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MULTIPLICITY OF POSITIVE SOLUTIONS FOR FOUR-POINT BOUNDARY VALUE PROBLEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. Using a fixed-point theorem in cones, we obtain sufficient conditions for the multiplicity of positive solutions for four-point boundary value problems of third-order impulsive differential equations with p-Laplacian.

1. Introduction

Recently, there has been much attention focused on the theory of impulsive differential equation as it is widely used in various areas such as mechanics, electromagnetism, chemistry. A lot of theories have been established to solve these problems, see [9], [3] and the references therein. Guo [4] obtained the existence of solutions, via cone theory, for second-order impulsive differential equation

$$\begin{gathered} x'' = f(t, x, Tx), \quad t \geq 0,\ t \neq t_k\ k = 1, 2, 3, \dots, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, 3, \dots, \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, 3, \dots, \\ x(0) = x_0, \quad x'(0) = x_0^*. \end{gathered}$$

In [1], using Leggett-Williams fixed point theorem, authors studied the multiplicity result for second order impulsive differential equations

$$\begin{gathered} y'' + \phi(t) f(y(t)) = 0, \quad t \in (0, 1) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta y(t_k) = I_k(y(t_k^-)),\ k = 1, 2, 3, \dots, m, \\ \Delta y'(t_k) = J_k(y(t_k^-)),\ k = 1, 2, 3, \dots, m, \\ y(0) = y(1) = 0. \end{gathered}$$

Kaufmann [8] studied a second-order nonlinear differential equation on an unbounded domain with solutions subject to impulsive conditions and the Sturm-Liouville type boundary conditions. In [5]-[7], the authors studied positive solutions of multiple points boundary value problems for second order impulsive differential equations.

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All the works above concern boundary value problems with second-order impulsive equations, and there are just a few works that consider multiplicity of positive solutions for third-order impulsive equations with p-Laplacian.

Motivated by all the works above, we concentrate on getting multiple positive solutions for four-point boundary value problems of third-order impulsive differential equations with p-Laplacian

$$\begin{gathered}
(\phi_p(u''(t)))' = f(t,u(t),u'(t)), \quad t \in (0,1)\backslash\{t_1,t_2,\dots,t_m\}, \\
\Delta u''(t)|_{t=t_k} = 0, \quad k=1,2,\dots,m, \\
\Delta u'(t)|_{t=t_k} = I_k(u(t_k)), \quad k=1,2,\dots,m, \\
\Delta u(t)|_{t=t_k} = J_k(u(t_k)), \quad k=1,2,\dots,m, \\
u''(0)=0, \quad u'(0)=\alpha u'(\xi)+\beta u'(\eta), \quad u(1)=\delta u(0),
\end{gathered} \tag{1.1}$$

where ϕ_p is p-Laplacian operator

$$\phi_p(s)=|s|^{p-2}s, p>1, \quad (\phi_p)^{-1}=\phi_q, \quad \frac{1}{p}+\frac{1}{q}=1,$$

t_k, $k=0,1,2,\dots,m,m+1$, are constants which satisfy

$$0=t_0<t_1<t_2<\dots<t_k<\dots<t_m<t_{m+1}=1,$$

$\Delta u|_{t=t_k} = u(t_k^+)-u(t_k^-)$, in which $u(t_k^+)$ ($u(t_k^-)$ respectively) denote the right limit (left limit respectively) of $u(t)$ at $t=t_k$, and $\alpha,\beta>0$, $\alpha+\beta<1$; $0<\xi$, $\eta<1$; $\xi,\eta\neq t_k$ ($k=1,2,\dots,m$); $\delta>1$; $f\in C([0,1]\times[0,+\infty)\times\mathbb{R},[0,+\infty))$, $I_k,J_k\in C([0,+\infty),[0,+\infty))$.

