\overline{u}(0) = \frac{\beta}{2n} + k\int_0^1 t^{1-n}\int_0^t \tau^{n-1}\overline{u}(\tau)\,d\tau\,dt.$$

Since $\overline{u}$ is strictly decreasing in Ω [8, Theorem 2.1], this implies that $\overline{u}(0) \geq \overline{u}(\tau)$, for all $\tau \in (0, 1)$ and

$$\overline{u}(0) \leq \frac{\beta}{2n} + k\int_0^1 t^{1-n}\int_0^t \tau^{n-1}\overline{u}(0)\,d\tau\,dt.$$

Clearly,

$$\overline{u}(0) \leq \frac{\beta}{2n - k}.$$

Let u be a solution of (1.1) with $h = 0$ and let $v = \overline{u} - u$. Therefore,

$$\begin{aligned} -\Delta v &= -\Delta\overline{u} + \Delta u \\ &= k\overline{u} + \beta - f(u) \\ &\geq k\overline{u} + \beta - (ku + \beta) \\ &\geq kv. \end{aligned}$$

The function v satisfies

$$\begin{gathered} -\Delta v - kv \geq 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Using the fact that $k < \lambda_1$ and the maximum principle, we obtain that $v \geq 0$ in Ω; see for instance [12].

Note that in the case $-\Delta v = kv$, necessarily $v = 0$ and therefore $u = \overline{u}$ in Ω which gives the desired result. So

$$0 < u(x) \leq \frac{\beta}{2n - k}, \quad \text{for all } x \in \Omega,$$

as required. □

Proof of the Proposition 2.1. The eventual solution u and its gradient are continuous in the whole domain Ω and the partial differential equation is satisfied in the classical sense respectively in the region $u < \mu$ or $u > \mu$. We look for the free boundary in the form

$$\Gamma = \{(r_0, \theta), \theta \in S\} \quad \text{for some } r_0 \in (0, 1).$$

Let $w := \{(r,\theta) \in \Omega; 0 \le r < r_0; \theta \in S\}$. We obtain a solution of (1.1) with $h = 0$ by finding a radial function $u(r)$ and a value r_0 such that

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } w, \\ -\Delta u &= 0 \quad \text{in } \Omega \setminus \overline{w}. \end{aligned} \tag{2.3}$$

Indeed: If u is a solution of (2.3) with $u(r_0) = \mu$, then by the maximum principle $u(r) > \mu$ for $0 \le r < r_0$. For $r > r_0$, u is a harmonic function, then its maximum occurs on the boundary. The maximum μ is taken only at the free boundary Γ. Hence $u < \mu$ and u satisfies (1.1) with $h = 0$.

In the region w, the solution u exists and is radial (see [2, Proposition 3.1]). The maximum of u is achieved at $r = 0$ (see [8, Theorem 1]). Hence $d := \max_{\overline{w}} u = u(0) \ge \mu$ and $f(d) \ge f(\mu)$ (since f is non-decreasing).

Now, in w, the function u satisfies

$$r^{1-n}\partial/\partial r(r^{n-1}\partial u/\partial r) = f(u).$$

And we will have

$$\begin{aligned} \frac{\partial u}{\partial r}(r_0 - 0) &= -r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds \\ &\ge -r_0^{1-n} \int_0^{r_0} s^{n-1} f(d) ds \\ &= -\frac{r_0 f(d)}{n} \end{aligned}$$

where $\frac{\partial u}{\partial r}(r_0 - 0)$ denotes the left derivative of u at the value $r = r_0$.

In the region, $u \ge \mu$, we conclude that $f(u) \ge f(\mu)$ and

$$\begin{aligned} \frac{\partial u}{\partial r}(r_0 - 0) &= -r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds \\ &\le -r_0^{1-n} \int_0^{r_0} s^{n-1} f(\mu) ds \\ &= -\frac{r_0 f(\mu)}{n}. \end{aligned}$$

Therefore,

$$-\frac{r_0 f(d)}{n} \le \frac{\partial u}{\partial r}(r_0 - 0) \le -\frac{r_0 f(\mu)}{n}.$$

By Lemma 2.2, $d \in (0, \frac{\beta}{2n-k}]$. Since f is constant on this interval, $f(d) = f(\mu)$. We deduce that

$$\frac{\partial u}{\partial r}(r_0 - 0) = -\frac{r_0 f(\mu)}{n} \tag{2.4}$$

By solving the differential equation in the region $\Omega \setminus \overline{w}$, we obtain that

$$u(r) = \frac{\mu r^{2-n}}{r_0^{2-n} - 1} - \frac{\mu}{r_0^{2-n} - 1}.$$

This implies that

$$\frac{\partial u}{\partial r}(r_0 + 0) = \frac{(2-n)\mu}{r_0 - r_0^{n-1}}. \tag{2.5}$$

