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REMARKS ON AN EIGENVALUE PROBLEM ASSOCIATED WITH THE p-LAPLACE OPERATOR

ALIN GĂLĂŢAN, CEZAR LUPU, FELICIAN PREDA

Dedicated to Professor Gheorghe Moroşanu on his 60-th birthday

Abstract. In this article we study eigenvalue problems involving p-Laplace operator and having a continuous family of eigenvalues and at least one isolated eigenvalue.

1. Introduction and statement of main results

Eigenvalue problems have been studied in various settings lately. The leading example of linear eigenvalue problem is to find all non-trivial solutions of the equation $\Delta u + \lambda u = 0$ with boundary values zero in a given bounded domain in $\mathbb{R}^N$. This is called a Dirichlet boundary-value problem.

In this article we study the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function and λ is a real number. The operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is called p-harmonic, and appears in many contexts in physics reaction-diffusion problems, non-linear elasticity, etc. The p-harmonic operator is defined as

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-4}\Big(|\nabla u|^{p-2}\Delta u + (p-2)\sum \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\Big),$$

where $1 < p < N$.

Definition 1.1. We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is an eigenfunction of (1.1), if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \lambda \int_\Omega f(x,u)v\,dx,$$

for all $v \in W_1^p(\Omega)$. The corresponding real number λ is called the eigenvalue of (1.1).

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The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\| = \Big(\int_\Omega (|\varphi|^p + |\operatorname{div}\varphi|^p)dx\Big)^{1/p}.$$

As usual, the space $C_0^\infty(\Omega)$ is the class of smooth functions with compact support in Ω. By standard elliptic regularity theory an eigenfunction is continuous. The smallest eigenvalue of (1.1) can be characterized by the minimum of Rayleigh quotient,

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega u^p dx}.$$

The study of eigenvalues involving Laplace and p-Laplace operators starts with the following basic problem, which represents a particular case of (1.1),

$$\begin{gathered} -\Delta u = \lambda u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.2}$$

As mentioned in [15] problem (1.2) goes back to the Riesz-Fredholm theory for compact operators on Hilbert spaces, where it is proved that it has an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_n \dots$. Also, in [15] other eigenvalue problems are mentioned; for example we have problems involving $p(x)$-Laplace operator in the case when $f(x,u) = |u|^{p(x)-2}u$ where we obtain the nonlinear model equation

$$\begin{gathered} -\Delta_{p(x)} u = \lambda|u|^{p(x)-2}u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.3}$$

where $p(\cdot) : \overline{\Omega} \to (1, 2^*)$ is a given continuous function and 2^* denotes the critical Sobolev exponent,

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3 \\ +\infty & \text{if } N \in \{1,2\}. \end{cases}$$

By specific methods of nonlinear analysis (Ekeland variational principle, mountain pass theorem, etc) many properties are established about problem (1.3). For further discussions of this problem as well as generalizations and extensions we refer to [5, 14, 15]. In the particular case, when $f(x,u) = |u|^{p-2}u$ we obtain the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda|u|^{p-2}u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.4}$$

which was introduced by Lieb [11] in 1983 and then studied by Lindqvist in [12], [9] and a modified eigenvalue problem (1.4) involving the weight function $V(\cdot)$ which changes sign and has nontrivial positive part by Cuesta in [3]. Inspired by the work of Mihăilescu and Rădulescu from [15], we study (1.1) in the case when

$$f(x,t) = \begin{cases} h(x,t) & \text{if } t \ge 0 \\ t & \text{if } t < 0, \end{cases} \tag{1.5}$$

where $h : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following properties

- (P1) there exists a positive constant $k \in (0,1)$ such that $|h(x,t)| \le k \cdot t^{p-1}$, for all $t \ge 0$ and a.e. $x \in \Omega$;
- (P2) there exists $t_0 > 0$ such that $H(x,t_0) = \int_0^{t_0} h(x,s)ds > 0$ for a.e. $x \in \Omega$;

(P3) $\lim_{t\to\infty} \frac{h(x,t)}{t^{p-1}} = 0$, uniformly in x.

These assumptions are related to those used by Diaz and Saa [4], to deduce an existence and uniqueness result for a quasilinear problem with Dirichlet boundary condition (see Brezis and Oswald [2] for the semilinear case).

Examples of functions satisfying properties (P1), (P2) and (P3) are mentioned in [15]. Regarding (1.1), we also point out the recent work of Pucci and Rădulescu [17] in which they study the problem for polyharmonic operator provided that f satisfies the same conditions as those in [15].

The main result of this article establishes a property of the (1.1) provided that f is defined as above and satisfies (P1), (P2) and (P3). It and shows that (1.1) has both isolated eigenvalues and a continuous spectrum in a neighborhood of the origin.