2. Preliminaries

Let $J=[0,1]\backslash\{t_1,t_2,\dots,t_m\}$, $PC[0,1]=\{u:[0,1]\to R,\ u$ is continuous at $t\neq t_k,\ u(t_k^+),\ u(t_k^-)$ exist, and $u(t_k^-)=u(t_k), k=1,2,\dots,m\}$, $PC^1[0,1]=\{u\in PC[0,1]\mid u'$ is continuous at $t\neq t_k$, $u'(t_k^+),\ u'(t_k^-)$ exist, $k=1,2,\dots,m\}$, with the norm

$$\|u\|_{PC}=\sup_{t\in J}|u(t)|, \quad \|u\|_{PC^1}=\max_{t\in J}\{\|u\|_{PC},\|u'\|_{PC}\}.$$

Obviously $PC[0,1]$ and $PC^1[0,1]$ are Banach spaces.

Lemma 2.1. *$u\in PC^1[0,1]\bigcap C^3[J]$ is a solution of (1.1) if and only if*

$$\begin{aligned}
u(t) = {}& u(0)+u'(0)t+\int_0^t(t-s)\phi_q\Big(\int_0^s f(r,u(r),u'(r))dr\Big)ds \\
&+\sum_{t_k<t}(t-t_k)I_k(u(t_k))+\sum_{t_k<t}J_k(u(t_k)),
\end{aligned} \tag{2.1}$$

where

$$
\begin{aligned}
u(0) = &\frac{\alpha\int_0^\xi \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds + \beta\int_0^\eta \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds}{(\delta-1)(1-\alpha-\beta)}\\
&+ \frac{\alpha\sum_{t_k<\xi} I_k(u(t_k)) + \beta\sum_{t_k<\eta} I_k(u(t_k))}{(\delta-1)(1-\alpha-\beta)}\\
&+ \frac{\int_0^1\int_0^s \phi_q(\int_0^r f(w,u(w),u'(w))dw)\,dr\,ds}{\delta-1}\\
&+ \frac{1}{\delta-1}\sum_{k=1}^m \big((1-t_k)I_k(u(t_k)+J_k(u(t_k))\big),
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
u'(0) = &\frac{\alpha\int_0^\xi \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds + \beta\int_0^\eta \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds}{1-\alpha-\beta}\\
&+ \frac{\alpha\sum_{t_k<\xi} I_k(u(t_k)) + \beta\sum_{t_k<\eta} I_k(u(t_k))}{1-\alpha-\beta}.
\end{aligned} \tag{2.3}
$$

Proof. Suppose $u \in PC^1[0,1]\bigcap C^3[J]$ is a solution of (1.1), for all $k = 1, 2, \dots, m$, from Lagrange's mean value theorem we have

$$u(t_k) - u(t_k - h) = u'(\xi_k)h, \quad 0 < h < t_k - t_{k-1},\ \xi_k \in (t_k - h, t_k),$$

because $u'(t_k^-)$ exists, we get

$$u'_-(t_k) = \lim_{h\to 0^+} \frac{u(t_k) - u(t_k - h)}{h} = \lim_{\xi_k \to t_k^-} u'(\xi_k) = u'(t_k^-).$$

Let $u'(t_k) = u'_-(t_k) = u'(t_k^-)$, $k = 1, 2, \dots, m$. We use Lagrange's mean value theorem again and obtain

$$u'(t_k) - u'(t_k - h) = u''(\eta_k)h, \quad 0 < h < t_k - t_{k-1},\ \eta_k \in (t_k - h, t_k),$$

we can get $u''_-(t_k)$ exists from $\Delta u''(t)|_{t=t_k} = u''(t_k^+) - u''(t_k^-) = 0$, and

$$u''_-(t_k) = \lim_{h\to 0^+} \frac{u'(t_k) - u'(t_k - h)}{h} = \lim_{\xi_k \to t_k^-} u''(\xi_k) = u''(t_k^-).$$

