*Electronic Journal of Differential Equations*, Vol. 2010(2010), No. 50, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# ANTI-PERIODIC SOLUTIONS FOR RECURRENT NEURAL NETWORKS WITHOUT ASSUMING GLOBAL LIPSCHITZ CONDITIONS

HONG ZHANG, YUANHENG WU

Abstract. In this paper we study recurrent neural networks with time-varying delays and continuously distributed delays. Without assuming global Lipschitz conditions on the activation functions, we establish the existence and local exponential stability of anti-periodic solutions.

## 1. Introduction

We consider the following model for recurrent neural networks(RNNs) with time-varying delays and continuously distributed delays

$$\begin{aligned} x_i'(t) &= -c_i(t)\alpha_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\tilde{g}_j(x_j(t-\tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(x_j(t-u))du + I_i(t), \quad i=1,2,\dots,n, \end{aligned} \tag{1.1}$$

in which $n$ corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the $i$th unit at the time $t$, $c_i(t) > 0$ represents the rate with which the $i$th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time $t$. $a_{ij}(t)$ and $b_{ij}(t)$ are the connection weights at the time $t$, $\tau_{ij}(t) \ge 0$ corresponds to the transmission delay of the $i$th unit along the axon of the $j$th unit at the time $t$, and $I_i(t)$ denote the external inputs at time $t$. $\tilde{g}_j$ and $g_j$ are activation functions of signal transmission.

As we know, RNNs is very general and includes Hopfield neural networks, cellular neural networks and BAM neural networks. The RNNs have been successfully applied to signal and image processing, pattern recognition and optimization. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic and almost periodic solutions for RNNs in the

2000 *Mathematics Subject Classification.* 34C25, 34D40.
*Key words and phrases.* Recurrent neural networks; anti-periodic; exponential stability; delay.

Submitted December 11, 2009. Published April 9, 2010.
Supported by grants 06JJ2063, 07JJ6001 from the Scientific Research Fund of Hunan Provincial Natural Science Foundation, and 08C616, 09B072 from the Scientific Research Fund of Hunan Provincial Education Department of China.

literature. We refer the reader to [5, 6, 3, 10] and the references cited therein. Recently, [11, 12, 14] obtained some sufficient conditions for the existence and stability of the anti-periodic solutions of RNNs. Moreover, in the above-mentioned literature [11, 12, 14], we observe that the assumption

(T0) For each $j \in \{1, 2, \dots, n\}$, the activation function $\tilde{g}_j, g_j : \mathbb{R} \to \mathbb{R}$ is global Lipschitz with Lipschitz constants $\tilde{L}_j$ and $L_j$; i.e.,
$$|\tilde{g}_j(u_j) - \tilde{g}_j(v_j)| \le \tilde{L}_j |u_j - v_j|, |g_j(u_j) - g_j(v_j)| \le L_j |u_j - v_j|, \tag{1.2}$$
for all $u_j, v_j \in \mathbb{R}$.

has been considered as fundamental for the considered existence and stability of anti-periodic solutions of RNNs. However, to the best of our knowledge, few authors have considered the problems of anti-periodic solutions of RNNs without the assumptions (T0). Since the existence of anti-periodic solutions play a key role in characterizing the behavior of nonlinear differential equations (See [1, 2, 7, 8, 13, 15]). It is worth while to continue to investigate the existence and stability of anti-periodic solutions of RNNs.

The main purpose of this paper is to give the conditions for the existence and exponential stability of the anti-periodic solutions for system (1.1). We derive some new sufficient conditions ensuring the existence and local exponential stability of the anti-periodic solution for system (1.1), which are new and complement to previously known results. In particular, we do not need the assumption (T0) . Moreover, an example is also provided to illustrate the effectiveness of our results.

Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous in $t$. $u(t)$ is said to be $T$-anti-periodic on $\mathbb{R}$ if,
$$u(t + T) = -u(t) \quad \text{for all } t \in \mathbb{R}.$$
Throughout this article, for $i, j = 1, 2, \dots, n$, it will be assumed that $c_i, I_i, a_{ij}, b_{ij} : \mathbb{R} \to \mathbb{R}$ and $\tau_{ij} : \mathbb{R} \to [0, +\infty)$ are continuous $2T-$periodic functions, and
$$\begin{gathered} c_i(t + T)\alpha_i(u) = -c_i(t)\alpha_i(-u), \quad a_{ij}(t + T)\tilde{g}_j(u) = -a_{ij}(t)\tilde{g}_j(-u), \\ \forall t, u \in \mathbb{R}, \end{gathered} \tag{1.3}$$
$$\begin{gathered} b_{ij}(t + T) = -b_{ij}(t), \quad g_j(u) = g_j(-u)) \quad \text{or} \\ b_{ij}(t + T) = b_{ij}(t),\ g_j(u) = -g_j(-u), \quad \forall t, u \in \mathbb{R}, \end{gathered} \tag{1.4}$$
$$\tau_{ij}(t + T) = \tau_{ij}(t), \quad I_i(t + T) = -I_i(t), \quad \forall u \in \mathbb{R}\,. \tag{1.5}$$
Then, we can choose constants $\overline{I}$ and $\tau$ such that
$$\overline{I} = \max_{1 \le i \le n} \sup_{t \in \mathbb{R}} |I_i(t)|, \quad \tau = \max_{1 \le i,j \le n} \{ \max_{t \in [0,T]} \tau_{ij}(t)\}. \tag{1.6}$$
We also assume that the following conditions:

(H0) For each $i \in \{1, 2, \dots, n\}$, $\alpha_i : \mathbb{R} \to \mathbb{R}$ are continuous function, and there exist constants $\underline{d}_i$ such that $\alpha_i(0) = 0$, $\underline{d}_i |u - v| \le \operatorname{sign}(u - v)(\alpha_i(u) - \alpha_i(v))$, for all $u, v \in \mathbb{R}$.

(H1) For each $j \in \{1, 2, \dots, n\}$, there exist $\tilde{f}_j, \tilde{h}_j, f_j, h_j \in C(R, R)$ and constants $L^{\tilde{f}}_j, L^{\tilde{h}}_j, L^f_j, L^h_j \in [0, +\infty)$ such that the following conditions are satisfied:
  (1) $\tilde{f}_j(0) = 0$, $\tilde{h}_j(0) = 0$, $\tilde{g}_j(u) = \tilde{f}_j(u)\tilde{h}_j(u)$, for all $u \in \mathbb{R}$;
  (2) $|\tilde{f}_j(u) - \tilde{f}_j(v)| \le L^{\tilde{f}}_j |u - v|, |\tilde{h}_j(u) - \tilde{h}_j(v)| \le L^{\tilde{h}}_j |u - v|$, for all $u, v \in \mathbb{R}$.
  (3) $f_j(0) = 0$, $h_j(0) = 0$, $g_j(u) = f_j(u)h_j(u)$, for all $u \in \mathbb{R}$;
  (4) $|f_j(u) - f_j(v)| \le L^f_j |u - v|, |h_j(u) - h_j(v)| \le L^h_j |u - v|$, for all $u, v \in \mathbb{R}$.

- (H2) For $i, j \in \{1, 2, \dots, n\}$, the delay kernels $K_{ij} : [0, \infty) \to \mathbb{R}$ are continuous, integrable.
- (H3) There exist constants $\eta > 0, \lambda > 0$ and $\xi_i > 0, i = 1, 2, \dots, n$, such that for all $t > 0$, there holds $0 < \xi_i \frac{\overline{I}}{\eta} \le 1$ and
$$\begin{aligned}&(\lambda - c_i(t))\underline{d}_i \xi_i + \sum_{j=1}^n 3|a_{ij}(t)| e^{\lambda\tau} L_j^{\tilde{f}} L_j^{\tilde{h}} \xi_j \\ &+ \sum_{j=1}^n 3|b_{ij}(t)| \int_0^\infty |K_{ij}(u)| e^{\lambda u} du L_j^f L_j^h \xi_j < -\eta.\end{aligned}$$

We will use the following notation: $x = \{x_j\} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$, and define $\|x\| = \max_{1 \le i \le n} |x_i|$.

The initial conditions associated with system (1.1) are of the form
$$x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0],\ i = 1, 2, \dots, n, \tag{1.7}$$
where $\varphi_i(\cdot)$ denotes real-valued bounded continuous function defined on $(-\infty, 0]$.

The remaining part of this paper is organized as follows. In Section 2, we shall derive new sufficient conditions for checking the existence of bounded solutions of (1.1). In Section 3, we present some new sufficient conditions for the existence and local exponential stability of the anti-periodic solution of (1.1). In Section 4, we shall give an example and some remarks to illustrate our results obtained in the previous sections.

## 2. Preliminary Results

The following lemmas will be used to prove our main results in Section 3.

**Lemma 2.1.** *Let* (H0)–(H3) *hold. Suppose that* $\widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), \dots, \widetilde{x}_n(t))^T$ *is a solution of* (1.1) *with initial conditions*
$$\widetilde{x}_i(s) = \widetilde{\varphi}_i(s), \quad |\widetilde{\varphi}_i(s)| < \xi_i \frac{\overline{I}}{\eta} \le 1, \quad s \in (-\infty, 0],\ i = 1, 2, \dots, n. \tag{2.1}$$
*Then*
$$|\widetilde{x}_i(t)| < \xi_i \frac{\overline{I}}{\eta}, \quad \textit{for all } t \ge 0,\ i = 1, 2, \dots, n. \tag{2.2}$$

