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GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF SLOW GROWTH

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Abstract. In this article, we investigate the hyper order of solutions of higher-order linear differential equations with entire coefficients of slow growth. We assume that the lower order of the dominant coefficient in the high-order linear equations is less than 1/2, and obtain some results which extend the results in [6, 13, 14, 19].

1. Introduction and statement of results

We shall assume that readers are familiar with the fundamental results and the standard notation of the Nevanlinna's theory of meromorphic functions [10, 18]. We use $\sigma(f)$ and $\mu(f)$ to denote the order and low order of meromorphic function $f(z)$ respectively. We use $\sigma_2(f)$ and $\mu_2(f)$ to denote the hyper order and hyper lower order of $f(z)$, which are defined as [21]

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log_2 T(r,f)}{\log r},$$
$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log_2 T(r,f)}{\log r}.$$

The hyper exponent of convergence of zeros and distinct zeros of $f(z)$ are respectively defined to be (see [5])

$$\lambda_2(f) = \limsup_{r\to\infty} \frac{\log_2 N(r,f)}{\log r}, \quad \overline{\lambda_2}(f) = \limsup_{r\to\infty} \frac{\log_2 \overline{N}(r,f)}{\log r}.$$

It is easy to see that $\sigma(f) = \infty$ if $\sigma_2(f) > 0$. We denote the linear measure of a set $E \subset [1,\infty)$ by $mE = \int_E dt$ and the logarithmic measure of E by $m_l E = \int_E \frac{dt}{t}$.

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The upper and lower logarithmic density of E are defined by (see [3])

$$\overline{\log\text{dens}}(E) = \limsup_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r},$$
$$\underline{\log\text{dens}}(E) = \liminf_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}.$$

Nevanlinna's value distribution theory has become a very useful tool in investigating the growth of solutions of linear differential equations. By the definition of hyper order, the growth of infinite order solutions of the differential equations can be estimated more precisely. In recent years, many papers began to investigate the hyper order of the infinite order solutions of the linear differential equations (see e.g. [5, 6, 19]).

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ are entire functions, it is well known that every nonconstant solution f of (1.1) has infinite order if $\sigma(A) < \sigma(B)$ or $A(z)$ is a polynomial and $B(z)$ is transcendental. Gundersen [9] proved the following result.

Theorem 1.1 ([9]). *Let $A(z)$ and $B(z)$ be entire functions such that*

(i) $\sigma(B) < \sigma(A) < 1/2$ *or*
(ii) $A(z)$ *is transcendental with* $\sigma(A) = 0$ *and* $\sigma(B)$ *is a polynomial.*

Then every nonconstant solution of (1.1) *has infinite order.*

In 1991, Hellerstein, Miles and Rossi improved Theorem 1.1 by proving the following result.

Theorem 1.2 ([13]). *If $A(z)$ and $B(z)$ are entire functions with $\sigma(B) < \sigma(A) \le \frac{1}{2}$, then any nonconstant solution of* (1.1) *has infinite order.*

In 1992, Hellerstein, Miles and Rossi improved Theorem 1.2 and proved the following result.

Theorem 1.3 ([14]). *$A_0,\dots,A_{k-1},F$ be entire functions. Suppose that there exists an A_s $(0\le s\le k-1)$ such that*

$$b = \max\{\sigma(F), \sigma(A_j)(j\ne s)\} < \sigma(A_s) \le \frac{1}{2}.$$

Then every solution of the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_s f^{(s)} + \dots + A_0 f = F \tag{1.2}$$

is either a polynomial or an entire function of infinite order.

In 2000, Chen and Yang [6] gave a more precise estimate of the growth of the solutions of (1.2) and its homogeneous differential equation and obtained the following results.

Theorem 1.4 ([6]). *Let $A_0,\dots,A_{k-1},F \not\equiv 0$ be entire functions, such that there exists an A_s $(0\le s\le k-1)$ satisfying*

$$b = \max\{\sigma(F), \sigma(A_j)(j\ne s)\} < \sigma(A_s) < 1/2.$$

Then every transcendental solution of (1.2) *satisfies* $\sigma_2(f) = \sigma(A_s)$.

Theorem 1.5 ([6]). *Let $A_j(z)$ $(j = 1, \dots, k-1)$ be entire functions such that $\max\{\sigma(A_j), j = 1, \dots, k-1\} < \sigma(A_0) < \infty$, then every nontrivial solution f of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_s f^{(s)} + \dots + A_0 f = 0 \tag{1.3}$$

satisfies $\sigma_2(f) = \sigma(A_0)$.

In 2008, Tu and Deng [19] investigated the growth of solutions of (1.3) and obtained the following results.