Let $u''(t_k) = u''(t_k^-)$, $k = 1, 2, \dots, m$. Integrating the differential equation (1.1) we have

$$\phi_p(u''(t)) - \phi_p(u''(0)) = \int_0^t f(s, u(s), u'(s))ds, \quad 0 \le t \le t_1.$$

By $u''(0) = 0$, we have

$$u''(t) = \phi_q(\int_0^t f(s, u(s), u'(s))ds);$$

that is,

$$u''(t) = \phi_q(\int_0^t f(s, u(s), u'(s))ds),$$

and

$$u''(t_1) = \phi_q(\int_0^{t_1} f(s, u(s), u'(s))ds).$$

Since $\Delta u''(t)|_{t=t_1} = u''(t_1^+) - u''(t_1^-) = 0$, for $t_1 < t \le t_2$, we obtain

$$u''(t) = \phi_q(\int_0^t f(s, u(s), u'(s))ds).$$

Similarly, by $\Delta u''(t)|_{t=t_k} = u''(t_k^+) - u''(t_k^-) = 0,\ k = 1, 2, \dots, m$, we can show for all $t \in [0, 1]$,

$$u''(t) = \phi_q(\int_0^t f(s, u(s), u'(s))ds). \tag{2.4}$$

For each $t \in (0, 1)$, there exist $0 \le t_k < t_{k+1} \le 1$, such that $t_k < t \le t_{k+1}$, by integrating both sides of (2.4), we obtain

$$\begin{gathered}
u'(t_1^-) - u'(0) = \int_0^{t_1} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds,\\
u'(t_2^-) - u'(t_1^+) = \int_{t_1}^{t_2} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds,\\
\dots\\
u'(t_k^-) - u'(t_{k-1}^+) = \int_{t_{k-1}}^{t_k} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds,\\
u'(t) - u'(t_k^+) = \int_{t_k}^{t} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds.
\end{gathered}$$

Hence,

$$u'(t) = u'(0) + \int_0^t \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds + \sum_{t_k<t} I_k(u(t_k)).$$

We have

$$\begin{gathered}
\alpha u'(\xi) = \alpha u'(0) + \alpha \int_0^{\xi} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds + \alpha \sum_{t_k<\xi} I_k(u(t_k)),\\
\beta u'(\eta) = \beta u'(0) + \beta \int_0^{\eta} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds + \beta \sum_{t_k<\eta} I_k(u(t_k)).
\end{gathered}$$

It follows that

$$\begin{aligned}
u'(0) &= \frac{\alpha \int_0^{\xi} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds + \beta \int_0^{\eta} \phi_q(\int_0^s f(r, u(r), u'(r))dr)ds}{1-\alpha-\beta}\\
&\quad + \frac{\alpha \sum_{t_k<\xi} I_k(u(t_k)) + \beta \sum_{t_k<\eta} I_k(u(t_k))}{1-\alpha-\beta}
\end{aligned}$$

from $u'(0) = \alpha u'(\xi) + \beta u'(\eta)$.

Similarly, we get the results as follows with the method above

$$\begin{aligned}
u(t) &= u(0) + u'(0)t + \int_0^t (t-s)\phi_q\Big(\int_0^s f(r, u(r), u'(r))dr\Big)ds\\
&\quad + \sum_{t_k<t} (t-t_k) I_k(u(t_k)) + \sum_{t_k<t} J_k(u(t_k)).
\end{aligned}$$

Note that by the boundary condition $u(1) = \delta u(0)$,

$$\begin{aligned} u(0) = & \frac{\alpha \int_0^\xi \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds + \beta \int_0^\eta \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds}{(\delta-1)(1-\alpha-\beta)} \\ & + \frac{\alpha \sum_{t_k<\xi} I_k(u(t_k)) + \beta \sum_{t_k<\eta} I_k(u(t_k))}{(\delta-1)(1-\alpha-\beta)} \\ & + \frac{\int_0^1 \int_0^s \phi_q(\int_0^r f(w,u(w),u'(w))dw)\,dr\,ds}{\delta-1} \\ & + \frac{1}{\delta-1}\sum_{k=1}^m [(1-t_k)I_k(u(t_k)+J_k(u(t_k))]. \end{aligned}$$