Now, a radial solution is obtained if u verifies the transmission conditions on the free boundary, i.e, there exists $r_0 \in (0,1)$ such that $u(r_0) = \mu$ and

$$\frac{\partial u}{\partial r}(r_0 - 0) = \frac{\partial u}{\partial r}(r_0 + 0). \tag{2.6}$$

Using (2.4), (2.5) and (2.6), one has

$$\frac{f(\mu)}{\mu} = \frac{n(n-2)}{r_0^2 - r_0^n}. \tag{2.7}$$

It is apparent that the function $r_0 \to \frac{n(n-2)}{r_0^2 - r_0^n}$ has a unique minimum M_n achieved at the point $r_0 = (\frac{2}{n})^{\frac{1}{n-2}}$. Since by the assumption (F3), there exists μ^* such that

$$\frac{f(\mu^*)}{\mu^*} = M_n,$$

it follows that equation (2.7) has only one root $r_0 = (\frac{2}{n})^{\frac{1}{n-2}}$ and we obtain the desired solution u with a free boundary which is a sphere of radius r_0.

Now, for the case (b), if

$$\frac{f(\mu)}{\mu} > M_n.$$

Equation (2.7) has two roots r_1, r_2 different from $(\frac{2}{n})^{\frac{1}{n-2}}$. The proof of Proposition 2.1 is complete. □

Note that (1.1) with $h = 0$ can have other solutions; see Theorem 3.5 below.

3. Main results

Let r_0 denote one of the values r_1 and r_2 of Proposition 2.1(b), then $r_0 \neq (\frac{2}{n})^{\frac{1}{n-2}}$. When $h \neq 0$, we look for the free boundary in the form $r_0 + b(\theta)$, $\theta \in S$, where $b(\theta)$ is the perturbation caused by h. Consider

$$B = \{b \in C(S, \mathbb{R}) : 0 \leq r_0 + b(\theta) < 1, \theta \in S\}$$

We seek a solution in $W^{2,p}(\Omega), p > 1$, then the boundary value function h which is a trace of $W^{2,p}$ function will be taken in the set

$$A = \{h \in W^{2-\frac{1}{p},p}(S, \mathbb{R}) : p > n\}.$$

Note that $W^{2-\frac{1}{p},p}(S) \subset W^{1,p}(S)$; see [1]. For $p > n$, we have $W^{1,p} \subset L^\infty$; see [1], [3]. Let

$$w = \{(r, \theta) \in (0,1) \times S : 0 \leq r < r_0 + b(\theta), \theta \in S\}.$$

We recall some results obtained in [2] which will be needed in the rest of this paper. We omit the proofs since they are similar to those given in [2].

Proposition 3.1. [2] *Assume that* (F1) *is satisfied. Then the problem*

$$\begin{gathered} -\Delta u = f(u) \quad \text{in } w, \\ u = \mu \quad \text{on } \partial w. \end{gathered} \tag{3.1}$$

has a unique solution $u^* \in H^1(w)$, for $\mu > 0$.

Now, we denote by χ_w the characteristic function of w. In the following proposition, we formulate a nonlinear equation for b and prove that by solving it, we can solve the problem (1.1).

Proposition 3.2. [2] *Assume* (F1). *Let* $v = \begin{cases} u^* & \text{in } w, \\ \mu & \text{in } \Omega \setminus \overline{w} \end{cases}$. *Then the problem*

$$\begin{gathered} -\Delta u = f(v)\chi_w(r,\theta) \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega \end{gathered} \tag{3.2}$$

has a unique solution $u_0 \in W^{2,p}(\Omega) \subset C^{1,\alpha}(\overline{\Omega}, \mathbb{R})$ *with* $\alpha = 1 - \frac{n}{p}$. *Moreover if* $u_0(r_0 + b(\theta), \theta) = \mu$ *with* $\|h\|_\infty < \mu$, *then* u_0 *is a solution of* (1.1).

Lemma 3.3. *Assume* (F1). *Then the function* u^* *satisfies*

$$\mu \le u^* \le \frac{\beta}{2n-k}.$$

Proof. Firstly, the function u^* is the solution of the problem

$$\begin{gathered} -\Delta u = f(u) \quad \text{in } w, \\ u = \mu \quad \text{on } \partial w. \end{gathered} \tag{3.3}$$