Theorem 1.6 ([19]). *Let A_j $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists some $s \in \{1, \dots, k-1\}$ such that $\max\{\sigma(A_j) : j \neq 0, s\} < \sigma(A_0) \le 1/2$ and that $A_s(z)$ has a finite deficient value, then every solution $f \not\equiv 0$ of (1.3) satisfies $\sigma(A_0) \le \sigma_2(f) \le \sigma(A_s)$.*

Theorem 1.7 ([19]). *Let A_j $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists some $s \in \{1, \dots, k-1\}$ such that $\max\{\sigma(A_j) : j \neq 0, s\} < \sigma(A_0) < 1/2$. Suppose that $A_s(z)$ is an entire function of genus $q \ge 1$, and that all the zeros of $A_s(z)$ lie in the angular sector $\theta_1 \le \arg z \le \theta_2$ satisfying*

$$\theta_2 - \theta_1 \le \frac{\pi}{q+1}.$$

Then every solution $f \not\equiv 0$ of (1.3) satisfies $\sigma(A_0) \le \sigma_2(f) \le \sigma(A_s)$.

Most of the above theorems are related to the problem: Does every transcendental solution of (1.1)-(1.3) have infinite order when the order of any one of the middle coefficients is greater than others? From Theorems 1.1–1.4, we know that the answer is affirmative when the fastest growing entire coefficient satisfies $\sigma(A_s) \le 1/2$ $(s \in \{1, \dots, k-1\})$. It is mentioned that Theorem 1.2 and Theorem 1.3 also hold under the hypothesis $\sigma(B) < \mu(A) \le 1/2$ in (1.1) or $\max\{\sigma(F), \sigma(A_j)(j \neq s)\} < \mu(A_s) \le 1/2$ in (1.2) (see [13, 14]). However the proof of the case $\sigma(A_s) = \frac{1}{2}$ or $\mu(A_s) = 1/2$ is more complicated. In the Theorem 1.8 below, we estimate the hyper order of the transcendental solutions of (1.1)-(1.3) under the assumption that the lower order of the dominant coefficient in (1.1)-(1.3) is less than 1/2.

Theorem 1.8. *Let $A_0, \dots, A_{k-1}, F$ be entire functions such that there exists an A_s $(1 \le s \le k-1)$ satisfying*

$$b = \max\{\sigma(F), \sigma(A_j)(j \neq s)\} < \mu(A_s) < 1/2.$$

Then every transcendental solution of (1.2) satisfies $\mu(A_s) \le \sigma_2(f) \le \sigma(A_s)$. Furthermore, if $F \not\equiv 0$, then every transcendental solution of (1.2) satisfies

$$\mu(A_s) \le \overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_s).$$

Corollary 1.9. *If $s = 1$, then every non-constant solution f of (1.2) satisfies $\mu(A_1) \le \sigma_2(f) \le \sigma(A_1)$. Furthermore, if $F \not\equiv 0$, then every non-constant solution f of (1.2) satisfies $\mu(A_1) \le \overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_1)$.*

Corollary 1.10. *If $A(z), B(z)$ are entire functions with $\sigma(B) < \mu(A) < 1/2$, then every solution $f \not\equiv 0$ of (1.1) satisfies $\mu(A) \le \sigma_2(f) \le \sigma(A)$.*

Corollary 1.11. *Under the hypotheses of Theorem 1.8, if $\varphi(z)$ is a transcendental entire function with $\sigma(\varphi) < \infty$, then every transcendental solution f of (1.2) or (1.3) satisfies*

$$\mu(A_s) \le \overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) \le \sigma(A_s).$$

Remark 1.12. Theorem 1.8 is an extension of Theorems 1.2–1.4. If $\mu(A_s) = \sigma(A_s) < 1/2$, by Theorem 1.4, we have that every transcendental solution of (1.2) satisfies $\sigma_2(f) = \mu(A_s) = \sigma(A_s)$, then Theorem 1.8 holds. Therefore, we only need to prove that Theorem 1.8 holds in the case $\mu(A_s) < 1/2$ and $\mu(A_s) < \sigma(A_s)$. In Theorem 1.8, if $s = 0$, we remove the restriction $\mu(A_0) < 1/2$ and have the following result.

Theorem 1.13. *Let A_j $(j = 0, \dots, k-1)$ be entire functions satisfying*

$$\max\{\sigma(A_j) : j \neq 0\} < \mu(A_0) \leq \sigma(A_0) < \infty,$$

then every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) = \mu_2(f) \leq \sigma_2(f) = \sigma(A_0)$.

