Electronic Journal of Differential Equations, Vol. 2010(2010), No. 55, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PAINLEVÉ'S DETERMINATENESS THEOREM EXTENDED TO PROPER COVERINGS

CLAUDI MENEGHIN

ABSTRACT. We extend Painlevé's determinateness theorem to first-order ordinary differential equations in the complex domain, with known terms allowed be multivalued in the dependent variable. The multivaluedness is supposed to be resolved by proper coverings.

1. INTRODUCTION

What is generally referred to as *Painlevé's Determinateness Theorem* [3, Thm. 3.3.2] for first order ordinary differential equations in the complex domain is stated as follows:

> If $F(z, w)$ is a rational function of w with coefficients which are algebraic functions of z, then any movable singularities of the solutions to the first order ODE $w' = F(z, w)$ are poles and/or algebraic branch points.

The so called "known term" F is required to be single-valued in the dependent variable w; the goal of this note is to allow F to be multivalued in w, and conclude that none of the movable singularities are essential notwithstanding. The multivaluedness of F will be resolved by passing to a Riemann domain (Δ, p) over $\mathbb{C}^2$ minus a complex-analytic curve, with the main assumption that (Δ, p) be a *proper cover* of $p(\Delta)$. This hypothesis cannot be dropped in general, as example (3.1) shows. More singularities (not necessarily poles) for F will be allowed on a complex-analytic curve in Δ.

Previous statements of Painlevé's theorem [5, p. 38], [6, p. 327-328], [4, p. 292], [3, Thm. 3.3.1], read as follows:

> If a solution of the first-order ODE $w' = F(z, w)$ is continued analytically along a rectifiable arc from $z = z_0$ to $z = z_1$ avoiding the set S of fixed singularities, and if $z_1 \notin S$, then the solution tends to a definite limit, finite or infinite, as $z \to z_1$.

The two statements of Painlevé's determinatess theorem are equivalent under the assumption about the known term F (see the references here above); since we make a broader hypothesis, we can no longer take equivalence for granted; in particular,

---

2000 *Mathematics Subject Classification.* 34M35, 34M45.
Key words and phrases. Painlevé's determinateness theorem; complex differential equations; Cauchy's problem; analytic continuation; singularities; multi-valued function; Riemann surface.

Submitted January 23, 2009. Published April 19, 2010.

logarithmic branch points in the solutions cannot be excluded, as pointed out in [3, sec. 3.1] (see also example 3.4 below). In this note we generalize this latter version, but the final remark shows that movable essential singularities can be ruled out anyway; the question whether or not, in our broader setting, natural boundaries could arise will be the object of future investigations.

This note does not require Bieberbach's precise definition of a fixed or of a movable singularity ([1], quoted in [7]); we will use these notions in an informal fashion (as in [3, 4, 5, 6]), within the examples only.

Before stating and proving our main theorem, we now introduce some terminology and discuss some examples.

2. Terminology

A *Riemann domain* over a region $\mathcal{U} \subset \mathbb{C}^n$ is a complex manifold Δ with an everywhere maximum-rank holomorphic surjective mapping $p : \Delta \to \mathcal{U}$; Δ is *proper* provided that so is p (see [2, p. 43]); a curve $\mathcal{S} \subset \mathbb{C}^n$ is *complex-analytic* provided that it is the common zero set of $N-1$ complex-analytic functions on $\mathbb{C}^n$; when $n = 1$ we talk about *Riemann surfaces*.

Definition 2.1. Let M be a complex manifold, U an open set in $\mathbb{C}^n$, $f : U \to M$ an holomorphic mapping: a regular analytic continuation of the holomorphic mapping element (U, f), is a quadruple (S, π, j, F) such that:

1. S is a connected Riemann domain over a region in $\mathbb{C}^n$;
2. $\pi : S \to \mathbb{C}^n$ is an everywhere nondegenerate holomorphic mapping such that $U \subset \pi(S)$;
3. $j : U \to S$ is a holomorphic immersion such that $\pi \circ j = id|_U$;
4. $F : S \to M$ is a holomorphic mapping such that $F \circ j = f$.