Theorem 1.2. *Assume that f is defined by the relation* (1.5) *and satisfies properties* (P1), (P2), (P3)*. Then the eigenvalue λ_1 defined by the Rayleigh quotient is isolated, and the corresponding set of eigenvectors form a cone. Moreover, there is no eigenvalue $\lambda \in (0, \lambda_1)$, but there exists $\mu_1 > \lambda_1$ such that any $\lambda > \mu_1$ is an eigenvalue of* (1.1)*.*

2. Proof of the main result

We shall use the method of Stamppachia and for any $u \in W_0^{1,p}(\Omega)$ we denote $u_\pm = \max\{\pm u(x), 0\}$, for all $x \in \Omega$. Then $u_+, u_- \in W_0^{1,p}(\Omega)$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if } u \le 0 \\ \nabla u, & \text{if } u > 0\,, \end{cases} \quad \nabla u_- = \begin{cases} 0, & \text{if } u \ge 0 \\ \nabla u, & \text{if } u < 0\,, \end{cases}$$

It follows that, with f given by (1.5), (1.1) becomes

$$\begin{gathered} -\Delta_p u = \lambda[h(x, u_+) - u_-], \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.1}$$

and $\lambda > 0$ is an eigenvalue of (2.1) if there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \nabla v\,dx - \int_\Omega |\nabla u_-|^{p-2}\nabla u_- \nabla v\,dx - \lambda\int_\Omega [h(x,u_+) - u_-]v\,dx = 0\,, \tag{2.2}$$

for all $v \in W_0^{1,p}(\Omega)$.

To prove the main result, Theorem 1.2, we shall begin with the following lemmata.

Lemma 2.1. *There are no eigenvalues of* (2.1) *in the interval* $(0, \lambda_1)$*.*

Proof. Assume that $\lambda > 0$ is an eigenvalue of (2.1) and u is its corresponding eigenfunction. We put $v = u_+$ and $v = u_-$ in (2.2) and we infer that

$$\int_\Omega |\nabla u_+|^p\,dx = \lambda\int_\Omega h(x, u_+)u_+ dx \tag{2.3}$$

and

$$\int_\Omega |\nabla u_-|^p dx = \lambda\int_\Omega u_-^p\,dx. \tag{2.4}$$

By property (P1) and relations (2.3) and (2.4), we obtain

$$\lambda_1\int_\Omega u_+^p dx \le \int_\Omega |\nabla u_+|^p dx = \lambda\int_\Omega h(x,u_+)u_+ dx \le \lambda\int_\Omega u_+^p dx$$

and

$$\lambda_1 \int_\Omega u_-^p\, dx \le \int_\Omega |\nabla u_-|^p dx = \lambda \int_\Omega u_-^p\, dx.$$

If λ is an eigenvalue of problem (2.1), then the corresponding eigenvector u is not null and thus, at least one of the eigenfunctions u_+ and u_- is not the zero function. This means that λ is an eigenvalue of (2.1), and by the definition of the Rayleigh quotient, $\lambda \ge \lambda_1$. □

Lemma 2.2. *λ_1 is an eigenvalue of* (2.1)*, and is isolated. Moreover, the set of eigenvectors corresponding to λ_1 form a cone.*

Proof. Indeed, as we already pointed out, λ_1 is the smallest eigenvalue of (1.2), it is simple, that is, all the associated eigenfunctions are merely multiples of each other (see, e.g., Gilbarg and Trudinger [8]) and the corresponding eigenfunctions of λ_1 never change signs in Ω. In other words, there exists $e_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$, with $e_1(x) < 0$ for any $x \in \Omega$ such that

$$\int_\Omega |\nabla e_1|^{p-2} \nabla e_1 \nabla v\, dx - \lambda_1 \int_\Omega e_1 v\, dx = 0\,,$$

for any $v \in W_0^{1,p}(\Omega)$. Thus, we have $(e_1)_+ = 0$ and $(e_1) = -e_1$ and we deduce that relation (2.2) holds with $u = e_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\lambda = \lambda_1$. In other words, λ_1 is an eigenvalue of (1.1) and the set of its corresponding eigenvectors lies in a cone of $W_0^{1,p}(\Omega)$. Now, we prove that λ_1 isolated in the set of eigenvalues of problem (2.1). Indeed, by the Lemma 2.1 we have that there does not exist an eigenvalue of (2.1) in the interval $(0, \lambda_1)$. On the other hand it is clear that if λ is also an eigenvalue of (2.1) for which u_+ is not identically zero, then we have

$$\lambda_1 \int_\Omega u_+^p\, dx \le \int_\Omega |\nabla u_+|^p\, dx = \lambda \int_\Omega h(x, u_+) u_+\, dx \le \lambda k \int_\Omega u_+^p\, dx\,,$$

and thus since $k \in (0,1)$ we have $\lambda \ge \frac{\lambda_1}{k} > \lambda_1$. This means that for any eigenvalue $\lambda \in (0, \lambda_1/k)$ of (2.1) we must have $u_+ = 0$. It follows that λ is an eigenvalue of (1.2) with the corresponding eigenfunction negative in Ω. As it has been already noticed, the set of eigenvalues of (1.2) is discrete and $\lambda_1 < \lambda_2$. Now, let us consider $\epsilon = \min\{\lambda_1/k, \lambda_2\}$ and we have that $\epsilon > \lambda_1$ and any $\lambda \in (\lambda_1, \epsilon)$ cannot be an eigenvalue of (1.2) and (2.1) and thus λ_1 is isolated in the set of eigenvalues of (2.1). □

Next, we show that there exists $\mu_1 > 0$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of (2.1). With that end in view, we consider the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda h(x, u_+), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.5}$$

We say that λ is an eigenvalue of (2.5) if there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v\, dx - \lambda \int_\Omega h(x, u_+) v\, dx = 0\,,$$

for any $v \in W_0^{1,p}(\Omega)$.

