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GREEN FUNCTION AND FOURIER TRANSFORM FOR O-PLUS OPERATORS

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Abstract. In this article, we study the o-plus operator defined by

$$\oplus^k = \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^4 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^4\Big)^k,$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $p + q = n$, and k is a nonnegative integer. Firstly, we studied the elementary solution for the $\oplus^k$ operator and then this solution is related to the solution of the wave and the Laplacian equations. Finally, we studied the Fourier transform of the elementary solution and also the Fourier transform of its convolution.

1. Introduction

Consider the ultra-hyperbolic operator iterated k times,

$$\Box^k = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}\Big)^k. \tag{1.1}$$

Trione [6] showed that the generalized function $R^H_{2k}(x)$, defined by (2.1) below, is the unique elementary solution for the $\Box^k$ operator, that is $\Box^k R^H_{2k}(x) = \delta$ for $x \in \mathbb{R}^n$, the n-dimensional Euclidian space.

Kananthai [2] studied the Diamond operator, iterated k times,

$$\diamondsuit^k = \Big(\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k, \tag{1.2}$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, where $p + q = n$, n is the dimension of Euclidean space $\mathbb{R}^n$, and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k \tag{1.3}$$

where $\triangle^k$ is the Laplacian operator iterated k times,

$$\triangle^k = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}\Big)^k. \tag{1.4}$$

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Kananthai [2] showed that the function $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the unique elementary solution for the operator $\diamondsuit^k$, where $*$ indicates convolution, and $R^e_{2k}(x)$, $R^H_{2k}(x)$ are defined by (2.5) and (2.2) with $\alpha = 2k$ respectively; that is,

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta\,. \tag{1.5}$$

Furthermore, The operator $\oplus^k$ was first studied by Kananthai, Suantai and Longani [4]. The $\oplus^k$ operator can be expressed in the form

$$\begin{aligned}\oplus^k = &\Big[\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big]^k \Big[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big]^k\\ &\cdot\Big[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big]^k.\end{aligned}$$

The purpose of this work is to study the operator

$$\begin{aligned}\oplus^k &= \Big(\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^4 - \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^4\Big)^k\\ &= \Big(\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k \cdot \Big(\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2 \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k.\end{aligned} \tag{1.6}$$

Let us denote the operator

$$\circledcirc^k = \Big(\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2 + \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k.$$

By (1.1) and (1.4) we obtain

$$\begin{aligned}\circledcirc^k &= \Big(\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2 + \Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big)^k\\ &= \Big[\Big(\frac{\triangle + \Box}{2}\Big)^2 + \Big(\frac{\triangle - \Box}{2}\Big)^2\Big]^k\\ &= \Big(\frac{\triangle^2 + \Box^2}{2}\Big)^k.\end{aligned} \tag{1.7}$$

Thus, (1.6) can be written as

$$\oplus^k = \diamondsuit^k \circledcirc^k\ . \tag{1.8}$$

For $k = 1$ the operator $\diamondsuit$ can be expressed in the form $\diamondsuit = \triangle\Box = \Box\triangle$ where $\Box$ is the Ultra-hyperbolic operator

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}. \tag{1.9}$$

where $p + q = n$ and $\triangle$ is the Lapacian operator

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}. \tag{1.10}$$

By putting $p = 1$ and $x_1 = t(t = time)$ in (1.9), we obtain the wave operator

$$\Box = \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \dots - \frac{\partial^2}{\partial x_n^2}\,. \tag{1.11}$$

From (1.6) with $q = 0$ and $k = 1$, we obtain

$$\oplus = \triangle_p^4 \tag{1.12}$$

where

$$\triangle_p = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}. \tag{1.13}$$

Firstly, we can find the elementary solution $G(x)$ of the operator $\oplus^k$; that is,

$$\oplus^k G(x) = \delta, \tag{1.14}$$

where δ is the Dirac-delta distribution. Moreover, we can find the relationship between $G(x)$ and the elementary solution of the wave operator defined by (1.11) depending on the conditions of p, q and k of (1.6) with $p = 1, q = n-1, k = 1$ and $x_1 = t$(t is time) and also we found that $G(x)$ relates to the elementary solution the Laplacian operator defined by (1.12) and (1.13) depending on the conditions of q and k of (1.6) with $q = 0$ and $k = 1$. In finding the elementary solution of (1.6), we use the method of convolutions of the generalized function. Finally, we study the Fourier transform of the elementary solution of the $\oplus^k$ operator and also study their convolution.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denoted by

$$\upsilon = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

the non-degenerated quadratic form, where $p + q = n$ is the dimension the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ be the interior of a forward cone and $\overline{\Gamma}_+$ denotes it closure. For any complex number α, define the function

$$R_\alpha^H(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}. \tag{2.3}$$

The function $R_\alpha^H(\upsilon)$ is called the Ultra-hyperbolic kernel of Marcel Riesz and was introduced by Nozaki [5, p.72]. It is well known that $R_\alpha^H(\upsilon)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $\mathrm{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(\upsilon)$ denote the support of $R_\alpha^H(\upsilon)$ and suppose $\operatorname{supp} R_\alpha^H(\upsilon) \subset \bar{\Gamma}_+$, that is $\operatorname{supp} R_\alpha^H(\upsilon)$ is compact.