We remark that $\underline{u} = \mu$ is a subsolution of problem (3.3). In other part, we show that problem (3.3) has a supersolution $\overline{u} \in C^2(w)$. is fact, as $k \in (0, \lambda_1)$, then the linear problem

$$\begin{gathered} -\Delta \overline{u} = k\overline{u} + \beta \quad \text{in } \Omega \\ \overline{u} = \mu \quad \text{on } \partial\Omega \end{gathered} \tag{3.4}$$

has a unique solution $\overline{u} \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$. See [9, p. 107]. By the maximum principle [12, Theorem 2.5], we deduce that $\overline{u} > \mu$ in Ω. consequently,

$$\overline{u} > \mu \quad \text{in } \overline{w} \subset \Omega.$$

Then

$$\begin{gathered} -\Delta \overline{u} = k\overline{u} + \beta \quad \text{in } w \\ \overline{u} > \mu \quad \text{on } \partial w. \end{gathered}$$

This implies that $\overline{u}$ is a supersolution for the problem (3.3). Now a classical result in [7] shows that (3.3) has a solution u^* with $\mu \le u^* \le \overline{u}$.

We remark that in the proof of lemma 2.2, the function $\overline{u}$ is radial and satisfies

$$\overline{u}(r) \le \frac{\beta}{2n-k} \quad \text{for } r \in (0,1).$$

Hence, by uniqueness of the solution u^* in the region w, we have

$$\mu \le u^* \le \frac{\beta}{2n-k}.$$

This completes the proof. □

Now, it is easy to see that

$$\mu \le v \le \frac{\beta}{2n-k}.$$

Hence, the problem (3.2) can be written as

$$\begin{gathered} -\Delta u = f(\mu)\chi_w(r,\theta) \quad \text{in } \Omega \\ u = h \quad \text{on } \partial\Omega. \end{gathered} \tag{3.5}$$

Now, using Green's representation formula for u_0, we obtain a nonlinear integral equation for $b \in C(S)$. Then the solution of (1.1) can be recovered from the knowledge of b. The solution u_0 corresponding to (3.2) has an integral representation which is well defined [2, Theorem 4.1], given by

$$u_0(r,\theta) = \int_S P(r,\theta,\theta')h(\theta')d\theta' - f(\mu)\int_S d\theta' \int_0^r (r')^{n-1}\chi_w(r',\theta')G(r,\theta,r',\theta')dr'$$

where P is the Poisson kernel and G is the Green function for the Laplacian in Ω.

Now, we define the operator $F : A \times \mathbb{R}^+ \times B \to C(S,\mathbb{R})$ by

$$F(h,\mu,b)(\theta) = u_0(r_0 + b(\theta),\theta) - \mu;$$

i.e.,

$$\begin{aligned} F(h,\mu,b)(\theta) = &\int_S P(r_0 + b(\theta),\theta,\theta')h(\theta')d\theta' \\ &- f(\mu)\int_S d\theta' \int_0^{r_0+b(\theta')} (r')^{n-1}G(r_0 + b(\theta),\theta,r',\theta')dr' - \mu\,. \end{aligned}$$

The main results of this section are stated in the following theorems.

Theorem 3.4. *Assume that* (F1), (F2) *are satisfied and suppose that there exists* $\mu > 0$ *such that*

$$\frac{f(\mu)}{\mu} > M_n, \quad M_n = \frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}, \quad \textit{forn} \geq 3.$$

Let $\|h\|_\infty = \max_{x\in\partial\Omega} |h(x)|$. *If* h *is small enough with* $\|h\|_\infty < \mu$, *then* (1.1) *has at least two positive solutions and the free boundaries are analytic hypersurfaces.*

Theorem 3.5. *Assume that* (F1), (F2), (F3) *are satisfied.*
Let $Z = \{\xi \in C(S), \int_S \xi(y)dy = 0\}$. *Then there exist:*

(i) *an interval* $I =]-\varepsilon, +\varepsilon[, \varepsilon > 0$*;*
(ii) *a continuous functions* $\phi : I \to \mathbb{R}$ *and* $\psi : I \to Z$ *with* $\phi(0) = \mu^*$ *and* $\psi(0) = 0$*;*
(iii) *a neighborhood* V *of* $(\mu^*, 0)$ *in* $\mathbb{R} \times C(S)$ *such that for all* $s \in I$, *the following pair is a solution of* $F(0,\mu,b) = 0$ *in* V

$$(\mu,b) = (\phi(s), s\phi_{00} + s\psi(s))$$

where ϕ_{00} *is a given constant.*

The proofs will be given in several steps.

Proof of Theorem 3.4. We will give the main steps of the proof and we refer the reader to [2] when appropriate to avoid unnecessary duplication of arguments. We deal with the resolution of the integral equation $u_0(r_0 + b(\theta),\theta) - \mu = 0$ with respect to $b \in C(S)$. The result is described by the following proposition from which Theorem 3.4 follows immediately.

Proposition 3.6. *If* h *is small enough with* $\|h\|_\infty < \mu$, *then there exists a neighborhood* V *of* 0 *in* A *and a unique function* $b : V \to B$ *differentiable such that*

(i) $b(0) = 0$
(ii) $F(h,\mu,b(h)) = 0$ *for* $h \in V$.