*Proof.* Assume, by way of contradiction, that (2.2) does not hold. Then, there must exist $i \in \{1, 2, \dots, n\}$ and $\rho > 0$ such that
$$|\widetilde{x}_i(\rho)| = \xi_i \frac{\overline{I}}{\eta}, \quad |\widetilde{x}_j(t)| < \xi_j \frac{\overline{I}}{\eta} \quad \text{for all } t \in (-\infty, \rho),\ j = 1, 2, \dots, n. \tag{2.3}$$
Calculating the upper left derivative of $|\widetilde{x}_i(t)|$, together with (H0), (H1), (H2) and (H3), (2.3) implies
$$\begin{aligned}&0 \le D^+(|\widetilde{x}_i(\rho)|) \\ &= -c_i(\rho)\alpha_i(\widetilde{x}_i(\rho)) \operatorname{sgn}(\widetilde{x}_i(\rho)) + \Big[\sum_{j=1}^n a_{ij}(\rho) \tilde{g}_j(\widetilde{x}_j(\rho - \tau_{ij}(\rho)))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^n b_{ij}(\rho)\int_0^\infty K_{ij}(u)g_j(\widetilde{x}_j(\rho-u))du+I_i(\rho)\Big]\operatorname{sgn}(\widetilde{x}_i(\rho))\\
&\le -c_i(\rho)|\alpha_i(\widetilde{x}_i(\rho))|+\Big|\sum_{j=1}^n a_{ij}(\rho)\tilde{g}_j(\widetilde{x}_j(\rho-\tau_{ij}(\rho)))\\
&+\sum_{j=1}^n b_{ij}(\rho)\int_0^\infty K_{ij}(u)g_j(\widetilde{x}_j(\rho-u))du+I_i(\rho)\Big|\\
&\le -c_i(\rho)\underline{d}_i\xi_i\frac{\overline{I}}{\eta}+\Big|\sum_{j=1}^n a_{ij}(\rho)\tilde{f}_j(\widetilde{x}_j(\rho-\tau_{ij}(\rho)))\tilde{h}_j(\widetilde{x}_j(\rho-\tau_{ij}(\rho)))\\
&+\sum_{j=1}^n b_{ij}(\rho)\int_0^\infty K_{ij}(u)f_j(\widetilde{x}_j(\rho-u))h_j(\widetilde{x}_j(\rho-u))du+I_i(\rho)\Big|\\
&\le -c_i(\rho)\underline{d}_i\xi_i\frac{\overline{I}}{\eta}+\sum_{j=1}^n |a_{ij}(\rho)|L_j^{\tilde{f}}\xi_j\frac{\overline{I}}{\eta}L_j^{\tilde{h}}\xi_j\frac{\overline{I}}{\eta}\\
&+\sum_{j=1}^n |b_{ij}(\rho)|\int_0^\infty |K_{ij}(u)|duL_j^f\xi_j\frac{\overline{I}}{\eta}L_j^h\xi_j\frac{\overline{I}}{\eta}+|I_i(\rho)|\\
&\le -c_i(\rho)\underline{d}_i\xi_i\frac{\overline{I}}{\eta}+\sum_{j=1}^n |a_{ij}(\rho)|L_j^{\tilde{f}}L_j^{\tilde{h}}\xi_j\frac{\overline{I}}{\eta}\\
&+\sum_{j=1}^n |b_{ij}(\rho)|\int_0^\infty |K_{ij}(u)|duL_j^fL_j^h\xi_j\frac{\overline{I}}{\eta}+|I_i(\rho)|\\
&\le [-c_i(\rho)\underline{d}_i\xi_i+\sum_{j=1}^n |a_{ij}(\rho)|L_j^{\tilde{f}}L_j^{\tilde{h}}\xi_j+\sum_{j=1}^n |b_{ij}(\rho)|\int_0^\infty |K_{ij}(s)|dsL_j^fL_j^h\xi_j]\frac{\overline{I}}{\eta}\\
&+|I_i(\rho)|\\
&\le \Big[-c_i(\rho)\underline{d}_i\xi_i+\sum_{j=1}^n |a_{ij}(\rho)|e^{\lambda\tau}|L_j^{\tilde{f}}L_j^{\tilde{h}}\xi_j\\
&+\sum_{j=1}^n |b_{ij}(\rho)|\int_0^\infty |K_{ij}(s)|e^{\lambda s}dsL_j^fL_j^h\xi_j\Big]\frac{\overline{I}}{\eta}+|I_i(\rho)|\\
&< -\eta\times\frac{\overline{I}}{\eta}+|I_i(\rho)|\le 0,
\end{aligned}
$$

which is a contradiction and implies that (2.2) holds. The proof is complete. □

**Remark 2.2.** In view of the boundedness of this solution, from [9, Theorems 2.3-2.4], it follows that $\widetilde{x}(t)$ can be defined on $(-\infty,\infty)$.