Corollary 1.14. *Let A_j $(j = 0, \dots, k-1)$ be entire functions satisfying*

$$\max\{\sigma(A_j) : j \neq 0\} < \mu(A_0) = \sigma(A_0) < \infty,$$

then every solution $f \not\equiv 0$ of (1.3) satisfies $\mu_2(f) = \sigma_2(f) = \sigma(A_0)$.

The following theorem studies the case when there are two dominant coefficients in (1.3).

Theorem 1.15. *Let A_j $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists some $s \in \{1, \dots, k-1\}$ such that $\max\{\sigma(A_j) : j \neq 0, s\} < \mu(A_0) < 1/2$ and that $A_s(z)$ has a finite deficient value, then every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) \leq \sigma_2(f) \leq \max\{\sigma(A_0), \sigma(A_s)\}$.*

Corollary 1.16. *Let A_j $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists some $s \in \{1, \dots, k-1\}$ such that $\max\{\sigma(A_j) : j \neq 0, s\} < \mu(A_0) < 1/2$. Suppose that $A_s(z)$ is an entire function of genus $q \geq 1$, and that all the zeros of $A_s(z)$ lie in the angular sector $\theta_1 \leq \arg z \leq \theta_2$ satisfying*

$$\theta_2 - \theta_1 \leq \frac{\pi}{q+1}.$$

Then every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) \leq \sigma_2(f) \leq \max\{\sigma(A_0), \sigma(A_s)\}$.

Corollary 1.17. *Let A_j $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists some $s \in \{1, \dots, k-1\}$ such that $\max\{\sigma(A_j) : j \neq 0, s\} < \mu(A_0) < 1/2$ and $A_s(z) = h_s(z)e^{a_s z}$, where $\sigma(h_s) < 1$ and $a_s \neq 0$ is a complex number, then every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) \leq \sigma_2(f) \leq \max\{\sigma(A_0), 1\}$.*

Remark 1.18. Theorem 1.13 extends Theorem 1.5. The meaning of Corollary 1.14 is that all solutions of (1.3) are regular growing when the dominant coefficient A_0 is regular growing. However, we can not give any information about $\mu_2(f)$ in Theorem 1.8 and Theorem 1.15. Theorem 1.15 is a supplement to Theorems 1.8–1.13 and an improvement of Theorem 1.6. Corollaries 1.16–1.17 are the immediate conclusions of Theorem 1.15, since $A_s(z)$ in Corollary 1.16 has zero as a finite deficient value [17] and $A_s(z) = h_s(z)e^{a_s z}$ in Corollary 1.17 also has zero as a finite deficient value. If $A_s(z)$ in Theorem 1.15 has a finite deficient value, then $\sigma(A_s) > 1/2$ [19].

Open problems. Do Theorems 1.8 and 1.15 hold under the hypothesis $\mu(A_s) = \sigma(A_s) = \frac{1}{2}$? Does Theorem 1.4 hold under the hypothesis $\sigma(A_s) = \frac{1}{2}$?

2. Lemmas for the proofs of main results

Lemma 2.1 ([8]). *Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant, for any given $\varepsilon > 0$,*

(i) there exist a set $E_1 \subset [1, \infty)$ that has finite logarithmic measure and a constant $B > 0$ that depends only on α and $(m, n)(m, n \in \{0, \dots, k\}$ with $m < n)$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$|\frac{f^{(n)}(z)}{f^{(m)}(z)}| \le B\Big(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\Big)^{n-m}. \tag{2.1}$$

(ii) there exist a set $E_1 \subset [0, 2\pi)$ that has linear measure zero and a constant $B > 0$ that depends only on α and $(m, n)(m, n \in \{0, \dots, k\}$ with $m < n)$ such that for all $z = re^{i\theta}$ satisfying $\theta \in [0, 2\pi)\backslash E$ and for sufficiently large $|z| = r$, (2.1) holds.

Lemma 2.2 ([1]). *Let $f(z)$ be an entire function of $\sigma(f) = \sigma < 1/2$ and denote $A(r) = \inf_{|z|=r} \log |f(z)|$, $B(r) = \sup_{|z|=r} \log |f(z)|$. If $\sigma < \alpha < 1$, then*

$$\underline{\log\text{dens}}\{r : A(r) > (\cos \pi\alpha)B(r)\} > 1 - \frac{\sigma}{\alpha}. \tag{2.2}$$

Lemma 2.3 ([2]). *Let $f(z)$ be entire with $\mu(f) = \mu < 1/2$ and $\mu < \sigma = \sigma(f)$. If $\mu \le \delta < \min(\sigma, \frac{1}{2})$ and $\delta < \alpha < 1/2$, then*

$$\overline{\log\text{dens}}\{r : A(r) > (\cos \pi\alpha)B(r) > r^\delta\} > C(\sigma, \delta, \alpha), \tag{2.3}$$

where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ and α.