If $p = 1$, then (2.2) reduces to the function

$$M_\alpha^H(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where $\upsilon = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{n-1}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2}) \Gamma(\frac{\alpha}{2})$. The function $M_\alpha^H(\upsilon)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$ and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$. The elliptic kernel of Marcel Riesz is defined as

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{2.5}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.6}$$

with α a complex parameter and n the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \triangle^k \delta(x)$ where $\triangle^k$ is defined by (1.4). It follows that $R^e_0(x) = \delta(x)$; see [3]. The function $R^e_{2k}(x)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is a distribution of α for $\mathrm{Re}(\alpha) < n$.

Definition 2.3. Let $f(x) \in L_1(\mathbb{R}^n)$ (the space of integrable function in $\mathbb{R}^n$). The Fourier transform of $f(x)$ is defined as

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx \tag{2.7}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \dots dx_n$. The inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \widehat{f(\xi)} d\xi. \tag{2.8}$$

If f is a distribution with compact supports by [8, Theorem 7.4-3], Equation (2.8) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi\cdot x}\rangle. \tag{2.9}$$

Lemma 2.4. *The function $R^H_{2k}(\upsilon)$ and $(-1)^k R^e_{2k}(x)$ are the elementary solutions of the operator $\Box^k$ and $\triangle^k$ respectively, where $\Box^k$ and $\triangle^k$ are defined by (1.4) and (1.3) respectively. The function $R^H_{2k}(\upsilon)$ defined by (2.2)with $\alpha = 2k$ and $R^e_{2k}(x)$ defined by (2.5) with $\alpha = 2k$.*

Proof. We have to show that $\Box^k R^H_{2k}(\upsilon) = \delta(x)$ and that $\triangle^k((-1)^k R^e_{2k}(x) = \delta(x)$. The first part follows from [7, Lemma 2.4], and the second part from [2, p.31]. $\square$

Lemma 2.5. *The convolution $R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)$ is an elementary solution for the operator $\diamondsuit^k$ iterated k times and is defined by (1.1)*

For the proof of the above lemma, see [2, p.33].

Lemma 2.6. *The function $R^H_\alpha(x)$ and $R^e_\alpha(x)$, defined by (2.2) and (2.5) respectively, for $\mathrm{Re}(\alpha)$ are homogeneous distribution of order $\alpha - n$ and also a tempered distribution.*

Proof. Note taht $R^H_\alpha(x)$ and $R^e_\alpha(x)$ satisfy the Euler equation; that is,

$$(\alpha - n)R^H_\alpha(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R^H_\alpha(x), (\alpha - n)R^e_\alpha(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R^e_\alpha(x).$$

Then $R^H_\alpha(x)$ and $R^e_\alpha(x)$ are homogeneous distributions of order $\alpha - n$. Since Donoghue [1, pp.154-155] proved the every homogeneous distribution is a tempered distribution, the proof is complete. $\square$

Lemma 2.7 (Convolution of tempered distributions). *$R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choose $\operatorname{supp} R^H_\alpha(x) = K \subset \Gamma_+$ where K is a compact set. Then $R^H_\alpha(x)$ is a tempered distribution with compact support. By Donoghue [1, pp.156-159], $R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution. $\square$

Lemma 2.8. *The functions $R^H_{-2k}(x)$ and $(-1)^k R^e_{-2k}(x)$ are the inverse in the convolution algebra of $R^H_{2k}(x)$ and $(-1)^k R^e_{2k}(x)$, respectively. That is,*

$$R^H_{-2k}(x) * R^H_{2k}(x) = R^H_{-2k+2k}(x) = R^H_0(x) = \delta(x),$$
$$(-1)^k R^e_{-2k}(x) * (-1)^k R^e_{2k}(x) = (-1)^{2k} R^e_{-2k+2k}(x) = R^e_0(x) = \delta(x)$$

For the proof of the above lemma, see [7, p.123], [1, p.118, p.158], [6, p.10].