Note that Proposition 3.6 shows that the dependence of the free boundary on the boundary data h is continuously differentiable.

Proof of Proposition 3.6. We denote by D_jF the partial derivative of F with respect to the j-th variable. Let r_0 be one of the values r_1 or r_2 obtained in Proposition 2.1, then $r_0 \neq (\frac{2}{n})^{\frac{2}{n-2}}$. Let K be the compact operator defined on $C(S)$ by

$$K\beta(\theta) = \int_S G(r_0, \theta, r_0, \theta')\beta(\theta')d\theta'.$$

Claim 3.7. [2] *The eigenvalues of the operator K are*

$$\sigma_l = -\frac{1}{r_0^{n-2}} \frac{1 - r_0^{2l+n-2}}{2l + n - 2}, \quad \textit{for } l \in \mathbb{N}.$$

Following the same argument as in [2], the expression of the operator D_3F is given by

$$\begin{aligned} D_3F(h, \mu, b)\beta(\theta) = \Big[&\int_S d\theta' \frac{\partial P}{\partial r}(r_0 + b(\theta), \theta, \theta')h(\theta') \\ &- f(\mu) \int_S d\theta' \int_0^{r_0 + b(\theta')} (r')^{n-1} dr' \frac{\partial G}{\partial r}(r_0 + b(\theta), \theta, r', \theta')\Big]\beta(\theta) \\ &- f(\mu) \int_S d\theta' (r_0 + b(\theta'))^{n-1} G(r_0 + b(\theta), \theta, r_0 + b(\theta'), \theta')\beta(\theta'). \end{aligned}$$

This operator is a continuous mapping of a neighborhood of $(0, \mu, 0)$ in $A \times \mathbb{R}^+ \times B$ into $C(S)$. In fact the operator D_3F can be written as

$$\frac{\partial u}{\partial r}(r_0 + b(\theta), \theta)\beta(\theta) - (\phi\beta)(\theta)$$

where $(\phi\beta)(\theta) = f(\mu) \int_S d\theta'(r_0 + b(\theta'))^{n-1} G(r_0 + b(\theta), \theta, r_0 + b(\theta'), \theta')\beta(\theta')$. The solution u depends continuously in the norm of $C^{1,\alpha}$ on (h, μ, b) and since the singularity of the Green function is integrable, then ϕ is a continuous mapping. Hence, the operator $D_3F(0, \mu, 0)$ can be written in form

$$D_3F(0, \mu, 0)\beta(\theta) = \frac{\partial u}{\partial r}(r_0, \theta)\beta(\theta) - r_0^{n-1} f(\mu) K\beta(\theta).$$

The implicit function theorem can be applied if $D_3F(0, \mu, 0)$ is invertible. It follows from the expression of $D_3F(0, \mu, 0)$ that this is the case if

$$r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds + r_0^{n-1} f(\mu)\sigma_l \neq 0 \tag{3.6}$$

for σ_l any eigenvalue of K.

If $l \geq 1$, it is clear that (3.6) is satisifed. If $l = 0$, then since $r_0 \neq (\frac{2}{n})^{\frac{1}{n-2}}$, it follows that (3.6) is satisfied. This proves Proposition 3.6. □

For the regularity of the free boundary, see the section 4 below. Now, it is easy to see that Theorem 3.4 is a consequence of Proposition 3.6.

For the proof of Theorem 3.5, we need some preparations. From the above computations, the operator $D_3F(0, \mu^*, 0)$ is invertible since (3.6) is satisfied when

$l \geq 1$. If $l = 0$, we have

$$
\begin{aligned}
r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds + r_0^{n-1} f(\mu^*)\sigma_0 &= r_0^{1-n} \int_0^{r_0} s^{n-1} f(\mu^*) ds + r_0^{n-1} f(\mu^*)\sigma_0 \\
&= \frac{r_0}{n} f(\mu^*) + r_0^{n-1} f(\mu^*)\sigma_0 \\
&= r_0 f(\mu^*)(\frac{1}{n} - \frac{1-r_0^{n-2}}{n-2}) = 0.
\end{aligned}
$$

Since in this case u is the solution of the reduced problem and by hypothesis (F2), it follows that $f(u) = f(\mu^*)$. Hence, the operator $D_3F(0, \mu^*, 0)$ is not invertible, the implicit function theorem fails and a phenomenon of bifurcation appears. In what follows, we apply a bifurcation theorem of Crandall-Rabinowitz [13, Theorem 2.2.1] to show the emergence of bifurcated solutions of reduced problem (1.1) with $h = 0$.