On the other hand, let $u \in PC^1[0,1] \bigcap C^3[J]$ be a solution of (2.1), differentiate (2.1) when $t \neq t_k$, we have

$$u''(t) = \phi_q(\int_0^t f(s,u(s),u'(s))ds);$$

that is,

$$\phi_p(u''(t)) = \int_0^t f(s,u(s),u'(s))ds\,.$$

Differentiating again,

$$(\phi_p(u''(t)))' = f(t,u(t),u'(t)).$$

By (2.1), we can easily get

$$\begin{gathered} \Delta u''(t)|_{t=t_k} = 0, \quad k=1,2,\dots,m, \\ \Delta u'(t)|_{t=t_k} = I_k(u(t_k)), \quad k=1,2,\dots,m, \\ \Delta u(t)|_{t=t_k} = J_k(u(t_k)), \quad k=1,2,\dots,m, \\ u''(0)=0, \quad u'(0)=\alpha u'(\xi)+\beta u'(\eta), \quad u(1)=\delta u(0). \end{gathered}$$

□

Next, we give the Bai-Ge fixed point theorem which is used in the proof of our main result. Let E be a Banach space, $P \subset E$ be a cone, θ, $\psi : P \to [0,+\infty)$ be nonnegative convex functions which satisfy

$$\|x\| \le k \max\{\theta(x),\psi(x)\}, \quad \text{for all } x \in P, \tag{2.5}$$

where k is a positive constant.

$$\Omega = \{x \in P : \theta(x) < r, \psi(x) < L\} \neq \phi, \text{ where } r>0, L>0. \tag{2.6}$$

Let $r > a > 0, L > 0$ be constants, θ, $\psi : P \to [0,+\infty)$ be two nonnegative continuous convex functions which satisfy (2.5) and (2.6), and γ be a nonnegative concave function on P. We define convex sets as follows

$$\begin{gathered} P(\theta,r;\psi,L) = \{x \in P : \theta(x) < r, \psi(x) < L\}, \\ \overline{P}(\theta,r;\psi,L) = \{x \in P : \theta(x) \le r, \psi(x) \le L\}, \\ P(\theta,r;\psi,L;\gamma,a) = \{x \in P : \theta(x) < r, \psi(x) < L, \gamma(x) > a\}, \\ \overline{P}(\theta,r;\psi,L;\gamma,a) = \{x \in P : \theta(x) \le r, \psi(x) \le L, \gamma(x) \ge a\}. \end{gathered}$$

Lemma 2.2 ([2]). *Let E be Banach space, $P \subset E$ be a cone and $r_2 \geq d > b > r_1 > 0, L_2 \geq L_1 > 0$ be constants. Assume $\theta, \psi : P \to [0, +\infty)$ are nonnegative continuous convex functions which satisfy* (2.5) *and* (2.6). *γ is a nonnegative concave function on P such that for all x in $\overline{P}(\theta, r_2; \psi, L_2)$ satisfies $\gamma(x) \leq \theta(x)$. $T : \overline{P}(\theta, r_2; \psi, L_2) \to \overline{P}(\theta, r_2; \psi, L_2)$ is a completely continuous operator. Suppose*

- (C1) *$\{x \in \overline{P}(\theta, d; \psi, L_2; \gamma, b) : \gamma(x) > b\} \neq \phi$, and $\gamma(Tx) > b$, for $x \in \overline{P}(\theta, d; \psi, L_2; \gamma, b)$;*
- (C2) *$\theta(Tx) < r_1$, $\psi(Tx) < L_1$, for $x \in \overline{P}(\theta, r_1; \psi, L_1)$;*
- (C3) *$\gamma(Tx) > b$, for $x \in \overline{P}(\theta, r_2; \psi, L_2; \gamma, b)$ with $\theta(Tx) > d$.*