Lemma 2.9 (Convolution of $R^e_\alpha(x)$ and $R^H_\alpha(x)$). *Let $R^e_\alpha(x)$ and $R^H_\alpha(x)$ defined by* (2.5) *and* (2.2) *respectively, then we obtain:*

1. *$R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x)$ where α and β are complex parameters;*
2. *$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$ for α and β are both integers and except only the case both α and β are both integers.*

Proof. Part (1) can be found in [1, p.158]. For the second formula, when α and β are both even integers, see [3]. When α is odd and β is even, or α is even and β is odd, from Trione [7] we have

$$\square^k R^H_\alpha(x) = R^H_{\alpha-2k}(x), \tag{2.10}$$
$$\square^k R^H_{2k}(x) = \delta(x), \quad k = 0, 1, 2, 3, \dots \tag{2.11}$$

where $\square^k$ is the Ultra-hyperbolic operator iterated k-times defined by

$$\square^k = \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \Big)^k.$$

Now let m be an odd integer, we have

$$\square^k R^H_m(x) = R^H_{m-2k}(x),$$
$$R^H_{2k}(x) * \square^k R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x)$$

or

$$(\square^k R^H_{2k}(x)) * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x), \delta * R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_m(x) = R^H_{2k}(x) * R^H_{m-2k}(x).$$

Since m is odd, hence $m - 2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k$, $\beta = m - 2k$, we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$$

for α is a nonnegative even and β is odd.

For the case α is a negative even and β is odd, by (2.8) we have

$$\square^k R^H_0(x) = R^H_{-2k}(x),$$

$$\Box^k \delta = R^H_{-2k}(x),$$

where $R^H_0(x) = \delta$. Now for m is odd,

$$R^H_{-2k}(x) * \Box^k R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x)$$

or

$$\big(\Box^k \delta\big) * \Box^k R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x),$$
$$\delta * \Box^{2k} R^H_m(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Thus

$$R^H_{m-2(2k)}(x) = R^H_{-2k}(x) * R^H_{m-2k}(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x).$$

This completes the proof. $\square$

Lemma 2.10. *Given the equation*

$$\circledcirc^k H(x) = \delta(x) \tag{2.12}$$

for $x \in R^n$, where $\circledcirc^k$ is the operator iterated $k-$ times is defined by (1.7) *and $\triangle^k$ is the Laplace operator iterated k times defined by* (1.4) *and $\Box^k$ is Ultra-hyperbolic operator iterated $k-$ times is defined by* (1.1). *Then we obtain $H(x)$ is an elementary solution of* (2.12)*, where*

$$H(x) = \big(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} \tag{2.13}$$

where

$$C(x) = \frac{1}{2} R^H_4(x) + \frac{1}{2}(-1)^2 R^e_4(x)\,. \tag{2.14}$$

Here $C^{*k}(x)$ denotes the convolution of $C(x)$ itself k times, $\big(C^{*k}(x)\big)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $H(x)$ is a tempered distribution.

Proof. We have

$$\circledcirc^k H(x) = \big(\frac{\triangle^2 + \Box^2}{2}\big)^k H(x) = \delta(x)$$

or we can write

$$\big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big)\big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big)^{k-1} H(x) = \delta(x).$$

Convolving both sides of the above equation by $R^H_4(x) * (-1)^2 R^e_4(x)$,

$$\begin{aligned}&\big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big) * \big(R^H_4(x) * (-1)^2 R^e_4(x)\big)\big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big)^{k-1} H(x)\\ &= \delta(x) * R^H_4(x) * (-1)^2 R^e_4(x)\end{aligned}$$

or

$$\begin{aligned}&\big(\frac{1}{2}\triangle^2(R^H_4(x) * (-1)^2 R^e_4(x)) + \frac{1}{2}\Box^2(R_4(x) * (-1)^2 S_4(x))\big) * \big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big)^{k-1} H(x)\\ &= \delta(x) * R^H_4(x) * (-1)^2 R^e_4(x).\end{aligned}$$

By properties of convolutions,

$$\big(\frac{1}{2}\triangle^2((-1)^2 R^e_4(x)) * R^H_4(x) + \frac{1}{2}\Box^2(R^H_4(x))$$

$$\begin{aligned}&* (-1)^2 R_4^e(x)\big) * \big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\big)^{k-1} H(x)\\ &= \delta(x) * R_4^H(x) * (-1)^2 R_4^e(x)\end{aligned}$$

By Lemmas 2.4 and 2.5, we obtain

$$\Big(\frac{1}{2}\delta * R_4^H(x) + \frac{1}{2}\delta * (-1)^2 R_4^e(x)\Big) * \Big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\Big)^{k-1} H(x) = R_4^H(x) * (-1)^2 R_4^e(x)$$

or

$$\Big(\frac{1}{2}R_4^H(x) + \frac{1}{2}(-1)^2 R_4^e(x)\Big) * \Big(\frac{1}{2}\triangle^2 + \frac{1}{2}\Box^2\Big)^{k-1} H(x) = R_4^H(x) * (-1)^2 R_4^e(x)$$

keeping on convolving both sides of the above equation by $R_4^H(x) * (-1)^2 R_4^e(x)$ up to $k-1$ times, we obtain

$$C^{*k}(x) * H(x) = \big(R_4^H(x) * (-1)^2 R_4^e(x)\big)^{*k} \tag{2.15}$$

the symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha^H(x)$ and $R_\alpha^e(x)$ in Lemma 2.6, we have

$$\big(R_4^H(x) * (-1)^2 R_4^e(x)\big)^{*k} = R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x).$$