Proof of Theorem 3.5. We shall explore the situation when $r_0 = (\frac{2}{n})^{\frac{1}{n-2}}$. As already mentioned, in this case the operator $D_3F(0, \mu^*, 0)$ is not invertible. The conditions needed to prove Theorem 3.5 are established in the next lemmas.

Lemma 3.8. *Let $\phi_{00} := 1/(nw_n)$, where w_n is the volume of the unit ball in $\mathbb{R}^n$. For $\mu^* > 0$, the operator $D_3F(0, \mu^*, 0)$ has a one dimensional null space spanned by ϕ_{00}, while its range has codimension one coinciding with the null space of the continuous linear functional*

$$
\Phi(\xi) = \int_S \xi(y)\phi_{00} dy.
$$

Proof. Initially, note that for $\beta \in C(S)$, we have

$$
D_3F(0, \mu^*, 0)\beta(\theta) = r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds \beta(\theta) + r_0^{n-1} f(\mu^*) K\beta(\theta).
$$

Since

$$
r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s)) ds + r_0^{n-1} f(\mu^*)\sigma_0 = 0,
$$

then the operator $D_3F(0, \mu^*, 0)$ is not invertible. Obviously,

$$
D_3F(0, \mu^*, 0)\phi_{00} = r_0^{1-n} \int_0^{r_0} s^{n-1} f(u(s))\phi_{00} ds + r_0^{n-1} f(\mu^*)\phi_{00} = 0.
$$

This gives that the kernel of $D_3F(0, \mu^*, 0)$ is a one dimensional space spanned by ϕ_{00}. The function ϕ_{00} is the first eigenfunction corresponding to the eigenvalue σ_0 (see [2, p. 2342]). Since the operator K is compact, the equation

$$
D_3F(0, \mu^*, 0)\beta(\theta) = \xi(\theta)
$$

has a solution if ξ is orthogonal to ϕ_{00}. Let

$$
\Phi(\xi) = \int_S \xi(\theta)\phi_{00} d\theta,
$$

it becomes apparent that

$$
\operatorname{Im} D_3F(0, \mu^*, 0) = \ker \Phi.
$$

This concludes the proof □

Lemma 3.9. [2] *The mixed derivative $D_2D_3F(0, \mu, b)$ exists and is continuous in a neighborhood of $(\mu^*, 0)$.*

Now, we can state the following lemma.

Lemma 3.10. *$D_2D_3F(0,\mu^*,0)\phi_{00}$ does not belong to the range of $D_3F(0,\mu^*,0)$.*

Proof. When $h=0$ and $b=0$, we have

$$D_2D_3F(0,\mu^*,0)\phi_{00}=\frac{\partial}{\partial\mu}(\frac{\partial u}{\partial r}(r_0+b(\theta),\theta))\phi_{00}\mid_{(h=0,\mu=\mu^*,b=0)}$$

Now, it easy to see that the partial derivative

$$\frac{\partial}{\partial\mu}(\frac{\partial u}{\partial r}(r_0))\neq 0,$$

this completes the proof. □

Now, let

$$Z=\{\xi\in C(S),\int_S\xi(y)\phi_{00}dy=0\}=\{\xi\in C(S),\int_S\xi(y)dy=0\}.$$

To conclude the proof of Theorem 3.5, we remark that all the hypothesis of bifurcation's theorem of Crandall-Rabinowitz [13, Theorem 2.2.1] are satisfied. Then there exists a solution with the desired properties (i), (ii) and (iii).

4. Regularity of the free boundary

In this section, we discuss regularity of the free boundary under the conditions of the Theorem 3.4.

Proposition 4.1. *Under the conditions of the theorem 3.4, the free boundary is an analytic hypersurface.*

The proof of this proposition is obtained with the aid of a suitably constructed mapping which transforms the two different regions of problem (1.1) separated by the free boundary Γ to the same half space. The partial differential equations in w and $\Omega\setminus\overline{w}$ are transformed into other equations in half space and we then apply the known regularity theorem for elliptic systems to obtain the desired result.

Proof of Proposition 4.1. First, under the condition of Theorem 3.4, we have that $r_0\neq(\frac{2}{n})^{\frac{1}{n-2}}$. Let u be a solution of the problem (1.1), and let

$$\Gamma:=\{(r,\theta):u(r,\theta)=\mu\}=\{(r_0+b(\theta),\theta),\theta\in S\}$$

be the free boundary. We have the following result.