Let $\gamma : I \to \mathbb{C}^n$ be an arc (with $I = [0, 1]$ or $I = [0, 1)$) such that $\gamma(0) = X$; a regular analytic continuation along γ of (U, f) is a regular analytic continuation (S, π, j, F) of (U, f) such that there exists an arc $\widetilde{\gamma} : I \to S$ with $\pi \circ \widetilde{\gamma} = \gamma$.

2.1. Cauchy's problems with multivalued known terms.

Let us now focus on differential equations whose "known terms" are defined on Riemann domains rather than just on open sets in $\mathbb{C}^2$. Introduce the following concepts:

- a complex-analytic curve $\mathcal{B} = \{(z, w) \in \mathbb{C}^2 : B(z, w) = 0\}$, with B holomorphic on $\mathbb{C}^2$, called the **branch locus** of the differential equation, and a proper Riemann domain (Δ, p) over $\mathbb{C}^2 \setminus \mathcal{B}$. Note that we do not require $\mathcal{B}$ to be algebraic.
- a complex one-dimensional submanifold $\Sigma \subset \Delta$, such that $p(\Sigma)$ is included in a complex-analytic curve $\Lambda(z, w) = 0$ in $\mathbb{C}^2$; $p(\Sigma)$ will be referred to as the **singularity locus**;
- the branch and the singularity loci will be collectively referred to as **the singularities of the differential equation**;
- a holomorphic function F on $\Delta \setminus \Sigma$, called the **known term**;
- a point $X_0 \in \Delta \setminus \Sigma$, with $(z_0, w_0) := p(X_0)$; we will refer to w_0 as the **initial value** of the Cauchy problem and to z_0 as the **initial point**; we will also refer to (z_0, w_0) collectively as the **initial values**;
- a local inverse η of p, defined in a bidisc $\mathbb{D}_1 \times \mathbb{D}_2$ around (z_0, w_0).

The above definitions are meant to be referred to a differential equation (or to an associated Cauchy problem) and not to its solutions.

3. Examples

In the realm of practice, the usual symbols of multivalued functions such as log or $\sqrt{\ }$ will go on to be used as well as the attributes *multi-valued* or *single-valued*. This is perfectly rigorous (even by a geometric point of view), inasmuch as the underlying machinery of analytic continuation is understood; in particular, a branch of the multivalued known term will always have to be specified alongside the initial conditions; i.e., a local inverse η of the covering map $p : \Delta \to \mathbb{C}^2$ will have to be explicitly chosen there.

3.1. **Attaining singularities of the known term.**

Consider the Cauchy problem

$$w'(z) = \sqrt{(1 - w^2(z))}\ \frac{w(z)}{\sin z}$$
$$w(\pi/4) = \sqrt{2}/2.$$

Here we understand the choice of the positive branch of the square root corresponding to the initial values $(\pi/4, \sqrt{2}/2)$.

In the terminology of section 2.1, we have:

- the branch locus is $\mathcal{B} = \{(z, w) \in \mathbb{C}^2 : w = \pm 1\}$, the Riemann domain of the known term is $\Delta = \{(z, w, y) \in \mathbb{C}^3 : w^2 + y^2 = 1,\ y \neq 0\}$, with the projection mapping $p(z, w, y) = (z, w)$, a twofold covering, hence a proper mapping;
- the singularity locus is $\Sigma = \{(z, w, y) \in \Delta : z = k\pi, k \in \mathbb{Z}\}$;
- the known term $F : \Delta \setminus \Sigma \to \mathbb{C}$ is defined by $F(z, w, y) = yw/\sin(z)$;
- the lifted initial point is $X_0 = (\pi/4, \sqrt{2}/2, \sqrt{2}/2) \in \Delta\setminus\Sigma$; note that $p(X_0) = (z_0, w_0) = (\pi/4, \sqrt{2}/2)$;
- $\eta(z, w) = (z, w, \sqrt{1 - w^2})$, where the positive branch of the square root has been chosen.

The singularities of the equation in the underlying $\mathbb{C}^2$ lie on $\{z = k\pi\} \cup \{w = \pm 1\}$. The problem is solved by the entire function $w(z) = \sin(z)$ (clearly admitting analytic continuation and, a fortiori, limit, everywhere in $\mathbb{C}$).