Thus (2.15) becomes,

$$C^{*k}(x) * H(x) = \big(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)\big)$$

or

$$H(x) = \big(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)\big) * (C^{*k}(x))^{*-1} \tag{2.16}$$

is an elementary solution of (2.12). We consider the function $C^{*k}(x)$, since $R_4^H(x) * (-1)^2 R_4^e(x)$ is a tempered distribution. Thus $C(x)$ defined by (2.14) is a tempered distribution, we obtain $C^{*k}(x)$ is a tempered distribution.

Now, $R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x) \in \mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution. Thus $R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x) \in \mathcal{D}'_{\mathcal{R}}$. It follow that $R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence Zemanian [8], (2.16) has a unique solution

$$H(x) = \big(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)\big) * \big(C^{*k}(x)\big)^{*-1},$$

where $\big(C^{*k}(x)\big)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra. $H(x)$ is called the Green function of the operator $\circledcirc^k$.

Since $R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)$ and $\big(C^{*k}(x)\big)^{*-1}$ are lies in $\mathcal{S}'$, then by [8, p.152] again, we have $\big(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x)\big) * \big(C^{*k}(x)\big)^{*-1} \in \mathcal{S}'$. Hence $H(x)$ is a tempered distribution. $\Box$

Lemma 2.11. *The Fourier transform of* $\oplus^k \delta$ *is*

$$\mathcal{F} \oplus^k \delta = \frac{1}{(2\pi)^{n/2}} \Big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4\Big]^k$$

where $\mathcal{F}$ *is the Fourier transform defined by* (2.7) *and if the norm of* ξ *is given by* $\|\xi\| = \big(\xi_1^2 + \xi_2^2 + \dots + \xi_n^2\big)^{1/2}$ *then*

$$\mathcal{F} \oplus^k \delta \le \frac{M}{(2\pi)^{n/2}} \|\xi\|^{8k}.$$

Since M is constant thus is $\mathcal{F} \oplus^k \delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by (2.8),

$$\oplus^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \Big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4 \Big]^k .$$

Proof. By (2.10)

$$\begin{aligned}
\mathcal{F} \oplus^k \delta &= \frac{1}{(2\pi)^{n/2}} \langle \oplus^k \delta, e^{-i\xi\cdot x} \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \delta, \oplus^k e^{-i\xi\cdot x} \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \delta, \diamondsuit^k \circledcirc^k e^{-i\xi\cdot x} \rangle \quad \text{by (1.6)} \\
&= \frac{1}{(2\pi)^{n/2}} \big\langle \delta, \diamondsuit^k \Big(\frac{1}{2} \triangle^2 + \frac{1}{2} \Box^2 \Big)^k e^{-i\xi\cdot x} \big\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \big\langle \delta, \diamondsuit^k \frac{(-1)^{2k}}{2} \big(\big(\xi_1^2 + \dots \xi_n^2 \big)^2 + \big(\xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots \\
&\quad - \xi_n^2 \big)^2 \big)^k e^{-i\xi\cdot x} \big\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \big\langle \delta, \frac{(-1)^{2k}}{2} \big(\big(\xi_1^2 + \dots + \xi_n^2 \big)^2 + \big(\xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots \\
&\quad - \xi_n^2 \big)^2 \big)^k \diamondsuit^k e^{-i\xi\cdot x} \big\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \delta, \big(\big(\xi_1^2 + \dots + \xi_p^2 \big)^2 + \big(\xi_{p+1}^2 + \dots + \xi_{p+q}^2 \big)^2 \big) \triangle^k \Box^k e^{-i\xi\cdot x} \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \big\langle \delta, \Big[\Big(\sum_{i=1}^{p} \xi_i^2 \Big)^2 + \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^2 \Big]^k (-1)^{2k} \Big[\Big(\sum_{i=1}^{p} \xi_i^2 \Big)^2 \\
&\quad - \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^2 \Big]^k e^{-i\xi\cdot x} \big\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \langle \delta, (-1)^{2k} \Big[\Big(\sum_{i=1}^{p} \xi_i^2 \Big)^4 - \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^4 \Big]^k e^{-i\xi\cdot x} \rangle \\
&= \frac{1}{(2\pi)^{n/2}} \Big[\Big(\sum_{i=1}^{p} \xi_i^2 \Big)^4 - \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^4 \Big]^k . \\
&= \frac{1}{(2\pi)^{n/2}} \big[\big(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2 \big)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 \big)^4 \big]^k .
\end{aligned}$$