Proposition 4.2. [2] *Let $b\in C(S)$ and if u is a solution of* (1.1) *such that $u(r_0+b(\theta),\theta)=\mu$, then $b\in C^{1,\alpha}(S)$, for some $\alpha\in(0,1)$.*

This proposition shows that Γ is $C^{1,\alpha}$-hypersurface with $\alpha\in(0,1)$. Now, one way to deal with the analyticity of Γ is to introduce an appropriate transformation. We proceed as follows: Consider a small ball B about a point $x_0=(r_0+b(\theta_0),\theta_0)\in\Gamma$ translating coordinates so that $x_0=0$. Using the rotational invariance of Laplacian and writing $v=u-\mu$, we have

$$\begin{gathered}-\Delta v+f(v+\mu)=0\quad\text{in }B^+=\{x=(x_1,\dots,x_n),x_n>0\},\\ -\Delta v=0\quad\text{in }B^-=\{x=(x_1,\dots,x_n),x_n<0\},\\ v=0\quad\text{on }\Gamma\end{gathered}\tag{4.1}$$

We know that $v \in C^{1,\alpha}(B^+ \cup \Gamma \cup B^-)$, and from the Hopf maximum principle, we have that $\frac{\partial v}{\partial \nu}(0) < 0$, where ν is the outer unit normal of Γ which is $\nu = (0,0,\dots,-1)$. We have

$$\frac{\partial v}{\partial \nu}(0) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_n}(0)\nu_i(0) = -\frac{\partial v}{\partial x_n}(0) < 0,$$

which implies that

$$\frac{\partial v}{\partial x_n}(0) > 0.$$

We introduce the zeroth order hodograph transformation [10] as

$$\begin{gathered} y_\sigma = x_\sigma \quad \sigma < n, \\ y_n = v(x) \quad x \in B, \end{gathered} \tag{4.2}$$

This definition transforms B^+, Γ, B^- into

$$U^+ = \{y \in U, y_n > 0\}, \quad \Sigma = \{y \in U, y_n = 0\}, \quad U^- = \{y \in U, y_n < 0\}$$

respectively. Define the inverse of (4.2) as

$$\begin{gathered} x_\sigma = y_\sigma \quad 1 \le \sigma \le n-1, \\ x_n = \psi(y) \quad y \in U, \end{gathered} \tag{4.3}$$

We remark that $\psi \in C^{1,\alpha}(U^+ \cup \Sigma \cup U^-)$. We denote by $v_i, 1 \le i \le n$, the partial derivative with respect to x_i, and $\psi_j, 1 \le j \le n$, the partial derivative with respect to y_j. □

One of the important properties of the hodograph transformation is that

$$\begin{aligned} dy_n = dv &= \sum_\sigma v_\sigma dx_\sigma + v_n dx_n \\ &= \sum_\sigma v_\sigma dx_\sigma + v_n d\psi \\ &= \sum_\sigma v_\sigma dx_\sigma + v_n(\sum_\sigma \psi_\sigma dy_\sigma + \psi_n dy_n) \\ &= \sum_\sigma (v_\sigma + v_n\psi_\sigma) dx_\sigma + v_n\psi_n dy_n \end{aligned}$$

This implies

$$\begin{gathered} \psi_n v_n = 1 \\ \psi_\sigma v_n + v_\sigma = 0 \end{gathered}$$

which in turn implies

$$\begin{gathered} \psi_n = \frac{1}{v_n} \\ \psi_\sigma = \frac{-v_\sigma}{v_n}\,. \end{gathered} \tag{4.4}$$

Using this property, it easy to see that

$$\begin{gathered} \frac{\partial}{\partial x_\sigma} = \partial_\sigma - \frac{\psi_\sigma}{\psi_n}\partial_n \quad 1 \le \sigma \le n-1, \\ \frac{\partial}{\partial x_n} = \frac{1}{\psi_n}\partial_n \quad \text{with } \partial_k = \frac{\partial}{\partial y_k}, \end{gathered} \tag{4.5}$$

From (4.4) and (4.5), we obtain the following property.

Claim 4.3.

$$v_{\sigma\sigma} = \frac{-\psi_{\sigma\sigma}}{\psi_n} + 2\frac{\psi_\sigma}{\psi_n^2}\psi_{\sigma n} - \frac{\psi_\sigma^2}{\psi_n^3}\psi_{nn}$$
$$v_{nn} = \frac{-1}{\psi_n^3}\psi_{nn}.$$

Proof. Using the properties (4.4), (4.5), for v_σ and v_n, we find

$$v_{\sigma\sigma} = \frac{\partial v}{\partial x_\sigma} = \frac{\partial v_\sigma}{\partial y_\sigma} - \frac{\psi_\sigma}{\psi_n}\frac{\partial v_\sigma}{\partial y_n}$$

We know that

$$\frac{\partial v_\sigma}{\partial y_\sigma} = -[v_n\frac{\partial \psi_\sigma}{\partial y_\sigma} + \psi_\sigma\frac{\partial v_n}{\partial y_\sigma}] = -[v_n\psi_{\sigma\sigma} - \psi_\sigma\frac{\psi_{n\sigma}}{\psi_n^2}],$$
$$\frac{\partial v_\sigma}{\partial y_n} - [\frac{\partial \psi_\sigma}{\partial y_n}v_n + \frac{\partial v_n}{\partial y_n}\psi_\sigma] = -v_n\psi_{n\sigma} + \psi_\sigma\frac{\psi_{nn}}{\psi_n^2}.$$