Note that the multivaluedness in w of the known term of this Cauchy problem makes it attain singularities along the graph of the solution, more precisely at $z = \frac{\pi}{2} + k\pi$. This fact does not affect the analytic continuation of the solution since the above singularities can be avoided by continuing along a suitable real arc; compare the argumentation following (4.5).

3.2. **A problem with essential singularities.**

The Cauchy problem

$$w'(z) = (e^{z\cdot w(z)} + 1)^{-1}(e^{-z\cdot w(z)} - w(z))e^{(e^{z\cdot w(z)} - z)^{-1} + 1}$$
$$w(2) = 0$$

is solved by $w(z) = [\log(z - 1)]/z$. The known term of this problem is single valued on $\mathbb{C}^2$, has poles on the complex-analytic curve $e^{wz} = -1$ and essential singularities on $e^{wz} = z$. No line $z = \text{const}$ (in particular $z = 1$) is a singularity. In view of theorem 4.3, note that w can be analytically continued along the arc γ defined on $[0, 1)$ by $\gamma(t) = 2 - t$ and there does exist $\lim_{t\to 1}[\log(1 - t)]/(2 - t) = \infty$.

3.3. **No limit.** Consider the Cauchy problem:

$$w'(z) = -\sqrt{1 - w^2(z)}/z^2$$
$$w((1+i)^{-1}) = \sin(1 + i + c).$$

Here we suppose $|c|$ small enough and understand the choice of the positive branch of the square root corresponding to the initial values $((1+i)^{-1}, \sin(1+i+c))$. In the terminology of section 2.1, we have:

- the branch locus is $\mathcal{B} = \{(z, w) \in \mathbb{C}^2 : w = \pm 1\}$, the Riemann domain of the known term is $\Delta = \{(z, w, y) \in \mathbb{C}^3 : w^2 + y^2 = 1,\ y \neq 0\}$, with the projection mapping $p(z, w, y) = (z, w)$, a twofold covering, hence a proper mapping;
- the singularity locus is $\Sigma = \{(z, w, y) \in \Delta : z = 0\}$;
- the known term $F : \Delta \setminus \Sigma \to \mathbb{C}$ is defined by $F(z, w, y) = -y/z^2$;
- the lifted initial point is $X_0 = (1/\pi, -\sin(c), -\cos(c)) \in \Delta \setminus \Sigma$; note that $p(X_0) = (z_0, w_0) = (1/\pi, -\sin(c))$;
- $\eta(z, w) = (z, w, -\sqrt{1 - w^2})$, where the positive branch of the square root has been chosen.

The singularities of the equation in the underlying $\mathbb{C}^2$ lie on $\{z = 0\} \cup \{w = \pm 1\}$. The problem is solved by $w(z) = \sin(1/z + c)$, showing an essential singularity at $z = 0$. Note that w can be analytically continued along the arc γ defined on $[0, 1)$ by $\gamma(t) = (1 - t)(1 + i)^{-1}$ and there does not exist $\lim_{t\to 1} \sin(1/\gamma(t) + c)$; in view of theorem 4.3, this should be compared with the fact that $\{z = 0\} \subset \mathbb{C}^2$ is a line of poles for the known term.

3.4. **Solution with logarithmic singularity.** In the Cauchy problem

$$w'(z) = e^{-w(z)}(1 + \sqrt[3]{e^{w(z)} - z + 1})/2$$
$$w(1) = 0,$$

we understand the choice of the positive branch of the cube root corresponding to the initial values $(1, 0)$. In the terminology of section 2.1, we have:

- the branch locus is $\mathcal{B} = \{(z, w) \in \mathbb{C}^2 : e^w - z + 1 = 0\}$, the Riemann domain of the known term is $\Delta = \{(z, w, y) \in \mathbb{C}^3 : e^w - z + 1 = y^3,\ y \neq 0\}$, with the projection mapping $p(z, w, y) = (z, w)$, a threefold covering, hence a proper mapping;
- the singularity locus Σ is empty, indeed the known term $F : \Delta \to \mathbb{C}$, defined by $F(z, w, y) = e^{-w}(1 + y)/2$ is holomorphic on the whole of Δ;
- the lifted initial point is $X_0 = (1, 0, 1) \in \Delta$; note that $p(X_0) = (z_0, w_0) = (1, 0)$;
- $\eta(z, w) = (z, w, \sqrt[3]{e^{w(z)} - z + 1})$, where the positive branch of the cube root has been chosen.