Next, we consider the boundness of $\mathcal{F} \oplus^k \delta$. Since

$$\begin{aligned}
\oplus^k &= \Big((\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2 \big)^4 \Big)^k \\
&= \Big(\Big(\xi_1^2 + \dots + \xi_p^2 \Big)^2 - \big(\xi_{p+1}^2 + \dots + \xi_n^2 \big)^2 \Big)^k \Big(\big(\xi_1^2 + \dots + \xi_p^2 \big)^2 \\
&\quad + \big(\xi_{p+1}^2 + \dots + \xi_n^2 \big)^2 \Big)^k \\
&= \big(\big(\xi_1^2 + \dots + \xi_n^2 \big) \big(\xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 \dots \xi_n^2 \big) \big)^k \Big(\big(\xi_1^2 + \dots + \xi_p^2 \big)^2
\end{aligned}$$

$$+\left(\xi_{p+1}^2+\cdots+\xi_n^2\right)^2\Big)^k$$

Thus

$$\begin{aligned}\mathcal{F}\oplus^k\delta &= \frac{1}{(2\pi)^{n/2}}\left(\left(\xi_1^2+\cdots+\xi_n^2\right)\left(\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2\cdots-\xi_n^2\right)\right)^k \\ &\quad\times\Big(\left(\xi_{p+1}^2+\cdots+\xi_n^2\right)^2+\left(\xi_{p+1}^2+\cdots+\xi_n^2\right)^2\Big)^k,\end{aligned}$$

$$\begin{aligned}|\mathcal{F}\oplus^k\delta| &= \frac{1}{(2\pi)^{n/2}}|\left(\xi_1^2+\cdots+\xi_n^2\right)\left(\xi_1^2+\cdots+\xi_{p+1}^2\cdots-\xi_n^2\right)|^k \\ &\quad\times\Big|\left(\xi_{p+1}^2+\cdots+\xi_n^2\right)^2+\left(\xi_{p+1}^2+\cdots+\xi_n^2\right)^2\Big|^k \\ &\le \frac{M}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\cdots+\xi_n^2\right)\right|^k\left|\left(\xi_1^2+\cdots+\xi_n^2\right)\right|^k\left|\left(\xi_1^2+\cdots+\xi_n^2\right)\right|^{2k}.\end{aligned}$$

It follows that

$$|\mathcal{F}\oplus^k\delta|\le\frac{M}{(2\pi)^{n/2}}\|\xi\|^{8k},$$

where M is constant and $\|\xi\|=\left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\dots,n)\in\mathbb{R}$. Hence we obtain $\mathcal{F}\oplus^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Since $\mathcal{F}$ is one-to-one transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.8)

$$\oplus\delta=\mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[(\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^4-\left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^4\right].$$

That completes the proof. □

3. Main Results

Theorem 3.1. *Given the equation*

$$\oplus^k G(x)=\delta(x), \tag{3.1}$$

where $\oplus^k$ is the Oplus operator iterated k times defined by (1.8), $\delta(x)$ is the dirac-delta distribution, $x\in R^n$ and k is a nonnegative integer. Then we obtain

$$G(x)=\left(R_{2k}^H(\upsilon)*(-1)^k R_{2k}^e(x)\right)*H(x) \tag{3.2}$$

or by (2.13) and Lemma 2.10, we obtain

$$G(x)=\left(R_{6k}^H(\upsilon)*(-1)^{3k}R_{6k}^e(x)\right)*\left(C^{*k}(x)\right)^{*-1} \tag{3.3}$$

is a Green's function or an elementary solution for the operator $\oplus^k$ iterated $k-$times where $\oplus^k$ is defined by (1.8), and $H(x)$ defined by (2.13).

For $q=0$, then(3.1) becomes

$$\triangle^{4k}G(x)=\delta(x) \tag{3.4}$$

By Lemma 2.4, we obtain

$$G(x)=(-1)^{4k}R_{8k}^e(x)=R_{8k}^e(x)$$

is an elementary solution of (3.4) where $\triangle_p^{4k}$ is the Laplacian of $p-$ dimension, iterated $4k-$ times and is defined by (1.13). Moreover, from (3.3), we obtain

$$R_{-4k}^H(\upsilon)*(-1)^{3k}R_{-6k}^e(x)*\left(C^{*k}(x)\right)*G(x)$$

$$
\begin{aligned}
&= \big(R^H_{4k}(\upsilon) * R^H_{-4k}(\upsilon)\big) * \big((-1)^{3k} R^e_{6k}(x) * (-1)^{3k} R^e_{-6k}(x)\big) \\
&\quad * \Big(\big(C^{*k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} \Big) * R^H_{2k}(\upsilon).
\end{aligned}
$$