Lemma 2.4 ([6]). *Let $f(z)$ be a transcendental entire function. Then there is a set $E_2 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin E_2$ and $|f(z)| = M(r, f)$, we have*

$$|\frac{f(z)}{f^{(s)}(z)}| \le 2r^s, \quad (s \in N). \tag{2.4}$$

Lemma 2.5 ([11, 15]). *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E_3 of r of finite logarithmic measure, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \Big(\frac{\nu_f(r)}{z}\Big)^k (1 + o(1)), \quad (k \in N, r \notin E_3), \tag{2.5}$$

where $\nu_f(r)$ is the central index of f.

Lemma 2.6 ([5, 20]). *Let $f(z)$ be an entire function of infinite order satisfying $\sigma_2(f) = \sigma$ and $\mu_2(f) = \mu$. Then*

$$\limsup_{r\to\infty} \frac{\log_2 \nu_f(r)}{\log r} = \sigma, \quad \liminf_{r\to\infty} \frac{\log_2 \nu_f(r)}{\log r} = \mu, \tag{2.6}$$

where $\nu_f(r)$ is the central index of f.

Lemma 2.7 ([18]). *Let $g : (0, +\infty) \to \mathbb{R}$,$h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that*

(i) $g(r) \le h(r)$ outside of an exceptional set E_4 of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.

(ii) $g(r) \le h(r)$ outside of an exceptional set E_4 of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(r^\alpha)$ for all $r > r_0$.

Lemma 2.8. *Let $f(z)$ be an entire function with $\mu(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_5 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $r \in E_5$, we have*

$$M(r, f) < \exp\{r^{\mu(f)+\varepsilon}\}. \tag{2.7}$$

Proof. By the definition of lower order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{n\to\infty} \frac{\log_2 M(r_n, f)}{\log r_n} = \sigma.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$, we have

$$M(r_n, f) \leq \exp\{r_n^{\mu(f)+\frac{\varepsilon}{2}}\}. \tag{2.8}$$

Let $E_5 = \bigcup_{n=n_1}^{\infty} [(\frac{n}{n+1})r_n, r_n]$, then for any $r \in E_5$, we have

$$M(r, f) \leq M(r_n, f) \leq \exp\{r_n^{\mu(f)+\frac{\varepsilon}{2}}\} \leq \exp\{[(1 + \frac{1}{n})r]^{\mu(f)+\frac{\varepsilon}{2}}\} \leq \exp\{r^{\mu(f)+\varepsilon}\}. \tag{2.9}$$

and $m_l E_5 = \sum_{n=n_1}^{\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty$. The proof is complete. □

Lemma 2.9. *Let A_j $(j = 0, \dots, k-1)$ be entire functions of finite order, then all solutions of (1.3) satisfies $\mu_2(f) \leq \max\{\mu(A_0), \sigma(A_j) : j = 1, \dots, k-1\}$.*

Proof. From (1.3), we have

$$|\frac{f^{(k)}(z)}{f(z)}| \leq |A_{k-1}||\frac{f^{(k-1)}(z)}{f(z)}| + \dots + |A_s||\frac{f^{(s)}(z)}{f(z)}| + \dots + |A_0|. \tag{2.10}$$

By Lemma 2.5, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r, f)$, we have

$$|\frac{f^{(j)}(z)}{f(z)}| = \Big(\frac{\nu_f(r)}{r}\Big)^j (1 + o(1)) \quad (j = 1, \dots, k-1). \tag{2.11}$$

Set $\max\{\mu(A_0), \sigma(A_j) : j = 1, \dots, k-1\} = a$, then for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_j(z)| < \exp\{r^{a+\varepsilon}\} \quad (j = 1, \dots, k-1). \tag{2.12}$$

By Lemma 2.8, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|A_0(z)| < \exp\{r^{a+\varepsilon}\}. \tag{2.13}$$