**Lemma 2.3.** *Suppose that* (H0), (H1), (H2), (H3) *are satisfied. Let* $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ *be the solution of* (1.1) *with initial value* $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T$, *where*

$$|\varphi_i^*(s)| < \xi_i\frac{\overline{I}}{\eta} \le 1, \quad s\in(-\infty,0],\ i=1,2,\dots,n. \tag{2.4}$$

*Then there exist constants $\lambda > 0$ and $M_\varphi > 1$ such that for every solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of (1.1) with initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in G_1(x^*)$,*

$$|x_i(t) - x_i^*(t)| \le M_\varphi \|\varphi - \varphi^*\|_1 e^{-\lambda t}, \quad \forall t > 0,\ i = 1, 2, \dots, n,$$

*where $\|\varphi - \varphi^*\|_1 = \sup_{-\infty \le s \le 0} \max_{1 \le i \le n} |\varphi_i(s) - \varphi_i^*(s)|$, and*

$$\begin{aligned} G_1(x^*) = \big\{\varphi \in C((-\infty, 0];\ R^n) : \sup_{-\infty \le s \le 0} |\varphi_i(t) - \varphi_i^*(t)| < \frac{\xi_i}{\max_{1 \le j \le n}\{\xi_j\}}, \\ i = 1, 2, \dots, n\big\}. \end{aligned}$$

*Proof.* In view (2.4) and Lemma 2.1,

$$|x_i^*(t)| < \xi_i \frac{\overline{I}}{\eta} \le 1, \quad \text{for all } t \in \mathbb{R},\ i = 1, 2, \dots, n. \tag{2.5}$$

Let $y(t) = \{y_j(t)\} = \{x_j(t) - x_j^*(t)\} = x(t) - x^*(t)$. Then

$$\begin{aligned} y_i'(t) &= -c_i(t)[\alpha_i(x_i(t)) - \alpha_i(x_i^*(t))] \\ &\quad + \sum_{j=1}^n a_{ij}(t)(\tilde{g}_j(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t))) - \tilde{g}_j(x_j^*(t - \tau_{ij}(t)))) \\ &\quad + \sum_{j=1}^n b_{ij}(t) \int_0^\infty K_{ij}(u)(g_j(y_j(t-u) + x_j^*(t-u)) - g_j(x_j^*(t-u)))du, \end{aligned} \tag{2.6}$$

where $i = 1, 2, \dots, n$. We consider the Lyapunov functional

$$V_i(t) = |y_i(t)| e^{\lambda t}, \quad i = 1, 2, \dots, n. \tag{2.7}$$

Calculating the left right derivative of $V_i(t)$ along the solution $y(t) = \{y_j(t)\}$ of system (2.6) with the initial value $\bar{\varphi} = \varphi - \varphi^*$, from (2.6), we have

$$\begin{aligned} &D^+(V_i(t)) \\ &\le -c_i(t)\underline{d}_i |y_i(t)| e^{\lambda t} + \sum_{j=1}^n |a_{ij}(t)(\tilde{g}_j(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t))) \\ &\quad - \tilde{g}_j(x_j^*(t - \tau_{ij}(t))))| e^{\lambda t} + \sum_{j=1}^n |b_{ij}(t) \int_0^\infty K_{ij}(u)(g_j(y_j(t-u) + x_j^*(t-u)) \\ &\quad - g_j(x_j^*(t-u)))du| e^{\lambda t} + \lambda |y_i(t)| e^{\lambda t} \\ &\le (\lambda - c_i(t)\underline{d}_i)|y_i(t)| e^{\lambda t} + \sum_{j=1}^n |a_{ij}(t)|(|\tilde{f}_j(y_j(t - \tau_{ij}(t)) \\ &\quad + x_j^*(t - \tau_{ij}(t)))\tilde{h}_j(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t))) \\ &\quad - \tilde{f}_j(x_j^*(t - \tau_{ij}(t)))\tilde{h}_j(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t)))| \\ &\quad + |\tilde{f}_j(x_j^*(t - \tau_{ij}(t)))\tilde{h}_j(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t))) \\ &\quad - \tilde{f}_j(x_j^*(t - \tau_{ij}(t))))\tilde{h}_j(x_j^*(t - \tau_{ij}(t)))|) e^{\lambda t} \\ &\quad + \sum_{j=1}^n |b_{ij}(t)| \int_0^\infty |K_{ij}(u)|(|f_j(y_j(t-u) + x_j^*(t-u))h_j(y_j(t-u) + x_j^*(t-u)) \end{aligned}$$

$$
\begin{aligned}
&- f_j(y_j(t-u)+x_j^*(t-u))h_j(x_j^*(t-u))| + |f_j(y_j(t-u) \\
&+ x_j^*(t-u))h_j(x_j^*(t-u)) - f_j(x_j^*(t-u))h_j(x_j^*(t-u))|)due^{\lambda t} \\
\le{}& (\lambda - c_i(t)\underline{d}_i)|y_i(t)|e^{\lambda t} + \sum_{j=1}^n |a_{ij}(t)|L_j^{\tilde f} L_j^{\tilde h}|y_j(t-\tau_{ij}(t))|(|y_j(t-\tau_{ij}(t)) \\
&+ x_j^*(t-\tau_{ij}(t))| + |x_j^*(t-\tau_{ij}(t))|)e^{\lambda t} \\
&+ \sum_{j=1}^n |b_{ij}(t)| \int_0^\infty K_{ij}(u) L_j^f L_j^h |y_j(t-u)|(|y_j(t-u)+x_j^*(t-u)| \\
&+ |x_j^*(t-u)|)due^{\lambda t}, \qquad (2.8)
\end{aligned}
$$