Then T has at least three fixed points x_1, x_2, x_3 in $\overline{P}(\theta, r_2; \psi, L_2)$. Further,

$$x_1 \in \overline{P}(\theta, r_1; \psi, L_1), \quad x_2 \in \{\overline{P}(\theta, r_2; \psi, L_2; \gamma, b) : \gamma(x) > b\},$$
$$x_3 \in \overline{P}(\theta, r_2; \psi, L_2) \backslash \big(\overline{P}(\theta, r_1; \psi, L_1) \cup \overline{P}(\theta, r_2; \psi, L_2; \gamma, b)\big).$$

3. Main results

Let closed cone P be defined by

$$P = \{u \in PC^1[0,1] : u(t) \geq 0\}.$$

Define operator $T : P \to PC^1[0,1]$ by

$$\begin{aligned} Tu(t) &= u(0) + u'(0)t + \int_0^t (t-s)\phi_q\Big(\int_0^s f(r, u(r), u'(r))dr\Big)ds \\ &\quad + \sum_{t_k < t}(t - t_k)I_k(u(t_k)) + \sum_{t_k < t} J_k(u(t_k)),\ t \in [0,1], \end{aligned}$$

which $u(0)$, $u'(0)$ are defined in (2.2), (2.3).

The nonnegative continuous convex functions θ, ψ, and nonnegative continuous concave function γ are defined by

$$\theta(u) = \sup_{0 \leq t \leq 1} u(t), \quad \psi(u) = \sup_{0 \leq t \leq 1} |u'(t)|, \quad \gamma(u) = \min_{t \in [a_m, b_m]} u(t),$$

for all $u \in P$, where $a_m = \frac{3t_m + t_{m+1}}{4}$, $b_m = \frac{t_m + 3t_{m+1}}{4}$. Let

$$l = \frac{\delta - 1}{\int_{a_m}^{b_m}(b_m - r)\phi_q(r - a_m)ds} = \frac{2^{q+1}q(q+1)(\delta-1)}{(1-t_m)^{q+1}},$$
$$I_u^R = \max\{I_1(u), I_2(u), \dots, I_m(u)\}, \quad u \in [0, R],$$
$$M_1 = \frac{1 - \alpha - \beta}{\int_0^1 \phi_q(s)ds + m(1-\alpha-\beta) + x\alpha + y\beta} = \frac{1-\alpha-\beta}{1/q + m(1-\alpha-\beta) + x\alpha + y\beta},$$

where x and y satisfy $t_x < \xi < t_{x+1}$, $t_y < \eta < t_{y+1}$.

Theorem 3.1. *Suppose there exist constants $r_2 \geq d \geq \delta b > b > r_1 > 0$, $L_2 \geq L_1 > 0$ such that*

$$r_2 \geq \frac{bl\delta(m + 1/q)}{(\delta - 1)(1 - \alpha - \beta)}, \quad L_2 \geq \frac{bl\delta(m+1/q)}{1 - \alpha - \beta},$$

and the following conditions hold

- (H1) *$f(t, u, v) < \phi_p(\min\{\frac{\delta-1}{\delta}M_1 r_1, M_1 L_1\})$, $(t, u, v) \in [0,1] \times [0, r_1] \times [-L_1, L_1]$;*
- (H2) *$\phi_p(bl) < f(t, u, v)$, $(t, u, v) \in [a_m, b_m] \times [b, d] \times [-L_2, L_2]$;*
- (H3) *$f(t, u, v) < \phi_p(\min\{\frac{\delta-1}{\delta}M_1 r_2, M_1 L_2\})$, $(t, u, v) \in [0,1) \times [0, r_2] \times [-L_2, L_2]$;*

(H4) $t_k I_k(u) > J_k(u)$ *for* $u \in [0, r_2]$, $I_u^{r_1} < \min\{\frac{\delta-1}{\delta} M_1 r_1, M_1 L_1\}$, $I_u^{r_2} < \min\{\frac{\delta-1}{\delta} M_1 r_2, M_1 L_2\}$.