Substituting (2.11)-(2.13) into (2.10), for any given $\varepsilon > 0$ and for sufficiently large $r \in E_5 \backslash E_3$ and $|f(z)| = M(r, f)$, we have

$$\Big(\frac{\nu_f(r)}{r}\Big)^k |1 + o(1)| \leq k \exp\{r^{a+\varepsilon}\} \Big(\frac{\nu_f(r)}{r}\Big)^{k-1} |1 + o(1)|, \tag{2.14}$$

then

$$\nu_f(r) \leq kr \exp\{r^{a+\varepsilon}\}. \tag{2.15}$$

Then by Lemma 2.6 we have $\mu_2(f) \leq a$. Thus, the proof is complete. □

Lemma 2.10 ([7]). *Let $f(z)$ be a meromorphic function of finite order σ. For any given $\zeta > 0$ and l, $0 < l < 1/2$, there exist a constant $K(\sigma,\zeta)$ and a set $E_\zeta \subset [0,\infty)$ of lower logarithmic density greater than $1-\zeta$ such that for all $r \in E_\zeta$ and for every interval J of length l*

$$r\int_J \Big|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\Big| d\theta < K(\sigma,\zeta)\Big(l\log\frac{1}{l}\Big)T(r,f). \tag{2.16}$$

Lemma 2.11 ([4, 12, 16]). *Let $A_j(z)$ $(j = 0,\dots,k-1)$ be entire functions with $\sigma(A_j) \le \sigma < \infty$, if $f(z)$ is a solution of (1.3), then $\sigma_2(f) \le \sigma$.*

2.1. **Proof of Theorem 1.8.** Assume that $f(z)$ is a transcendental solution of (1.2). From (1.2), we have

$$\begin{aligned} A_s &= \frac{F}{f^{(s)}} - \Big(\frac{f^{(k)}}{f^{(s)}} + \dots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \dots + A_0\frac{f}{f^{(s)}}\Big) \\ &= \frac{F}{f}\cdot\frac{f}{f^{(s)}} - \Big\{\frac{f^{(k)}}{f^{(s)}} + \dots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} \\ &\quad + \frac{f}{f^{(s)}}\cdot\Big(A_{s-1}\frac{f^{(s-1)}}{f} + \dots + A_0\Big)\Big\}. \end{aligned} \tag{2.17}$$

By Lemma 2.1(i), there exists a set $E_1 \subset (1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\Big|\frac{f^{(j)}(z)}{f^{(s)}(z)}\Big| \le M\cdot r^\alpha[T(2r,f)]^{2k}, \quad (j = s+1,\dots,k), \tag{2.18}$$

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le M\cdot r^\alpha[T(2r,f)]^{2k}, \quad (l = 1,\dots,s-1), \tag{2.19}$$

where $M > 0$ and $\alpha > 0$ are constants, not always the same at each occurrence. Since $0 < \mu(A_s) < 1/2, \mu(A_s) < \sigma(A_s)$, we choose ε, δ such that

$$b + \varepsilon < \mu(A_s) \le \delta < \min\{\sigma(A_s), \frac{1}{2}\}, \tag{2.20}$$

where $b = \max\{\sigma(F), \sigma(A_j)(j \ne s)\}$. For sufficiently large r, we have

$$|A_j(z)| \le \exp\{r^{b+\varepsilon}\}, \quad (j = 0,\dots,s-1,s+1,\dots,k-1), \tag{2.21}$$

$$|F(z)| \le \exp\{r^{b+\varepsilon}\}. \tag{2.22}$$

Since $M(r,f) > 1$ for sufficiently large r, by (2.22), we have

$$\frac{|F(z)|}{M(r,f)} \le |F(z)| \le \exp\{r^{b+\varepsilon}\}. \tag{2.23}$$

By Lemma 2.3 ($\mu(A_s) < \sigma(A_s)$), there exists a set $H_1 \subset (1,+\infty)$ having infinite logarithmic measure such that for all z satisfying $|z| = r \in H_1$, we have

$$|A_s(z)| > \exp\{r^\delta\}. \tag{2.24}$$

By Lemma 2.4, there is a set $E_2 \subset (1,+\infty)$ of finite logarithmic measure such that for a point z satisfying $|z| = r \notin [0,1) \cup E_2$ and $|f(z)| = M(r,f)$, we have

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \le 2r^s. \tag{2.25}$$

By (2.17)-(2.19) and (2.21)-(2.25), for all z satisfying $|z| = r \in H_1 - ([0,1] \cup E_1 \cup E_2)$ and $|f(z)| = M(r,f)$, we have

$$\exp\{r^{\delta}\} \le M \cdot \exp\{r^{b+\varepsilon}\} \cdot r^{\alpha} \cdot [T(2r,f)]^{2k}. \tag{2.26}$$

Again by (2.20) and (2.26), we see that for a point z satisfying$|z| = r \in H_1 - ([0,1] \cup E_1 \cup E_2)$ and $|f(z)| = M(r,f)$, we have

$$\exp\{r^{\delta}(1+o(1))\} \le [T(2r,f)]^{2k}. \tag{2.27}$$

Since δ is arbitrarily close to $\mu(A_s)$, from (2.27), we obtain

$$\limsup_{r\to\infty} \frac{\log_2 T(r,f)}{\log r} \ge \mu(A_s). \tag{2.28}$$

On the other hand, from Lemma 2.5, there is a set $E_3 \subset (1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{\nu_f(r)}{z}\Big)^j (1+o(1)), \quad (j = 1,\dots,k). \tag{2.29}$$