The singularities of the equation in the underlying $\mathbb{C}^2$ lie on the curve $e^w - z = -1$. The problem is solved by $w(z) = \log z$, which can be analytically continued along the arc γ defined on $[0, 1)$ by $\gamma(t) = 1 - t$. In view of theorem 4.3, note that the complex line $z = 0$ is not a singularity for the differential equation and there does exist $\lim_{t\to 1} \log(1 - t) = \infty$.

3.5. **A separable multivalued problem.** Consider the Cauchy problem

$$w'(z) = \sqrt{z}/\sqrt{w(z)}$$
$$w(1) = (1+c)^{2/3}.$$

We have supposed $|c|$ is positive, real and small enough; we have chosen the positive branches of the square and cube roots corresponding to the initial values $(1, (1+c)^{2/3})$.

As in the preceding examples, the Riemann domain of the known term $\sqrt{z}/\sqrt{w}$ is proper; the underlying singularities of the equation are on $\{z=0\} \cup \{w=0\}$.

The problem is solved by $w(z) = (z^{3/2}+c)^{2/3}$, showing a fixed algebraic branch point at $z=0$ and a movable one at $z^{3/2}+c=0$. Note that w can be analytically continued along the arc γ defined on $[0,1)$ by $\gamma(t) = 1-t$ (without stumbling on any of the movable branch points $z^{3/2}+c=0$ on this path, since $(1-t)^{3/2} > 0$ and $c > 0$) and $\lim_{t\to 1}[(1-t)^{3/2}+c)]^{2/3} = c^{2/3}$.

In a different fashion, w admits the following analytic continuation along a path pointing towards one of the movable branch points: consider the arc, defined on $[0,1]$:

$$\beta(t) := \begin{cases} e^{4\pi i t} & \text{if } 0 \le t \le 1/2 \\ 2(1-t) + c^{2/3}(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

After that t has run on $[0,1/2]$, the analytic continuation of $w(z) = (z^{3/2}+c)^{2/3}$ has changed sign from positive to negative; further running the parameter on $[1/2,1]$ makes z run into $c^{2/3}$, which is this time a branch point for $(z^{3/2}+c)^{2/3}$ on this path, since $[2(1-t)+c^{2/3}(2t-1)]^{3/2} < 0$ and $c > 0$. All the same, notice that $\lim_{t\to 1}\{[2(1-t)+c^{2/3}(2t-1)]^{3/2}+c\}^{2/3} = 0$ on this branch i.e., the analytic continuation of the solution of our Cauchy problem admits limit even in the above circumstance.

3.6. **Counterexample: A logarithmic known term.** Consider the autonomous Cauchy problem

$$\begin{gathered} w'(z) = -w(z)\log^2(w(z)) \\ w(0) = e^{-1/c}. \end{gathered} \tag{3.1}$$

This problem is solved by $w(z) = e^{1/(z-c)}$. Notice that the multivaluedness in w of the known term is logarithmic. In the terminology of section 2.1, we have:

- the branch locus is $\mathcal{B} = \{(z,w) \in \mathbb{C}^2 : w = 0\}$, the Riemann domain of the known term is $\Delta = \{(z,w,y) \in \mathbb{C}^3 : w = e^y\}$, with the projection mapping $p(z,w,y) = (z,w)$, which is not a proper covering;
- the singularity locus Σ is empty, indeed the known term $F : \Delta \to \mathbb{C}$, defined by $F(z,w,y) = -wy^2$ is holomorphic on the whole of Δ;
- the lifted initial point is $X_0 = (0, e^{-1/c}, -1/c) \in \Delta$; note that $p(X_0) = (z_0, w_0) = (0, e^{-1/c})$;
- $\eta(z,w) = (z, w, \log w)$, where the real branch of the logarithm has been chosen.

The singularities of the equation in the underlying $\mathbb{C}^2$ lie on the curve $w=0$. In view of theorem 4.3, note that the known term is not resolved by a proper cover and that the solution has a movable essential singularity; alternatively, it can be stated that there exists a path γ defined on $[0,1]$, joining 0 and c and such that w admits analytic continuation along $\gamma|_{[0,1)}$ but no continuous extension up to $\gamma(1)=c$.