Then boundary-value problem (1.1) *has at least three positive solutions* $u_1, u_2, u_3 \in \overline{P}(\theta, r_2; \psi, L_2)$ *which satisfy*

$$\sup_{0\le t\le 1} u_1(t) \le r_1, \quad \sup_{0\le t\le 1} |u_1'(t)| \le L_1;$$
$$b < \min_{t\in[a_m,b_m]} u_2(t) \le \sup_{0\le t\le 1} u_2(t) \le r_2, \quad \sup_{0\le t\le 1} |u_2'(t)| \le L_2;$$
$$\sup_{0\le t\le 1} u_3(t) \le \delta d, \quad \sup_{0\le t\le 1} |u_3'(t)| \le L_2.$$

Proof. We need to prove $\frac{\delta-1}{\delta} M_1 r_1 \ge bl$ in order to make sure that the theorem makes sense, since we have $r_2 \ge bl\delta \frac{m+1/q}{(\delta-1)(1-\alpha-\beta)}$, and

$$\begin{aligned}
\frac{\delta-1}{\delta} M_1 r_1 &= \frac{(\delta-1)(1-\alpha-\beta)}{\delta(1/q + m(1-\alpha-\beta) + x\alpha + y\beta)} r_1 \\
&\ge \frac{\delta-1}{\delta} \times \frac{1-\alpha-\beta}{1/q+m} \times \frac{m+1/q}{(\delta-1)(1-\alpha-\beta)} bl\delta \\
&= bl.
\end{aligned}$$

Similarly, we have $M_1 L_2 \ge M_1 r_2(\delta-1) \ge bl\delta > bl$, so there has no contradiction among conditions. It is easy to see that (1.1) has a solution if and only if

$$\begin{aligned}
Tu(t) &= u(0) + u'(0)t + \int_0^t (t-s)\phi_q\Big(\int_0^s f(r, u(r), u'(r))dr\Big) ds \\
&\quad + \sum_{t_k<t} (t-t_k) I_k(u(t_k)) + \sum_{t_k<t} J_k(u(t_k)), \quad t \in [0,1]
\end{aligned}$$

has a fixed point.

Next, we will check the conditions (C1), (C2) and (C3) of Lemma 2.2 are satisfied for the operator T.

Obviously, we can get $Tu(t) \ge 0$, $(Tu)'(t) \ge 0$, for all $t \in [0,1]$ and $u \in P$, that also means Tu is a monotone increasing function.

Firstly, we have $\theta(u) \le r_2$, $\psi(u) \le L_2$ for all $u \in \overline{P}(\theta, r_2; \psi, L_2)$. By the condition (H4) $t_k I_k(u) > J_k(u)$ and $I_u^{r_2} < \min\{\frac{\delta-1}{\delta} M_1 r_2, M_1 L_2\}$, we get

$$\sum_{k=1}^m \Big((1-t_k) I_k(u(t_k)) + J_k(u(t_k))\Big) \le \sum_{k=1}^m I_k(u(t_k)) \le m\frac{\delta-1}{\delta} M_1 r_2.$$

By condition (H3), $f(t,u,v) < \phi_p(\frac{\delta-1}{\delta} M_1 r_2)$, we obtain

$$\phi_q\Big(\int_0^s f(t, u(r), u'(r))dr\Big) \le \frac{\delta-1}{\delta} M_1 r_2 \phi_q(s).$$