Combining the two previous results,

$$v_{\sigma\sigma} = -v_n\psi_{\sigma\sigma} + \psi_\sigma\frac{\psi_{n\sigma}}{\psi_n^2} - \frac{\psi_\sigma}{\psi_n}[-v_n\psi_{n\sigma} + \psi_\sigma\frac{\psi_{nn}}{\psi_n^2}]$$
$$v_{\sigma\sigma} = \frac{-\psi_{\sigma\sigma}}{\psi_n} + 2\frac{\psi_\sigma}{\psi_n^2}\psi_{\sigma n} - \frac{\psi_\sigma^2}{\psi_n^3}\psi_{nn},$$

and

$$v_{nn} = \frac{\partial v_n}{\partial x_n} = \frac{1}{\psi_n}\frac{\partial v_n}{\partial y_n} = -\frac{\psi_{nn}}{\psi_n^3}\,.$$

□

Now, since v satisfies $\Delta v = \sum_\sigma v_{\sigma\sigma} + v_{nn}$, it follows that ψ satisfies the nonlinear equation

$$g(\psi, D\psi, D^2\psi) + f(y_n + \mu) = 0 \quad \text{in } U^+,$$

where

$$g(\psi, D\psi, D^2\psi) = \frac{-1}{\psi_n}\sum_\sigma \psi_{\sigma\sigma} + \frac{2}{\psi_n^2}\sum_\sigma \psi_\sigma\psi_{\sigma n} - \frac{1}{\psi_n^3}(1 + \sum_\sigma \psi_\sigma^2)\psi_{nn}\,.$$

Moreover, ψ satisfies $g(\psi, D\psi, D^2\psi) = 0$ in U^-. Writing $y = (y_1, \dots, y_{n-1}, y_n) = (y', y_n)$. We define for $y \in U^+$, $\phi(y) = \psi(y', -y_n)$, then ϕ satisfies

$$g(\phi, D\phi, D^2\phi) = 0$$

in U^+. Hence, we obtain the system

$$g(\psi, D\psi, D^2\psi) + f(y_n + \mu) = 0 \quad \text{in } U^+, \tag{4.6}$$
$$g(\phi, D\phi, D^2\phi) = 0 \quad \text{in } \ U^+, \tag{4.7}$$

with the boundary conditions

$$\begin{aligned} \phi - \psi = 0 \quad &\text{on } \Sigma, \\ \phi_n + \psi_n = 0 \quad &\text{on } \Sigma\,. \end{aligned} \tag{4.8}$$

Claim 4.4. *The system* (4.6)-(4.7) *is elliptic and the boundary conditions* (4.8) *are coercive at a point* 0.

Proof. We can verify immediately that (4.6)-(4.7) is elliptic at the point 0. It remains to prove the coerciveness of (4.6)-(4.7)-(4.8). For that, we show that (4.6)-(4.7) with the boundary conditions admit no nontrivial bounded exponential solutions. First, by our choice of coordinates

$$\psi_n(0) = \frac{1}{v_n(0)} > 0 \quad \psi_\sigma(0) = -\frac{v_\sigma(0)}{v_n(0)} = 0, \quad 1 \le \sigma \le n-1$$

Hence, the linearized equations for (4.6)-(4.7) with the obvious weight $s_1 = s_2 = 0, t_1 = t_2 = 2$, are

$$\sum_\sigma \psi_{\sigma\sigma} + a^2\psi_{nn} = 0 \quad \text{in } \mathbb{R}^n_+, \tag{4.9}$$

$$\sum_\sigma \phi_{\sigma\sigma} + a^2\phi_{nn} = 0 \quad \text{in } \mathbb{R}^n_+, \tag{4.10}$$

where $a = v_n(0) > 0$. The linearized boundary conditions are

$$\begin{gathered} \phi - \psi = 0 \quad \text{on } \mathbb{R}^{n-1}, \\ \phi_n + \psi_n = 0 \quad \text{on } \mathbb{R}^{n-1} \end{gathered} \tag{4.11}$$