Remark This example shows that, in general, the hyphotesis in theorem 4.3 that the multivaluedness of the known term be resolved by a proper cover cannot be dropped; however, a first order ODE with nonproper covering associated to its known term can yield a family of functions free from movable singularities notwithstanding. For instance, the equation

$$w'(z)=\sqrt{1-w^2(z)}\ \frac{\arcsin(w(z))}{z}$$

admits the family of solutions $\{\sin(kz)\}_{k\in\mathbb{C}}$, which are free from any singularities at all.

4. Main result

Now we are ready to introduce the main issue of this paper: in the terminology of section 2.1, introduce the (well defined) Cauchy problem:

$$\begin{gathered} u'(v)=F\circ\eta(z,w(z))\\ w(z_0)=w_0. \end{gathered}\tag{4.1}$$

Note that this problem is of a classical type; i.e., the known term is defined on an open set $\mathcal{U}\subset\mathbb{C}^2$ and a solution is sought that be a holomorphic function on a one-dimensional complex disc and whose graph is contained in $\mathcal{U}$.

Thanks to the classical existence-and-uniqueness theorem (see [3, Thm. 2.2.2], [4, p.281-284], such a solution does exist. The problem of its analytic continuation is natural and settles (besides the usual matters dealing with the analytic continuation of a function of one complex variable) a supplementary question; i.e., what happens if the analytic continuation ω of the graph of the solution leads to singularities in the known term; i.e., for instance, points where $F\circ\eta$ is not holomorphic? Let γ be an arc defined on $[0,1]$: if the Riemann domain (Δ,p) resolving the multivaluedness of F is proper and the complex line $\{w=\gamma(1)\}$ is not a singularity, Theorem 4.3 answers that the feasibility of analytic continuation along γ restricted to the semi-open interval $[0,1)$ entails the existence of a (finite or infinite) limit for $\omega\circ\gamma$ as the arc parameter tends to 1.

Now we need a technical lemma.

Lemma 4.1. *Let X be a metric space, $\alpha:[0,1)\to X$ a continuous arc such that $\lim_{t\to1}\alpha(t)$ does not exist in X.*

- (A) *let $\{x_l\}\to x_\infty$ be an injective converging sequence in X: then there exists a sequence $\{t_k\}\to1$ and an open neighbourhood U of $\{x_l\}\cup\{x_\infty\}$ such that $\{\alpha(t_k)\}\subset X\setminus U$.*
- (B) *for every N-tuple $\{x_1\dots x_N\}\subset X$ there exists a sequence $\{t_i\}\to b$ and neighbourhoods U_k of x_k such that $\{\alpha(t_i)\}\subset X\setminus\bigcup_{k=1}^N U_k$.*

Proof. (A) Since none of the $\{x_l\}$'s ($l\in\mathbb{N}\cup\{\infty\}$) is $\lim_{t\to1}\gamma(t)$, we have that for every $l\in\mathbb{N}\cup\{\infty\}$ there exists an open neigbhourhoud V_l of x_l such that $\alpha([\lambda,1))\not\subset V_l$ for every $\lambda\in[0,1)$; moreover, up to shrinking V_∞, there exists $N>1$ such that $n>N$ implies $x_n\in V_\infty$ but $x_N\notin V_\infty$. Clearly we can choose the $\{V_l\}$'s in such a way that: $V_i\cap V_k=\emptyset$ if $i,k\in\mathbb{N}$ and $i\neq k$; $V_i\cap V_\infty=\emptyset$ if $l\leq N$. Let now $U:=(\bigcup_{l=1}^N V_l)\cup V_\infty$: by construction, U is disconnected. Since $\alpha([\lambda,1))$ is, by contrast, connected, and, by construction, $\alpha([\lambda,1))$ is not contained in a single connected component of U, we must have $\alpha([\lambda,1)) \not\subset U$; this entails that, for every $k>0$, the set $W_k := \alpha^{-1}(X\setminus U)\cap(1-1/k,1)$ is not empty; picking $t_k\in W_k$ ends the proof. The proof of (B) is analogous and will be omitted. $\square$

We also need to generalize Hille's theorem [3, Thm 3.2.1] to our broader setting. We will use once more the notation of section 2.1. Note that in the proof of this lemma we are forced to work first in the underlying environments $\mathbb{C}$ and $\mathbb{C}^2$ and to lift the results by local charts into the overlying Riemann surface R and domain Δ. This is why derivation is defined on $\mathbb{C}_z$ and the analytic continuation of the solution takes values in $\mathbb{C}_w$. Also, the Taylor developments (4.2) are feasible using local charts in $\mathbb{C}^2$ and not directly in the complex manifold Δ. A similar approach is implicit in Hille's proof.