Hence,

$$u(0) \le \frac{\alpha/q + \beta/q + x\alpha + y\beta}{\delta(1-\alpha-\beta)} M_1 r_2 + \frac{1}{q\delta} M_1 r_2 + \frac{m}{\delta} M_1 r_2.$$

Similarly,

$$u'(0) \le \frac{(\delta-1)(\alpha+\beta+q(x\alpha+y\beta))}{q\delta(1-\alpha-\beta)} M_1 r_2.$$

Therefore, we can show that

$$\begin{aligned}\theta(Tu) &= \sup_{0\le t\le 1}(u(0)+u'(0)t+\int_0^t (t-s)\phi_q\Big(\int_0^s f(r,u(r),u'(r))dr\Big)ds \\ &\quad +\sum_{t_k<t}(t-t_k)I_k(u(t_k))+\sum_{t_k<t}J_k(u(t_k))) \\ &\le \frac{\alpha/q+\beta/q+x\alpha+y\beta}{\delta(1-\alpha-\beta)}M_1r_2+\frac{1}{q\delta}M_1r_2+\frac{m}{\delta}M_1r_2 \\ &\quad +\frac{(\delta-1)(\alpha+\beta+q(x\alpha+y\beta))}{q\delta(1-\alpha-\beta)}M_1r_2+\frac{(\delta-1)(1+mq)}{q\delta}M_1r_2 \\ &= \frac{1/q+m(1-\alpha-\beta)+x\alpha+y\beta}{1-\alpha-\beta}M_1r_2.\end{aligned}$$

Since $M_1=\frac{1-\alpha-\beta}{1/q+m(1-\alpha-\beta)+x\alpha+y\beta}$, we have $\theta(Tu)\le r_2$.

Similarly, we have

$$\begin{aligned}\psi(Tu) &= \sup_{0\le t\le 1}|(Tu)'(t)| \\ &= \sup_{0\le t\le 1}|u'(0)+\int_0^t \phi_q(\int_0^s f(r,u(r),u'(r))dr)ds+\sum_{t_k<t}I_k(u(t_k))| \\ &\le \frac{1/q+m(1-\alpha-\beta)+x\alpha+y\beta}{1-\alpha-\beta}M_1L_2=L_2.\end{aligned}$$

Therefore, $T:\overline{P}(\theta,r_2;\psi,L_2)\to\overline{P}(\theta,r_2;\psi,L_2)$, and it is easy to see that T is a completely continuous operator.

The proof of the condition (C2) in Lemma 2.2 is similar to the one above.

To check condition (C1) of Lemma 2.2, we choose $u_0=d$. It is easy to see that $u_0\in\overline{P}(\theta,d;\psi,L_2)$ and $\gamma(u)=d>b$, so $\{x\in\overline{P}(\theta,d;\psi,L_2;\gamma,b):\gamma(x)>b\}\ne\phi$.

For $u\in\overline{P}(\theta,d;\psi,L_2;\gamma,b)$, we have $b\le u(t)\le d$, $|u'(t)|\le L_2$ for all $t\in[a_m,b_m]$. Since Tu is a monotone increasing function, and $(Tu)(t)\ge 0$, $t\in[0,1]$, we have

$$\begin{aligned}\gamma(Tu) &= \min_{t\in[a_m,b_m]}\big(u(0)+u'(0)t+\int_0^t (t-s)\phi_q(\int_0^s f(r,u(r),u'(r))dr)ds \\ &\quad +\sum_{t_k<t}[(t-t_k)I_k(u(t_k)+\sum_{t_k<t}J_k(u(t_k))\big) \\ &= Tu(a_m).\end{aligned}$$