By (2.14) *the above equation becomes*

$$
\Big(\frac{(-1)^{3k}}{2} * R^e_{-6k}(x) + R^H_{-4k}(\upsilon) * \frac{(-1)^{5k}}{2} R^e_{-2k}(x)\Big) * G(x) = R^H_{2k}(\upsilon) \tag{3.5}
$$

as an elementary solution of the operator k times is defined by (1.4) *In particular, if we put $p = 1$, $q = n-1$, $k = 1$ and $x_1 = t$ in* (1.4),(3.3)*, we obtain*

$$
\Big(\frac{(-1)^{3}}{2} R^e_{-6}(x) + M^H_{-2}(\upsilon) * \frac{(-1)^{5}}{2} R^e_{-6}(x)\Big) * G(x) = M^H_2(\upsilon), \tag{3.6}
$$

as an elementary solution of the wave operator defined by

$$
\square = \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \dots - \frac{\partial^2}{\partial x_n^2}, \tag{3.7}
$$

where $M^H_2(\upsilon)$ and $M^H_{-4}(\upsilon)$ defined by (2.4) *with $\alpha = 2$ and $\alpha = -4$ respectively, and $\upsilon = t_1^2 - x_2^2 - \dots - x_n^2$. The function $R^e_{-6}(x)$ is defined by* (2.5) *with $\alpha = -6$.*

Proof. From (3.1) and (1.8), we have

$$
\oplus^k G(x) = \big(\diamondsuit^k \circledcirc^k\big)\, G(x) = \delta(x). \tag{3.8}
$$

Convolving both sides of (3.8) by $\big(R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)\big) * H(x)$, we obtain

$$
\big(R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)\big) * H(x) * (\diamondsuit^k \circledcirc^k) G(x) = \delta * \big(R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)\big) * H(x)
$$

By properties of convolution

$$
\diamondsuit^k \,\big(R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)\big) * \circledcirc^k (H(x)) * G(x) = \big(R^H_{2k}(\upsilon) * (-1)^k R^e_{2k}(x)\big) * H(x).
$$

By Lemma 2.5 and 2.10, we obtain,

$$
\delta * \delta * G(x) = G(x) = \big(R^H_{2k}(x) * (-1)^k R^e_{2k}(\upsilon)\big) * H(x).
$$

By Lemma 2.9 and (2.13), we obtain

$$
G(x) = R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x) * \big(C^{*k}(x)\big)^{*-1} \tag{3.9}
$$

is an elementary solution or Green's function of $\oplus^k$ operator. Now, for $q = 0$ the (3.1) becomes

$$
\triangle_p^{4k} G(x) = \delta, . \tag{3.10}
$$

where $\triangle_p^{4k}$ is the Laplacian operator of p-dimension iterated $4k$ times. By Lemma 2.4, we have

$$
G(x) = (-1)^{4k} R^e_{8k}(x)
$$

is an elementary solution of (3.10).

On the other hand, we can also find $G(x)$ from (3.9). Since $q = 0$, we have $R^H_{2k}(\upsilon)$ reduces to $(-1)^k R^e_{2k}(x)$. Thus, by (3.9) for $q = 0$, we obtain

$$
\begin{aligned}
G(x) &= \big((-1)^{6k} R^e_{6k}(x) * R^e_{6k}(x)\big) * \big((-1)^{2k} R^e_{4k}(x)\big)^{*-1} \\
&= (-1)^{6k} R^e_{6k+6k}(x) \,\big((-1)^{2k} R^e_{4k}(x)\big)^{*-1} \\
&= R^e_{8k}(x).
\end{aligned}
$$

Now, consider the case of the wave equation. From (3.3), we have

$$
G(x) = \big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * \big(C^{*k}(x)\big)^{*-1}.
$$

Convolving the above equation by $R^H_{-4k}(\upsilon) * (-1)^{3k} R^e_{-6k}(x) * \big(C^{*k}(x)\big)$ and by Lemma 2.8, we obtain

$$\begin{aligned}
&R^H_{-4k}(\upsilon) * (-1)^{3k} R^e_{-6k}(x) * \big(C^{*k}(x)\big) * G(x) \\
&= \big(R^H_{4k}(\upsilon) * R^H_{-4k}(\upsilon)\big) * \big((-1)^{3k} R^e_{6k}(x) * (-1)^{3k} R^e_{-6k}(x)\big) \\
&\quad * \Big(\big(C^{*k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} \Big) * R^H_{2k}(\upsilon).
\end{aligned}$$