Lemma 4.2. *Let the Cauchy problem* (4.1) *be given. Suppose that* $\gamma:[0,1]\to\mathbb{C}$ *is an arc starting at the initial point* z_0 *and that an analytic continuation* (R,π,j,ω) *of the initial solution* w *can be carried out along* $\gamma|_{[0,1)}$*; let* $\widetilde{\gamma}:[0,1)\to R$ *be the lifted arc of* γ *with respect to the natural projection* π*. Consider the arc* $\theta:=\gamma\times[\omega\circ\widetilde{\gamma}]:[0,1)\to\mathbb{C}^2$*; suppose that the initial known term* $F\circ\eta$ *can be analytically continued along* θ *and let* $\widetilde{\theta}:[0,1)\to\Delta$ *be the lifted arc with respect to the natural projection* $p:\Delta\to\mathbb{C}^2$*. Finally, suppose that there exists a sequence* $\{t_k\}\to 1$ *such that* $\{(\widetilde{\theta}(t_k)\}$ *converges to* $\vartheta\in\Delta$ *and that the known term* F *is holomorphic at* ϑ*. Then the initial solution* w *admits an analytic continuation along* γ *up to the endpoint* $\gamma(1)$*.*

Proof. Let (z_k,w_k) be the coordinates of $p\circ\widetilde{\theta}(t_k)$ and (z_∞,w_∞) those of $p(\vartheta)$. Let η_∞ be the branch of p^{-1} such that $\eta_\infty(\vartheta)=\vartheta$ and let

$$F\circ\eta(z,w)=\sum_{r,s=0}^{\infty}c_{r,s}(z-z_k)^r(w-w_k)^s$$

be the Taylor development of $F\circ\eta$ in a bidisc $\mathbb{D}(z_\infty,w_\infty,\rho,\sigma)$ around (z_∞,w_∞).

The analytic continuation of $F\circ\eta$ along $\widetilde{\theta}$ can be concretely carried out in the underlying $\mathbb{C}^2$ by a chain of bidiscs and Taylor developments $\{(\mathcal{U}_k,F\circ\eta_k)\}_{k\in\mathbb{N}}$, where

$$F\circ\eta_k(z,w)=\sum_{r,s=0}^{\infty}c_{r,s,k}(z-z_k)^r(w-w_k)^s,\quad k\in\mathbb{N}, \tag{4.2}$$

$\gamma\times[\omega\circ\widetilde{\gamma}](t_k)\in\mathcal{U}_{N(k)}$ for every k and some strictly increasing function $N:\mathbb{N}\to\mathbb{N}$ (which we call the *counting function*) and, for each k: η_k is a local inverse of the projection mapping p and $\eta_{N(k)}:\mathcal{U}_{N(k)}\to\Delta$ is the local inverse of p such that $\eta_{N(k)}\circ p(\widetilde{\theta}(t_k))=\widetilde{\theta}(t_k)$.

By continuity, $\{c_{r,s,k}\}\to c_{r,s}$ for all r,s as $k\to\infty$, hence we can find $a>0$ and $b>0$ such that the developments in (4.2) converge absolutely and uniformly in the closed bidiscs $\overline{\mathbb{D}(u_k,v_k,a,b)}$. By Cauchy estimates, this implies that there exists $T\in\mathbb{R}^+$ such that $\sum_{r,s=0}^{\infty}|c_{r,s,k}|a^rb^s<T$ for all $k\in\mathbb{N}$; by classical complex analysis (see e.g., [3, Thm. 2.5.1]), all solutions to the Cauchy problems

$$\begin{gathered}\Omega_k'(z)=F\circ\eta_k(z,\Omega_k(z))\\ \Omega_k(z_k)=w_k,\end{gathered}\qquad k\in\mathbb{N}\cup\{\infty\} \tag{4.3}$$