By (H2) and $u(0),u'(0)$ defined before, we have

$$\phi_p(bl)<f(t,u,v),\quad (t,u,v)\in[a_m,b_m]\times[b,d]\times[-L_2,L_2],$$

$$
\begin{aligned}
Tu(a_m) &= u(0) + u'(0)a_m + \int_0^{a_m} (a_m - s)\phi_q(\int_0^s f(r, u(r), u'(r))dr)ds \\
&\quad + \sum_{t_k < a_m} (a_m - t_k)I_k(u(t_k)) + \sum_{t_k < a_m} J_k(u(t_k)) \\
&\geq u(0) \\
&\geq \frac{1}{\delta - 1} \int_0^1 \int_0^s \phi_q(\int_0^r f(w, u(w), u'(w))dw)\, dr\, ds \\
&> \frac{1}{\delta - 1} \int_{a_m}^{b_m} ds \int_{a_m}^s \phi_q(\int_{a_m}^r \phi_p(bl)dw)dr \\
&= \frac{bl}{\delta - 1} \int_{a_m}^{b_m} ds \int_{a_m}^s \phi_q(r - a_m)dr \\
&= \frac{bl}{\delta - 1} \int_{a_m}^{b_m} (b_m - r)\phi_q(r - a_m)dr = b.
\end{aligned}
$$

Thus $\gamma(Tu) > b$ and the condition (C1) of Lemma 2.2 also holds.

Finally to prove (C3) of Lemma 2.2, we check $\gamma(Tu) > b$ to be satisfied for all $u \in \overline{P}(\theta, r_2; \psi, L_2; \gamma, b)$ with $\theta(Tu) > d$. Since Tu is a nonnegative monotone increasing function, we can get

$$
\begin{gathered}
\theta(Tu) = \sup_{0 \leq t \leq 1} Tu(t) = Tu(1), \\
\gamma(Tu) = \min_{t \in [a_m, b_m]} Tu(t) = Tu(a_m), \\
Tu(a_m) \geq Tu(0) = \frac{1}{\delta} Tu(1) > \frac{d}{\delta} \geq b;
\end{gathered}
$$

that is, $\gamma(Tu) > b$.

We have checked Lemma 2.2 to make sure all the conditions are satisfied with the work we have done in the section above. Then T has at least three fixed points u_1, u_2, u_3 in $\overline{P}(\theta, r_2; \psi, L_2)$. Further,

$$
\begin{gathered}
u_1 \in \overline{P}(\theta, r_1; \psi, L_1), \quad u_2 \in \{\overline{P}(\theta, r_2; \psi, L_2; \gamma, b) : \gamma(x) > b\}, \\
u_3 \in \overline{P}(\theta, r_2; \psi, L_2) \backslash \{\overline{P}(\theta, r_1; \psi, L_1) \cup \overline{P}(\theta, r_2; \psi, L_2; \gamma, b)\}.
\end{gathered}
$$

Therefore (1.1) has at least three positive solutions u_1, u_2, u_3. From the boundary conditions we have $u_3(1) = \delta u_3(0)$ and u_3 is a monotone increasing function, so we have

$$
b > \gamma(u_3) = \min_{a_m \leq t \leq b_m} u_3(t) = u_3(a_m) \geq u_3(0) = \frac{1}{\delta} u_3(1) = \frac{1}{\delta} \theta(u_3),
$$

so $\theta(u_3) \leq \delta b$, that means $\sup_{0 \leq t \leq 1} u_3(t) \leq \delta d$, and u_1, u_2, u_3 satisfy

$$
\begin{gathered}
\sup_{0 \leq t \leq 1} u_1(t) \leq r_1, \quad \sup_{0 \leq t \leq 1} |u_1'(t)| \leq L_1; \\
b < \min_{t \in [a_m, b_m]} u_2(t) \leq \sup_{0 \leq t \leq 1} u_2(t) \leq r_2, \quad \sup_{0 \leq t \leq 1} |u_2'(t)| \leq L_2; \\
\sup_{0 \leq t \leq 1} u_3(t) \leq \delta d, \quad \sup_{0 \leq t \leq 1} |u_3'(t)| \leq L_2.
\end{gathered}
$$

□

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