For any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_s(z)| \le \exp\{r^{\sigma(A_s)+\varepsilon}\}. \tag{2.30}$$

Now we take a point z satisfying $|z| = r \notin [0,1] \cup E_3$ and $|f(z)| = M(r,f)$ and substitute (2.21)-(2.23),(2.29)-(2.30) into (1.2), then we obtain

$$\Big(\frac{\nu_f(r)}{|z|}\Big)^k |1+o(1)| \le (k+1)\Big(\frac{\nu_f(r)}{|z|}\Big)^{k-1} |1+o(1)| \exp\{r^{\sigma(A_s)+\varepsilon}\}. \tag{2.31}$$

This gives

$$\limsup_{r\to\infty} \frac{\log_2 \nu_f(r)}{\log r} \le \sigma(A_s) + \varepsilon. \tag{2.32}$$

Since ε is arbitrary, by Lemma 2.6 and (2.32), we have $\sigma_2(f) \le \sigma(A_s)$. Combining this and (2.28), we obtain

$$\mu(A_s) \le \sigma_2(f) \le \sigma(A_s).$$

Assume that if f is a transcendental solution of (1.2), then $\sigma(f) = \infty$ by (2.28). We next show that $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$ if $F \not\equiv 0$. By (1.2), it is easy to see that if f has a zero at z_0 of order more than k, then F must has a zero at z_0. Hence we have

$$N(r, \frac{1}{f}) \le k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}). \tag{2.33}$$

From (1.2), we have

$$\frac{1}{f} = \frac{1}{F}\Big(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_0\Big). \tag{2.34}$$

Hence

$$m(r, \frac{1}{f}) \le \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + \sum_{j=0}^{k-1} m(r, A_j) + m(r, \frac{1}{F}). \tag{2.35}$$

By (2.33) and (2.35), we obtain that

$$T(r,f) \le k\overline{N}(r, \frac{1}{f}) + M(\log(rT(r,f))) + T(r,F) + \sum_{j=0}^{k-1} T(r, A_j), \quad (r \notin E_4), \tag{2.36}$$

where $E_4 \subset (0,+\infty)$ is a set having finite linear measure. For sufficiently large r, we have

$$M(\log(rT(r,f))) \leq \frac{1}{2}T(r,f), \tag{2.37}$$

$$\sum_{j=0}^{k-1} T(r,A_j) + T(r,F) \leq (k+1)r^{\sigma(A_s)+\varepsilon}. \tag{2.38}$$

By (2.36)-(2.38), we have

$$T(r,f) \leq 2k\overline{N}(r,\frac{1}{f}) + 2(k+1)r^{\sigma(A_s)+\varepsilon}, \tag{2.39}$$

hence $\sigma_2(f) \leq \overline{\lambda}_2(f)$ by (2.39). Therefore, $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$. By $\mu(A_s) \leq \sigma_2(f) \leq \sigma(A_s)$, we have $\mu(A_s) \leq \overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \sigma(A_s)$.

2.2. **Proof of Corollaries.** Using the similar proof in Theorem 1.8, we can easily obtain the Corollaries 1.9–1.10.

Assume that $f(z)$ is a transcendental solution of (1.2) or (1.3), then we have $\mu(A_s) \leq \sigma_2(f) \leq \sigma(A_s)$. Set $g(z) = f(z) - \varphi$, then we have $\sigma_2(g) = \sigma_2(f)$, and $\overline{\lambda}_2(g) = \overline{\lambda}_2(f-\varphi)$. Substituting $f = g + \varphi$ into (1.2) or (1.3), we obtain

$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi) \tag{2.40}$$

or

$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0 g = -(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi). \tag{2.41}$$

If $F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi) \equiv 0$ or $\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi \equiv 0$, by Theorem 1.8, we have $\sigma(\varphi) = \infty$. This is a contradiction. Therefore, $\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi \not\equiv 0$ and $F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi) \not\equiv 0$. Using the similar proof in (2.33)-(2.39), we can easily obtain $\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g)$, therefore Corollary 1.11 holds.