have radii of convergence of at least $\sigma := a(1 - e^{-b/(2aT)})$, thus (keeping into account that the counting function N is strictly increasing) there exists $\ell \in \mathbb{N}$ such that $\ell \in N(\mathbb{N})$, $v_\infty \in \mathbb{D}(v_\ell, \sigma)$; by continuity, $\Omega_\ell(z_\infty) = w_\infty$. This means that Ω_ℓ admits analytic continuation along γ up to $z_\infty = \gamma(1)$; since, by the hypothesis of the existence of the analytic continuation of the initial solution w to the Cauchy problem (4.1), we can in turn construct Ω_ℓ by starting from w and carrying out analytic continuation along γ, we can conclude that w itself admits analytic continuation along γ up to $z_\infty = \gamma(1)$. $\square$

Finally, here is our main theorem (notation has been set up and discussed in section 2.1).

Theorem 4.3. *Let a Cauchy problem for a first order ordinary differential equation in the complex domain be given. Suppose the singularities of the differential equation to be contained in a complex-analytic curve $\mathcal{S} \subset \mathbb{C}^2$. Let $\gamma : [0,1] \to \mathbb{C}$ be an arc starting at the initial point z_0 such that the complex line $z = \gamma(1)$ is not contained in $\mathcal{S}$. Suppose that an analytic continuation (R, π, j, ω) of the initial solution w can be obtained along $\gamma|_{[0,1)}$; let $\widetilde{\gamma} : [0,1) \to R$ the lifted arc of γ with respect to the natural projection π: then there exists (finite or infinite) $\lim_{t\to 1} \omega \circ \widetilde{\gamma}(t)$.*

Proof. Let $\mathcal{B}$, (Δ, p), η, Σ, (z_0, w_0) have the same meaning as discussed in section 2.1. In particular, recall that η is a local inverse of p, defined in a bidisc $\mathbb{D}_1 \times \mathbb{D}_2$ around (z_0, w_0) and with value in the Riemann domain Δ resolving the multivaluedness of the known term; hence, viewed from the underlying $\mathbb{C}^2$, $F \circ \eta$ is a branch of the multivalued function F. Also, recall that our Cauchy problem is:

$$\begin{gathered} u'(v) = F \circ \eta(z, w(z)) \\ w(z_0) = w_0. \end{gathered}$$

Now $\mathcal{S}$ is complex-analytic and $\{z = \gamma(1)\} \not\subset \mathcal{S}$, so

$$P := \{w \in \mathbb{C} : (\gamma(1), w) \in \mathcal{S}\}$$

is discrete in $\mathbb{C}_w$; hence we can suppose it to be indexed over $\mathbb{N}$ or a finite subset.

Suppose now, by contradiction, that $\lim_{t\to 1} \omega \circ \widetilde{\gamma}(t)$ does not exist. By lemma 4.1 (A) or (B), according as P is finite or infinite, (with $X = \mathbb{P}^1$, $\alpha = \omega \circ \widetilde{\gamma}$, $\{x_k\} = P \cup \{\infty\}$), there exist: a sequence $\{t_k\} \to 1$, $r > 0$, $\varepsilon > 0$ and a finite subset $Q = \{\lambda_\nu\} \subset P$, such that

$$\{\omega \circ \widetilde{\gamma}(t_k)\} \subset \overline{\mathbb{D}(0,r)} \setminus \cup_{\lambda_\nu \in Q} \mathbb{D}(\lambda_\nu, \varepsilon).$$

Now, by continuity, there exists $\varrho > 0$ such that

$$z \in \mathbb{D}(\gamma(1), \varrho) \quad \text{implies} \quad \mathrm{pr}_{\mathbb{C}_w}\left[\mathcal{S} \cap (\{z\} \times \mathbb{C})\right] \subset \cup_{\lambda_\nu \in Q} \mathbb{D}(\lambda_\nu, \varepsilon);$$