Introduce $\psi(y', t) = e^{i\xi' y'} w(t)$ and $\phi(y', t) = e^{i\xi' y'} m(t)$, for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. We obtain by replacing ϕ, ψ in (4.9)-(4.10),

$$\begin{gathered} a^2 w''(t) - |\xi'|^2 w(t) = 0, \\ a^2 m''(t) - |\xi'|^2 m(t) = 0, \end{gathered}$$

with the conditions (4.11),

$$w(0) = m(0) = 0, \quad w'(0) + m'(0) = 0.$$

Let $X(t) = w(t) + m(t)$. Then

$$a^2 X''(t) - |\xi'|^2 X(t) = 0, \tag{4.12}$$

The boundary condition (4.11) imply $X(0) = 0$ and $X'(0) = 0$, which implies

$$\begin{gathered} c_1 + c_2 = 0 \\ c_2 - c_1 = 0\,. \end{gathered}$$

Then $c_1 = c_2 = 0$ which implies that $X(t) = 0$. Now, let $Y(t) = w(t) - m(t)$ with $Y(0) = 0$, we obtain

$$Y(t) = c_3(e^{-\frac{|\xi'|}{a}t} - e^{\frac{|\xi'|}{a}t}).$$

The function $Y(t)$ is bounded if and only if $c_3 = 0$. Hence, this conclude that $w(t) = m(t) = 0$. which implies that (4.9)-(4.10)-(4.11) admit no nontrivial bounded exponential solutions, then (4.9)-(4.10)-(4.11) is coercive at 0.

As in problem (P3), (see the proof of Lemma 3.3) the function f is constant, then f is analytic. It suffices to apply [10, Theorem 3.3] to show that the free boundary Γ is analytic. $\square$

4.1. **Conclusions and open problems.** (1) The regularity of the free boundary in the case when $r_0 = (\frac{2}{n})^{\frac{1}{n-2}}$ remains an open problem. We have shown only the existence of a continuous function b. It seems that it is possible to study the optimal regularity using the ideas introduced by Caffarelli [4, 5].

(2) For the sake of simplicity Theorem 3.4 and Theorem 3.5 are stated only for the case $n \geq 3$, it is not difficult to see that the same result holds for the case $n = 2$.

(3) We remark that the regularity of free boundary is preserved after perturbations. Hence for a small perturbation h and under a suitable conditions, the free boundary is analytic and does not develop singularities.

(4) The case of a general domain Ω is still unknown.

(5) In Theorem 3.5, if the boundary value $h \neq 0$, what happens to the bifurcated solutions?.

Acknowledgments. The authors express their thanks to the anonymous referee for valuable suggestions and comments. This work was completed when the second author was visiting the International Center of Theoretical Physics, ICTP of Trieste during the summer 2008. He expresses his thanks for financial support and warm hospitality.

References

[1] R. A. Adams; *Sobolev spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York-London, 1975.

[2] S. Bensid, S. M. Bouguima; On a free boundary problem, *Nonlinear Anal. T.M.A*,**68** (2008), 2328–2348.

[3] H. Brezis; *Analyse Fonctionnelle*, Masson, Paris, 1983.

[4] L. Caffarelli; The regularity of elliptic and parabolic free boundaries. *Bull. Amer. Math. Soc.* **82** (4) (1976), 616–618.

[5] L. Caffarelli; The regularity of free boundaries in higher dimensions. *Acta Math.* **139** (3-4) (1977), 155–184.

[6] P. Candito; Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities, *Proc. Edinb. Math. Soc., II. Ser.*,**45** (2) (2002), 397–409.

[7] L. C. Evans; *Partial Differential Equations*. Graduate Studies in Mathematics, **19**, Amer. Math. Soc, 1998.

[8] B. Gidas, W. N. Ni, L. Nirenberg; Symmetry and related properties via maximum principle. *Commun. Math. Phys.* **68**, (1979), 209–243.

[9] D. Gilbarg, N. Trudinger; *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, 1983.

[10] D. Kinderlehrer, G. Stampacchia; *An Introduction to Variational Inequalities and their Applications*. New York-London : Academic Press, 1980.

[11] J. Kolibal; Level set method for solving Poisson's equation with discontinuous nonlinearities *Electron. J. Diff. Equ.*, Vol (2005), No. 132, 1–12.

[12] J. López-Gómez; The maximum principle and the existence of principal eigenvalue for some linear weighted boundary value problems, *J. Differential Equations* **127** (1996), 263–294.

[13] J. López Gómez; *Spectral Theory and Nonlinear Functional Analysis*. Chapman and Hall/CRC Press, Research Notes in Mathematics, **426**, Boca Raton, Florida, 2001.

[14] S. A. Marano, D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian, *J. Differential Equations* **182** (1) (2002), 108–120.

Sabri Bensid
Department of Mathematics, Faculty of Sciences, University of Tlemcen, B.P. 119, Tlemcen 13000, Algeria
E-mail address: edp_sabri@yahoo.fr

Sidi Mohammed Bouguima
Department of Mathematics, Faculty of Sciences, University of Tlemcen, B.P. 119, Tlemcen 13000, Algeria
E-mail address: bouguima@yahoo.fr