3. Proof of Theorem 1.13

From Theorem 1.5, we know that every nontrivial solution f of (1.3) satisfies $\sigma_2(f) = \sigma(A_0)$. Then we only need to prove that every nontrivial solution f of (1.3) satisfies $\mu_2(f) = \mu(A_0)$. From (1.3), we have

$$-A_0 = \frac{f^{(k)}}{f} + \dots + A_1\frac{f'}{f}. \tag{3.1}$$

By this equality and the logarithmic derivative lemma, we have

$$m(r,A_0) \leq \sum_{j=1}^{k} m\left(r,\frac{f^{(k)}}{f}\right) + \sum_{j=1}^{k-1} m(r,A_j) \leq O\{\log rT(r,f)\} + \sum_{j=1}^{k-1} m(r,A_j), \tag{3.2}$$

where $r \notin E_7$, $E_7 \subset (1,\infty)$ is a set having finite linear measure, not necessarily the same at each occurrence. Set $\max\{\sigma(A_j) : j \neq 0\} = c$, then for any given $\varepsilon (0 < 2\varepsilon < \mu(A_0) - c)$ and for sufficiently large r, we have

$$m(r,A_0) > r^{\mu(A_0)-\varepsilon}, \quad m(r,A_j) < r^{c+\varepsilon} \quad j \neq 0. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$r^{\mu(A_0)-\varepsilon} \leq O\{\log T(r,f)\} + kr^{c+\varepsilon} \quad r \notin E_7. \tag{3.4}$$

By (3.4) and Lemma 2.7(ii), we have $\mu_2(f) \geq \mu(A_0)$. On the other hand, by Lemma 2.9, we have $\mu_2(f) \leq \mu(A_0)$, therefore every nontrivial solution f of (1.3) satisfies $\mu_2(f) = \mu(A_0)$. Thus Theorem 1.13 hold.

4. Proof of Theorem 1.15

Suppose that $A_s(z)$ has deficiency $\delta(a, A_s) = 2d > 0$ at $a \in C$ as stated in the hypothesis. Then it follows from the definition of deficiency that for all sufficiently large r, we have

$$m\Big(r, \frac{1}{A_s - a}\Big) \geq dT(r, A_s).$$

Hence, for any sufficiently large r, there exists a point $z_r = re^{i\theta_r}$ such that

$$\log |A_s(z_r) - a| \leq -dT(r, A_s). \tag{4.1}$$

Assume first that $A_s(z)$ has zero as a deficient value, that is, $a = 0$. By Lemma 2.10, for any given $l (0 < l < 1/2)$ and for sufficiently small $\zeta > 0$, there exists a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for all $r \in E_\zeta$ and for all $\theta \in [\theta_r - l, \theta_r + l]$, then we have

$$\log |A_s(re^{i\theta})| \leq 0.$$

In fact, if we choose l sufficiently small in (2.16), we have

$$\begin{aligned}
\log |A_s(re^{i\theta})| &= \log |A_s(re^{i\theta_r})| + \int_{\theta_r}^{\theta} \frac{d}{dt} \log |A_s(re^{it})| dt \\
&\leq -dT(r, A_s) + r \int_{\theta_r}^{\theta} |\frac{A_s'(re^{it})}{A_s(re^{it})}||dt| \\
&\leq (-d + \varepsilon_1) T(r, A_s) \leq 0,
\end{aligned}$$

where $0 < \varepsilon_1 < d$. In general, if $A_s(z)$ has a finite deficient value $a \in C$, then we can apply the same reasoning as above to the function $A_s(z) - a$ since it has zero as a deficient value. Hence, for sufficiently small $\zeta > 0$ and for sufficiently small $l > 0$, there exists a set $E_\zeta \subset [1, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for all $r \in E_\zeta$ and for all $\theta \in [\theta_r - l, \theta_r + l]$, we have

$$\log |A_s(re^{i\theta}) - a| \leq 0.$$

Thus for these r and θ, we have

$$|A_s(re^{i\theta})| \leq |a| + 1. \tag{4.2}$$

Since $0 < \mu(A_0) < 1/2$, we divide the proof into two cases: (i) $0 < \mu(A_0) = \sigma(A_0) < 1/2$; (ii) $0 < \mu(A_0) < 1/2, \mu(A_0) < \sigma(A_0)$.