By Lemma 2.8, we obtain

$$R^H_{-4k}(\upsilon) * (-1)^{3k} R^e_{-6k}(x) * \big(C^{*k}(x)\big) * G(x) = R^H_0(x) * R^e_0(x) * \delta(x) * R^H_{2k}(\upsilon)$$

or

$$R^H_{-4k}(\upsilon) * (-1)^{3k} R^e_{-6k}(x) * \big(C^{*k}(x)\big) * G(x) = \delta(x) * \delta(x) * \delta(x) * R^H_{2k}(\upsilon).$$

It follows that

$$R^H_{-4k}(\upsilon) * (-1)^{3k} R^e_{-6k}(x) * \big(C^{*k}(x)\big) * G(x) = R^H_{2k}(\upsilon) \tag{3.11}$$

as an elementary solution of the operator $\square^k$ iterated k times defined by (1.4). In particular , if we put $p = 1$, $q = n - 1$, $k = 1$ and $x_1 = t$ in (3.3) and (3.9) then R^H_{-4} reduces to $M^H_{-4}(\upsilon)$ and $R^H_2(\upsilon)$ reduces to $M^H_2(\upsilon)$where $M^H_{-4}(\upsilon)$ and $M^H_2(\upsilon)$ is defined by (2.4) with $\alpha = -4$, $\alpha = 2$ respectively. Thus (3.11) becomes

$$M^H_{-4}(\upsilon) * (-1)^3 R^e_{-6}(x) * \big(C^{*1}(x)\big) * G(x) = M^H_2(\upsilon) \tag{3.12}$$

by (2.14), we obtain

$$M^H_{-4}(\upsilon) * (-1)^3 R^e_{-6}(x) * \Big(\frac{1}{2} M^H_4(\upsilon) + \frac{(-1)^2}{2} R^e_4(x)\Big) * G(x) = M^H_2(\upsilon)$$

or

$$\begin{aligned}
&\Big(M^H_{-4}(\upsilon) * (-1)^3 R^e_{-6}(x) * \frac{1}{2} M^H_4(\upsilon) + M^H_{-4}(\upsilon) \\
&* (-1)^3 R^e_{-6}(x) * \frac{1}{2} (-1)^2 R^e_4(x)\Big) * G(x) \\
&= M^H_2(\upsilon).
\end{aligned}$$

By Lemma 2.8, we obtain

$$\Big(\frac{(-1)^3}{2} R^e_{-6}(x) + M^H_{-4}(\upsilon) * \frac{(-1)^5}{2} R^e_{-2}(x)\Big) * G(x) = M^H_2(\upsilon) \tag{3.13}$$

as an elementary solution of the wave operator defined by

$$\square = \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \dots - \frac{\partial^2}{\partial x_n^2},$$

and $R^e_{-6}(x)$ defined by (2.5) with $\alpha = -6$. This completes the proof. $\square$

Theorem 3.2.

$$\begin{aligned}
&\mathcal{F}\left(\big(R^H_{6k}(x) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}\right) \\
&= \frac{1}{(2\pi)^{n/2}} \big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4\,\big]^k
\end{aligned}$$

and

$$|\mathcal{F}\left(\big(R^H_{6k}(x) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}\right)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} M \tag{3.14}$$

for a large $\xi_i \in R$, where M is a constant and $C(x)$ is defined by (2.14). That is, $\mathcal{F}$ is bounded and continuous on the space S' of the tempered distributions.

Proof. By Theorem 3.1, we have

$$\oplus^k(\big(\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}\big) = \delta(x)$$

or

$$\big(\oplus^k \delta\big) * (\big(\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}\big) = \delta(x).$$

Taking the Fourier transform on both sides of the above equation, we obtain

$$\mathcal{F}(\big(\oplus^k \delta\big) * [\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$

By (2.10)

$$\frac{1}{(2\pi)^{n/2}}\langle(\big(\oplus^k \delta\big) * [\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]), e^{-i(\xi\cdot x)}\rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}}\langle(\big(\oplus^k \delta\big) * [\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]), e^{-i\xi.(x+r)}\rangle \\
&= \frac{1}{(2\pi)^{n/2}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}}\langle[\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]), e^{-i(\xi.r)}\rangle\langle(\big(\oplus^k \delta\big), e^{-i\xi\cdot x}\rangle \\
&= \frac{1}{(2\pi)^{n/2}},
\end{aligned}$$

$$\mathcal{F}([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}])\,(2\pi)^{\frac{n}{2}}\,\mathcal{F}\left(\oplus^k \delta\right) = \frac{1}{(2\pi)^{n/2}},$$