where $i = 1, 2, \dots, n$. In view of the definition of $\varphi \in G_1(Z^*)$,

$$
V_i(t) = |y_i(t)|e^{\lambda t} < \frac{\xi_i}{\max_{1\le j\le n}\{\xi_j\}}, \quad \text{for all } t \in (-\infty, 0],\ j = 1, 2, \dots, n.
$$

We claim that

$$
V_i(t) = |y_i(t)|e^{\lambda t} < \frac{\xi_i}{\max_{1\le j\le n}\{\xi_j\}}, \quad \text{for all } t > 0, \quad i = 1, 2, \dots, n. \qquad (2.9)
$$

Contrarily, there must exist $i \in \{1, 2, \dots, n\}$ and $t_i > 0$ such that

$$
V_i(t_i) = \frac{\xi_i}{\max_{1\le j\le n}\{\xi_j\}}, \quad V_j(t) < \frac{\xi_j}{\max_{1\le j\le n}\{\xi_j\}},
$$

for all $t \in (-\infty, t_i)$, $j = 1, 2, \dots, n$, which, together with $\varphi \in G_1(Z^*)$, implies

$$
V_i(t_i) - \frac{\xi_i}{\max_{1\le j\le n}\{\xi_j\}} = 0, \quad V_j(t) - \frac{\xi_j}{\max_{1\le j\le n}\{\xi_j\}} < 0, \qquad (2.10)
$$

for all $t \in (-\infty, t_i)$, $j = 1, 2, \dots, n$, and

$$
|y_j(t)| \le 1, \quad \forall t \in (-\infty,\ t_i),\ j = 1, 2, \dots, n. \qquad (2.11)
$$

Together with (2.5), (2.8), (2.10) and (2.11), we obtain

$$
\begin{aligned}
0 \le{}& D^+(V_i(t_i) - m\xi_i) \\
={}& D^+(V_i(t_i)) \\
\le{}& (\lambda - c_i(t_i)\underline{d}_i)|y_i(t_i)|e^{\lambda t_i} + \sum_{j=1}^n |a_{ij}(t_i)|L_j^{\tilde f} L_j^{\tilde h}|y_j(t_i-\tau_{ij}(t_i))|(|y_j(t_i-\tau_{ij}(t_i)) \\
&+ x_j^*(t_i-\tau_{ij}(t_i))| + |x_j^*(t_i-\tau_{ij}(t_i))|)e^{\lambda t_i} \\
&+ \sum_{j=1}^n |b_{ij}(t_i)| \int_0^\infty |K_{ij}(u)| L_j^f L_j^h |y_j(t_i-u)| \\
&\times \Big(|y_j(t_i-u)+x_j^*(t_i-u)| + |x_j^*(t_i-u)|\Big)due^{\lambda t_i} \\
\le{}& (\lambda - c_i(t_i)\underline{d}_i)|y_i(t_i)|e^{\lambda t_i} + \sum_{j=1}^n 3|a_{ij}(t_i)|L_j^{\tilde f} L_j^{\tilde h}|y_j(t_i-\tau_{ij}(t_i))|e^{\lambda t_i} \\
&+ \sum_{j=1}^n 3|b_{ij}(t_i)| \int_0^\infty |K_{ij}(u)| L_j^f L_j^h |y_j(t_i-u)|due^{\lambda t_i} \\
={}& (\lambda - c_i(t_i)\underline{d}_i)|y_i(t_i)|e^{\lambda t_i}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} 3|a_{ij}(t_i)|L_j^{\tilde f}L_j^{\tilde h}||y_j(t_i-\tau_{ij}(t_i))|e^{\lambda(t_i-\tau_{ij}(t_i))}e^{\lambda\tau_{ij}(t_i)}\\
&+\sum_{j=1}^{n} 3|b_{ij}(t_i)|\int_0^\infty |K_{ij}(u)|L_j^{f}L_j^{h}|y_j(t_i-u)|e^{\lambda(t_i-u)}e^{\lambda u}du\\
\le{}& \Big[(\lambda-c_i(t_i)\underline{d}_i)\xi_i+\sum_{j=1}^{n} 3|a_{ij}(t_i)|e^{\lambda\tau}L_j^{\tilde f}L_j^{\tilde h}\xi_j\\
&+\sum_{j=1}^{n} 3|b_{ij}(t_i)|\int_0^\infty |K_{ij}(u)|e^{\lambda u}duL_j^{f}L_j^{h}\xi_j\Big]\frac{1}{\max_{1\le j\le n}\{\xi_j\}}.
\end{aligned} \tag{2.12}
$$