Set

$$W := \overline{\mathbb{D}(\gamma(1), \varrho)} \times \left[\overline{\mathbb{D}(0,r)} \setminus \cup_{\lambda_\nu \in Q} \mathbb{D}(\lambda_\nu, \varepsilon)\right];$$

by construction W is compact in $\mathbb{C}^2$ and $W \cap \mathcal{S} = \emptyset$; also, we may suppose, without loss of generality, $\{\gamma(t_k)\} \subset \mathbb{D}(\gamma(1), \varrho)$, implying in turn that

$$\{\gamma \times [\omega \circ \widetilde{\gamma}](t_k)\} \subset W. \tag{4.4}$$

Let now A be the holomorphic function on $\mathbb{C}^2$ such that $\mathcal{S} = A^{-1}(0)$; the set

$$\mathcal{B} := \{\zeta \in R : A(\pi(\zeta), \omega(\zeta)) = 0\} \tag{4.5}$$

is discrete for otherwise we would have $A(\pi(\zeta), \omega(\zeta)) \equiv 0$ for all $\zeta \in R$ contradicting the hypothesis that $(z_0, w_0) \notin \mathcal{S}$. Thus, by continuity. $\widetilde{\gamma}^{-1}(\mathcal{B})$ is discrete and, by (4.4), $\widetilde{\gamma}^{-1}(\mathcal{B}) \cap \{t_k\}$ is finite. Therefore, by passing to a nearby homotopic arc if needed, we may suppose $\widetilde{\gamma}([0,1)) \cap \mathcal{B} = \emptyset$, implying

$$\gamma \times [\omega \circ \widetilde{\gamma}]([0,1)) \cap \mathcal{S} = \emptyset.$$

Hence $F \circ \eta$ admits regular analytic continuation along $\theta := \gamma \times [\omega \circ \widetilde{\gamma}] : [0,1) \to \mathbb{C}^2$. Let $\widetilde{\theta} : [0,1) \to \Delta$ be the lifted arc with respect to the projection mapping p. By construction and by (4.4) we have

$$\{\widetilde{\theta}(t_k)\} \subset p^{-1}(\{\gamma \times \omega \circ \widetilde{\gamma}(t_k))\}) \subset p^{-1}(W).$$

Since W is compact and p is proper, $p^{-1}(W)$ is compact; thus, by maybe passing to a subsequence, we may assume that $\{\widetilde{\theta}(t_k)\}$ converges to a limit $\vartheta \in p^{-1}(W) \subset \Delta \setminus \Sigma$; note that F is holomorphic at ϑ; now a direct application of lemma 4.2 allows us to conclude that w admits analytic continuation up to $\gamma(1)$. This fact contradicts the hypothesis that $\lim_{t\to 1} \omega \circ \widetilde{\gamma}(t)$ does not exist. □

Remark Theorem 4.3 implies that none of the singularities of the solution of the Cauchy problem (4.1) are essential, for if ζ were such a singularity, there would exists a path $\gamma : [0,1] \to \mathbb{C}$ connecting z_0 to ζ such that w admits analytic continuation along $\gamma|_{[0,1)}$ and no continuous extension up to $\gamma(1)$.

Acknowledgements. The author wishes to thank the anonymous referee for his or her valuable remarks and criticisms, which helped us improve this article. In particular, for suggesting the inclusion of counterexample 3.1.

References

[1] Ludwig Bieberbach; *Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt* Springer (Berlin), 1953

[2] Robert C. Gunning, Hugo Rossi; *Analytic functions of several complex variables* Prentice Hall, 1965

[3] Einar Hille; *Ordinary differential equations in the complex domain,* John Wiley & sons, 1976

[4] E. L. Ince, *Ordinary differential equations* Dover, 1956 (originally published in 1926)

[5] Paul Painlevé; *Sur les lignes singulires des fonctions analytiques* Annales de la faculté des sciences de Toulouse, Sér. 1, 2 (1888)

[6] E. Picard; *Traité d'analyse, Tome II* Gauthier-Villars et fils, Paris, 1893

[7] Norbert Steinmetz; *Book review of Painlevé differential equations in the complex plane, by V. I. Gromak, I. Laine, and S. Shimomura* Bulletin (new series) of the American Mathematical Society Volume 41, Number 4, Pages 523-528

Claudi Meneghin

(Isrc) - Fermo Posta Chiasso 1, CH-6830 Chiasso, Switzerland

E-mail address: claudi.meneghin@gmail.com