$$\begin{aligned}
&\mathcal{F}([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (S^{*k}(x))^{*-1}]) \\
&\cdot (-1)^{3k}\Big[(\xi_1^2+\xi_2^2+\dots+\xi_p^2)^4 - \big(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2\big)^4\Big]^k \\
&= \frac{1}{(2\pi)^{n/2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{F}([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]) \\
&= \frac{1}{(-1)^{4k}\,(2\pi)^{n/2}\Big[(\xi_1^2+\xi_2^2+\dots+\xi_p^2)^4 - \big(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2\big)^4\Big]^k}.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\big(\xi_1^2+\dots+\xi_p^2\big)^4 - \big(\xi_{p+1}^2+\dots+\xi_{p+q}^2\big)^4} \\
&= \frac{1}{\Big(\big(\xi_1^2+\dots+\xi_p^2\big)^2 + \big(\xi_{p+1}^2+\dots+\xi_{p+q}^2\big)^2\Big)} \\
&\quad\times \frac{1}{\big((\xi_1^2+\dots+\xi_n^2)\,\big(\xi_1^2+\dots+\xi_p^2-\xi_{p+1}^2-\dots-\xi_{p+q}^2\big)\big)}.
\end{aligned} \tag{3.15}$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.1. Then $(\xi_1^2 + \dots + \xi_p^2 + \xi_{p+1}^2 + \dots + \xi_{p+q}^2) > 0$ and for a large k, the right-hand side of (3.15) tend to zero. It follows that it is bounded by a positive constant M say, that is we obtain (3.15) as required and also by (3.13) $\mathcal{F}$ is continuous on the space S' of the tempered distribution. □

Theorem 3.3.

$$
\begin{aligned}
&\mathcal{F}\Big([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}] \\
&* [\big(R^H_{6m}(\upsilon) * (-1)^{3m} R^e_{6m}(x)\big) * (C^{*m}(x))^{*-1}]\Big) \\
&= (2\pi)^{n/2} \mathcal{F}\left([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]\right) \\
&\quad \times \mathcal{F}([\big(R^H_{6m}(\upsilon) * (-1)^{3m} R^e_{6m}(x)\big) * (C^{*m}(x))^{*-1}]) \\
&= \frac{1}{(2\pi)^{n/2} \Big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4\Big]^{k+m}},
\end{aligned}
$$

where k and m are nonnegative integer and $\mathcal{F}$ is bounded and continuous on the space S' of the tempered distribution.

Proof. Since $R^H_{6k}(\upsilon)$ and $R^e_{4k}(x)$ are tempered distribution with compact support,

$$
\begin{aligned}
&\big([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]\big) \\
&* \Big([\big(R^H_{6m}(\upsilon) * (-1)^{3m} R^e_{6m}(x)\big) * (C^{*m}(x))^{*-1}]\Big) \\
&= \big(R^H_{6k}(\upsilon) * R^H_{6m}(\upsilon)\big) * \Big((-1)^{3(k+m)} R^e_{6k}(x) * R^e_{6m}(x)\Big) \\
&\quad * \Big(\big(C^{*k}(x)\big)^{*-1} * ((C^{*m}(x))^{*-1}\Big) \\
&= \Big(R^H_{6(k+m)}(\upsilon) * (-1)^{3(k+m)} R^e_{6(k+m)}(x)\Big) * \big(C^{*(k+m)}(x)\big)^{*-1}
\end{aligned}
$$

by [7, pp.156-159] and [3, Lemma 2.5]. Taking the Fourier transform on both sides and using Theorem 3.2, we obtain

$$
\begin{aligned}
&\mathcal{F}\left[\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x)^{*-1}]\right) \\
&* \Big([\big(R^H_{6m}(\upsilon) * (-1)^{3m} R^e_{6m}(x)\big) * (C^{*m}(x))^{*-1}]\Big) \\
&= \frac{1}{(2\pi)^{n/2} \Big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4\Big]^{k+m}} \\
&= \frac{1}{(2\pi)^{n/2} \Big[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - \big(\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2\big)^4\Big]^{k}} \\
&\quad \times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2} \Big[\big(\xi_1^2 + \dots + \xi_p^2\big)^4 - \big(\xi_{p+1}^2 + \dots + \xi_{p+q}^2\big)^4\Big]^{m}} \\
&= (2\pi)^{n/2} \mathcal{F}\left([\big(R^H_{6k}(\upsilon) * (-1)^{3k} R^e_{6k}(x)\big) * (C^{*k}(x))^{*-1}]\right) \\
&\quad \times \mathcal{F}([\big(R^H_{6m}(\upsilon) * (-1)^{3m} R^e_{6m}(x)\big) * (C^{*m}(x))^{*-1}]).
\end{aligned}
$$

Since $(R^H_{6(k+m)}(\upsilon) * (-1)^{3(k+m)} R^e_{6(k+m)}(x)) * (C^{*(k+m)}(x))^{*-1} \in S'$, the space of tempered distribution and by Theorem 3.2, we obtain that $\mathcal{F}$ is bounded and continuous on S'. $\square$

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