Thus,

$$
\begin{aligned}
0\le (\lambda-c_i(t_i)\underline{d}_i)\xi_i&+\sum_{j=1}^{n} 3|a_{ij}(t_i)|e^{\lambda\tau}L_j^{\tilde f}L_j^{\tilde h}\xi_j\\
&+\sum_{j=1}^{n} 3|b_{ij}(t_i)|\int_0^\infty |K_{ij}(u)|e^{\lambda u}duL_j^{f}L_j^{h}\xi_j,
\end{aligned} \tag{2.13}
$$

which contradicts (H3). Hence, (2.9) holds. Letting

$$
\|\varphi-\varphi^*\|_1=\sup_{-\infty\le s\le 0}\max_{1\le j\le n}|\varphi_j(s)-\varphi_j^*(s)|>0,\quad i=1,2,\dots,n,
$$

and $M_\varphi>1$ such that

$$
\frac{\xi_i}{\max_{1\le j\le n}\{\xi_j\}}\le M_\varphi\|\varphi-\varphi^*\|_1,\quad i=1,2,\dots,n. \tag{2.14}
$$

In view of (2.9) and (2.13),

$$
|x_i(t)-x_i^*(t)|=|y_i(t)|\le \max_{1\le i\le n}\{m\xi_i\}e^{-\lambda t}\le M_\varphi\|\varphi-\varphi^*\|_1e^{-\lambda t},
$$

where $i=1,2,\dots,n$, $t>0$. This completes the proof. □

**Remark 2.4.** If $x^*(t)=(x_1^*(t),x_2^*(t),\dots,x_n^*(t))^T$ be the $T$-anti-periodic solution of system (1.1), it follows from Lemma 2.3 that $x^*(t)$ is locally exponentially stable.

## 3. Main Results

The following is our main result.

**Theorem 3.1.** *Suppose that* (H0)-(H3) *are satisfied. Then* (1.1) *has at least one* $T$*-anti-periodic solution* $x^*(t)$*. Moreover,* $x^*(t)$ *is locally exponentially stable.*

*Proof.* Let $v(t)=(v_1(t),v_2(t),\dots,v_n(t))^T$ be a solution of (1.1) with initial conditions

$$
v_i(s)=\varphi_i^v(s),\quad |\varphi_i^v(s)|<\xi_i\frac{\overline{I}}{\eta},\quad s\in(-\infty,0],\ i=1,2,\dots,n. \tag{3.1}
$$

By Lemma 2.1, the solution $v(t)$ is bounded and

$$
|v_i(t)|<\xi_i\frac{\overline{I}}{\eta},\quad \text{for all } t\in\mathbb{R},\ i=1,2,\dots,n. \tag{3.2}
$$

From (1.1)–(1.5),

$$
\begin{aligned}
&((-1)^{k+1}v_i(t+(k+1)T))' \\
&= (-1)^{k+1}v_i'(t+(k+1)T) \\
&= (-1)^{k+1}\{-c_i(t+(k+1)T)\alpha_i(v_i(t+(k+1)T)) \\
&\quad + \sum_{j=1}^{n} a_{ij}(t+(k+1)T)\tilde{g}_j(v_j(t+(k+1)T-\tau_{ij}(t+(k+1)T))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t+(k+1)T)\int_0^\infty K_{ij}(u)g_j(v_j(t+(k+1)T-u))du \\
&\quad + I_i(t+(k+1)T)\} \\
&= (-1)^{k+1}\{-c_i(t+(k+1)T)\alpha_i(v_i(t+(k+1)T)) \\
&\quad + \sum_{j=1}^{n} a_{ij}(t+(k+1)T)\tilde{g}_j(v_j(t+(k+1)T-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t+(k+1)T)\int_0^\infty K_{ij}(u)g_j(v_j(t+(k+1)T-u))du \\
&\quad + I_i(t+(k+1)T)\} \\
&= -c_i(t)\alpha_i((-1)^{k+1}v_i(t+(n+1)T)) \\
&\quad + \sum_{j=1}^{n} a_{ij}(t)\tilde{g}_j((-1)^{k+1}v_j(t+(k+1)T-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t)\int_0^\infty K_{ij}(u)g_j((-1)^{k+1}v_j(t+(k+1)T-u))du + I_i(t),
\end{aligned}
\tag{3.3}
$$

$i=1,2,\dots,n$. Thus, for any natural number $k$, $(-1)^{k+1}v(t+(k+1)T)$ are the solutions of (1.1) on $\mathbb{R}$. Then, by Lemma 2.3, there exists a constant $M>0$ such that

$$
\begin{aligned}
&|(-1)^{k+1}v_i(t+(k+1)T)-(-1)^k v_i(t+kT)| \\
&\le Me^{-\lambda(t+kT)}\sup_{-\infty\le s\le 0}\max_{1\le i\le n}|v_i(s+T)+v_i(s)| \\
&\le e^{-\lambda(t+kT)}M2\max_{1\le i\le n}\{\xi_i\frac{\overline{I}}{\eta}\}, \forall t+kT>0, \quad i=1,2,\dots,n.
\end{aligned}
\tag{3.4}
$$