We notice that if λ is an eigenvalue for (2.5) with the corresponding eigenfunction u, then taking $v = u_-$ in the above relation we deduce that $u_- = 0$, and thus, we find $u \ge 0$. In other words, the eigenvalues of (2.5) possesses nonnegative corresponding eigenfunctions. Moreover, the above discussion show that an eigenvalue of (2.5) is an eigenvalue of (2.1).

Now, for each $\lambda > 0$ we define the energy functional associated to (2.5) by $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega H(x,u_+)\,dx\,,$$

where $H(x,t) = \int_0^t h(x,s)\,ds$. Standard arguments show that $I_\lambda \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ with the derivative given by

$$\langle I'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx - \lambda\int_\Omega h(x,u_+)v\,dx\,,$$

for any $u,\, v \in W_0^{1,p}(\Omega)$. Thus, $\lambda > 0$ is an eigenvalue of (2.5) if and only if there exists a critical nontrivial point of functional I_λ.

Lemma 2.3. *The functional I_λ defined as above is bounded from below and coercive. Moreover, there exists $\lambda^\star > 0$ such that assuming that $\lambda \geq \lambda^\star$ we have $\inf_{W_0^{1,p}(\Omega)} I_\lambda < 0$.*

Proof. By (P3) we deduce that

$$\lim_{t\to\infty}\frac{H(x,t)}{t^p} = 0, \quad \text{uniformly in } \Omega\,.$$

Then for a given $\lambda > 0$ and λ_1 defined as the Rayleigh quotient, there exists a positive constant $C_\lambda > 0$ such that

$$\lambda H(x,t) \leq \frac{\lambda_1}{2p}t^p + C_\lambda, \quad \forall\, t \geq 0,\ \text{a.e. } x \in \Omega\,.$$

Thus, for any $u \in W_0^{1,p}(\Omega)$,

$$I_\lambda(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda_1}{2p}\int_\Omega u^p\,dx - C_\lambda|\Omega| \geq \frac{1}{2p}\|u\|^p - C_\lambda|\Omega|\,,$$

where by $\|\cdot\|_p$ is denoted the norm on $W_0^{1,p}(\Omega)$, that is $\|u\|_p = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$. This shows that I_λ is bounded from below and coercive. Now, we prove the second part of the lemma. We employ the property (P2) which states that there exists $t_0 > 0$ such that $H(x,t_0) > 0$ a.e. for all $x \in \overline{\Omega}$. Let us consider $\Omega_1 \subset \Omega$ be a sufficiently large compact subset and $u_0 \in C_0^1(\Omega) \subset W_0^{1,p}(\Omega)$ such that $u_0(x) = t_0$ for $x \in \Omega_1$ and $0 \leq u_0(x) \leq t_0$ for any $x \in \Omega - \Omega_1$. By (P1) we have

$$\int_\Omega H(x,u_0)dx \geq \int_{\Omega_1} H(x,t_0)dx - \int_{\Omega-\Omega_1} ku_0^p dx \geq \int_\Omega H(x,t_0)dx - kt_0^p|\Omega-\Omega_1| > 0.$$

This means that $I_\lambda(u_0) < 0$ for sufficiently large $\lambda > 0$ and thus, we obtain $\inf_{W_0^{1,p}(\Omega)} I_\lambda < 0$. □

By Lemma 2.3, the functional I_λ has a negative global minimum for $\lambda > 0$ sufficiently large and any large $\lambda > 0$ is an eigenvalue of (1.1) and thus is an eigenvalue of (2.1). By Lemma 2.1, the statement of Theorem 1.2 holds.

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Alin Gălăţan
University of Bucharest, Faculty of Mathematics, Str. Academiei 14, RO–70109, Bucharest, Romania
E-mail address: alin9d@yahoo.com

Cezar Lupu
University of Bucharest, Faculty of Mathematics, Str. Academiei 14, RO–70109, Bucharest, Romania
E-mail address: lupucezar@yahoo.com, lupucezar@gmail.com

Felician Preda
Institute of Mathematical Statistics and Applied Mathematics "Gheorghe Mihoc–Caius Iacob", Calea 13 Septembrie No. 13, Bucharest 5, RO-050711, Romania
E-mail address: felician.preda@yahoo.com