Thus, for any natural number $m$, we obtain

$$
(-1)^{m+1}v_i(t+(m+1)T) = v_i(t)+\sum_{k=0}^{m}[(-1)^{k+1}v_i(t+(k+1)T)-(-1)^k v_i(t+kT)]. \tag{3.5}
$$

Then,

$$
|(-1)^{m+1}v_i(t+(m+1)T)| \le |v_i(t)|+\sum_{k=0}^{m}|(-1)^{k+1}v_i(t+(k+1)T)-(-1)^k v_i(t+kT)|, \tag{3.6}
$$

where $i=1,2,\dots,n$.

In view of (3.4), we can choose a sufficiently large constant $N > 0$ and a positive constant $\alpha$ such that

$$|(-1)^{k+1}v_i(t+(k+1)T) - (-1)^k v_i(t+kT)| \le \alpha(e^{-\lambda T})^k, \quad \forall k > N,\ i = 1,2,\dots,n, \tag{3.7}$$

on any compact subset of $\mathbb{R}$. It follows from (3.5)–(3.7) that $\{(-1)^m v(t+mT)\}$ converges uniformly to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ on any compact subset of $\mathbb{R}$.

Now we show that $x^*(t)$ is $T$-anti-periodic solution of system (1.1). First, $x^*(t)$ is $T$-anti-periodic, since

$$\begin{aligned} x^*(t+T) &= \lim_{m\to\infty} (-1)^m v(t+T+mT) \\ &= -\lim_{(m+1)\to\infty} (-1)^{m+1} v(t+(m+1)T) = -x^*(t). \end{aligned}$$

Next, we prove that $x^*(t)$ is a solution of (1.1). In fact, together with the continuity of the right side of (1.1), (3.3) implies that $\{((-1)^{m+1}v(t+(m+1)T))'\}$ uniformly converges to a continuous function on any compact set of $\mathbb{R}$. Thus, letting $m \to \infty$, we obtain

$$\begin{aligned} \frac{d}{dt}\{x_i^*(t)\} &= -c_i(t)\alpha_i(x_i^*(t)) + \sum_{j=1}^n a_{ij}(t)\tilde{g}_j(x_j^*(t-\tau_{ij}(t))) \\ &\quad + \sum_{j=1}^n b_{ij}(t)\int_0^\infty K_{ij}(u) g_j(x_j^*(t-u))du + I_i(t). \end{aligned} \tag{3.8}$$

Therefore, $x^*(t)$ is a solution of (1.1). Finally, by Lemma 2.3, we can prove that $x^*(t)$ is globally exponentially stable. This completes the proof. □

## 4. An Example

In this section, we give an example to demonstrate the results obtained in previous sections. Consider the recurrent neutral network

$$\begin{aligned} x_1'(t) &= -(x_1(t) + x_1^3(t)) + \frac{1}{24}(\cos t)\tilde{g}_1(x_1(t-22)) + \frac{1}{108}(\cos t)\tilde{g}_2(x_2(t-11)) \\ &\quad + \frac{1}{24}(\sin t) g_1\Big(\int_0^\infty |\sin u| e^{-u} x_1(t-u)du\Big) \\ &\quad + \frac{1}{108}(\sin t) g_2\Big(\int_0^\infty |\cos u| e^{-u} x_2(t-u)du\Big) + I_1(t), \\ x_2'(t) &= -(x_2(t) + x_2^3(t)) + \frac{1}{12}(\cos t)\tilde{g}_1(x_1(t-6)) + \frac{1}{24}(\cos t)\tilde{g}(x_2(t-8)) \\ &\quad + \frac{1}{12}(\sin t) g_1\Big(\int_0^\infty |\cos u| e^{-u} x_1(t-u)du\Big) \\ &\quad + \frac{1}{24}(\sin t) g_2\Big(\int_0^\infty |\sin u| e^{-u} x_2(t-u)du\Big) + I_2(t), \end{aligned} \tag{4.1}$$

where $\tilde{g}_1(x) = \tilde{g}_2(x) = |\sin x| x, g_1(x) = g_2(x) = x^2 = x \times x,\ I_1(t) = \frac{1}{24}\cos t$, $I_2(t) = \frac{1}{24}\sin t$.

Noting that $c_1 = c_2 = L_j^{\tilde{f}} = L_j^{\tilde{h}} = L_j^f = L_j^h = 1$, $a_{11}^+ = b_{11}^+ = \frac{1}{24}$, $a_{12}^+ = b_{12}^+ = \frac{1}{108}$, $a_{21}^+ = b_{21}^+ = \frac{1}{12}$, $a_{22}^+ = b_{22}^+ = \frac{1}{24}$, $\underline{d}_i = 1$, $\int_0^\infty K_{ij}(s)ds \le 1$, where $i,j = 1,2, \beta^+ =$

$\sup_{t\in\mathbb{R}} \beta(t)$. Then

$$d_{ij} = c_i^{-1}\underline{d}_i(3a_{ij}^+ L_j^{\tilde{f}} L_j^{\tilde{h}} + 3b_{ij}^+ L_j^f L_j^h) \quad i,j = 1,2, \quad D = (d_{ij})_{2\times 2} = \begin{pmatrix} 1/4 & 1/18 \\ 1/2 & 1/4 \end{pmatrix}.$$

From the theory of $M$-matrices in [4], we can choose constants $\eta = \frac{1}{4}$ and $\xi_i = 1$ such that for all $t > 0$,

$$\begin{aligned} &- c_i\underline{d}_i\xi_i + \sum_{j=1}^{2} 3a_{ij}^+ L_j^{\tilde{f}} L_j^{\tilde{h}} \xi_j + \sum_{j=1}^{2} 3b_{ij}^+ \int_0^\infty K_{ij}(s) ds L_j^f L_j^h \xi_j \\ &< -c_i\underline{d}_i\xi_i + \sum_{j=1}^{2} a_{ij}^+ L_j^{\tilde{f}} L_j^{\tilde{h}} \xi_j + \sum_{j=1}^{2} b_{ij}^+ L_j^f L_j^h \xi_j < -\eta, \end{aligned}$$

where $i = 1,2$, which implies that (4.1) satisfy all the conditions in Theorem 3.1. Hence, (4.1) has at least one $\pi$-anti-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is locally exponentially stable, the domain of attraction of $Z^*(t)$ is the set $G_1(x^*)$.

**Remark 4.1.** System (4.1) is a very simple form of recurrent neural networks with mixed delays. Since $h_i(u) \neq u$, $i = 1,2$, $\tilde{g}_1(x) = \tilde{g}_2(x) = |\sin x|x$, $g_1(x) = g_2(x) = x^2 = x \times x$. One can observe that condition (T0) is not satisfied. Therefore, the results in in this article and their references can not be applied to (4.1).

## References


[1] A. R. Aftabizadeh, S. Aizicovici, N. H. Pavel; *On a class of second-order anti-periodic boundary value problems*, J. Math. Anal. Appl. 171 (1992) 301–320.

[2] S. Aizicovici, M. McKibben, S. Reich; *Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities*, Nonlinear Anal. 43 (2001) 233–251.

[3] B. Liu and L. Huang; *Positive almost periodic solutions for recurrent neural networks*, Nonlinear Analysis: Real World Applications,9 (2008) 830-841.

[4] A. Berman and R. J. Plemmons; *Nonnegative Matrices in the Mathematical Science*, Academic Press, New York, 1979.

[5] J. Cao and J. Wang; *Global exponential stability and periodicity of recurrent neural networks with time delays*, IEEE Trans. Circuits Syst.-I, 52(5) (2005) 920-931.

[6] J. Cao and J. Wang; *Global asymptotic and robust stability of recurrent neural networks with time delays*, IEEE Trans. Circuits Syst.-I, 52(2) (2005) 417-426.

[7] Y. Chen, J. J. Nieto and D. O'Regan; *Anti-periodic solutions for fully nonlinear first-order differential equations*, Mathematical and Computer Modelling, 46 (2007) 1183–1190

[8] F. J. Delvos, L. Knoche; *Lacunary interpolation by antiperiodic trigonometric polynomials*, BIT 39 (1999) 439–450.

[9] Jack K. Hale and Junji Kato; *Phase Space for Retarded Equations with Infinite Delay.* Funkcialaj Ekvacioj, 21 (1978), 11–41.

[10] H. Huang, J. Cao and J. Wang; *Global exponential stability and periodic solutions of recurrent cellular neural networks with delays*, Physics Letters A 298 (5-6) (2002) 393-404.

[11] Bingwen Liu; *An anti-periodic LaSalle oscillation theorem for a class of functional differential equations*, Journal of Computational and Applied Mathematics, 223(2) (2009) 1081-1086.

[12] Chunxia Ou; *Anti-periodic solutions for high-order Hopfield neural networks Computers & Mathematics with Applications*, 56(7) (2008) 1838-1844.

[13] H. Okochi; *On the existence of periodic solutions to nonlinear abstract parabolic equations*, J. Math. Soc. Japan 40 (3) (1988) 541–553.

[14] Jianying Shao; *An anti-periodic solution for a class of recurrent neural networks, Journal of Computational and Applied Mathematics*, 228 ( 1) (2009) 231-237.

[15] R. Wu; *An anti-periodic LaSalle oscillation theorem*, Applied Mathematics Letters (2007), 21 (9) (2008) 928-933 .

Hong Zhang
Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, China
*E-mail address*: hongzhang320@yahoo.com.cn

Yuanheng Wu
College of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, China
*E-mail address*: wyhcd2006@yahoo.com.cn