Case (i): $0 < \mu(A_0) = \sigma(A_0) < 1/2$. By Lemma 2.2, there exists a set $H_1 \subset [1, \infty)$ of lower logarithmic density greater than 0 such that for any given $\varepsilon_2 > 0$ and for all $r \in H_1$, we have

$$|A_0(z)| > \exp\{r^{\mu(A_0) - \varepsilon_2}\}. \tag{4.3}$$

Let $f \not\equiv 0$ be a solution of (1.3). From (1.3), we obtain

$$|A_0(z)| \leq |\frac{f^{(k)}(z)}{f(z)}| + \cdots + |A_s(z)\frac{f^{(s)}(z)}{f(z)}| + \cdots + |A_1(z)\frac{f'(z)}{f(z)}|. \tag{4.4}$$

By Lemma 2.1(ii), there exists a set $E_1 \subset [0, 2\pi)$ with linear measure zero such that for all $z = re^{i\theta}$ satisfying $\arg z = \theta \in [0, 2\pi)\backslash E_1$ and for all sufficiently large r, we have

$$|\frac{f^{(j)}(re^{i\theta})}{f(re^{i\theta})}| \leq r[T(2r, f)]^k, \quad (j = 1, \dots, k). \tag{4.5}$$

Furthermore, choosing ε_2 small enough such that $\max\{\sigma(A_j), j \neq 0, s\} = \beta < \mu(A_0) - 2\varepsilon_2$, then for sufficiently large r, we have

$$|A_j(z)| < \exp\{r^{\beta+\varepsilon_2}\}, \quad (j \neq 0, s). \tag{4.6}$$

Hence by (4.2)-(4.6), for all sufficiently large $r \in E_\zeta \bigcap H_1$ and for all $\theta \in [\theta_r - l, \theta_r + l]\backslash E_1$, we have

$$\exp\{r^{\mu(A_0)-\varepsilon_2}\} \leq kr \exp\{r^{\beta+\varepsilon_2}\}[T(2r, f)]^k. \tag{4.7}$$

Since ε_2 is arbitrarily small and $\beta + \varepsilon_2 < \mu(A_0) - \varepsilon_2$, by (4.7), we have $\sigma_2(f) \geq \mu(A_0)$. On the other hand, by Lemma 2.11, $\sigma_2(f) \leq \sigma(A_s) = \max\{\sigma(A_0), \sigma(A_s)\}$. Therefore, every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) \leq \sigma_2(f) \leq \sigma(A_s) = \max\{\sigma(A_0), \sigma(A_s)\}$.

Case (ii) $0 < \mu(A_0) < 1/2, \mu(A_0) < \sigma(A_0)$. By Lemma 2.3, there exists a set $H_2 \subset [1, \infty)$ of upper logarithmic density greater than 0 such that for any given $\delta(\mu(A_0) \leq \delta < \min(\sigma, \frac{1}{2}))$ and for all $r \in H_2$, we have

$$|A_0(z)| > \exp\{r^\delta\}. \tag{4.8}$$

Note that the set $E_\zeta \cap H_2$ has a positive upper logarithmic density. In fact, without loss of generality, set $\overline{\log\operatorname{dens}}(H_2) = 2\zeta > 0$, we have

$$\zeta \leq \overline{\log\operatorname{dens}}(H_2) + \underline{\log\operatorname{dens}}(E_\zeta) - \overline{\log\operatorname{dens}}(E_\zeta \cup H_2) \leq \overline{\log\operatorname{dens}}(E_\zeta \cap H_2).$$

By the same reasoning, we know that the set $E_\zeta \cap H_1$ in (4.7) also has a positive upper logarithmic density. Hence from (4.4)-(4.6) and (4.8), for all sufficiently large r in $E_\zeta \cap H_2$ and for all $\theta \in [\theta_r - l, \theta_r + l]\backslash E_1$, we have

$$\exp\{r^\delta\} \leq kr \exp\{r^{\beta+\varepsilon_2}\}[T(2r, f)]^k, \tag{4.9}$$

where $0 < \varepsilon_2 < \delta - \beta$. By (4.9), we get $\sigma_2(f) \geq \delta$, since δ is arbitrarily close to $\mu(A_0)$, we have $\sigma_2(f) \geq \mu(A_0)$. On the other hand, by Lemma 2.11, we have $\sigma_2(f) \leq \max\{\sigma(A_0), \sigma(A_s)\}$. Therefore, every solution $f \not\equiv 0$ of (1.3) satisfies $\mu(A_0) \leq \sigma_2(f) \leq \max\{\sigma(A_0), \sigma(A_s)\}$.

References

[1] P. D. Barry; *On a theorem of Besicovitch.* Quart. J. Nath. Oxford, Ser.(2), 14 (1963), 293-302.
[2] P. D. Barry; *Some theorems related to the cosπρ theorem.* Proc. London Math. Soc., (3), 21 (1970), 334-360.
[3] P. D. Barry; *The minimum modulus of small integral and subharmonic functions*, Proc. London Math. Soc. (3), 12, (1962), 445-495.
[4] L. G. Bernal; *On growth k-order of solutions of a complex homogeneous linear differential equations.* Proc Amer. Math. Soc., 101 (1987), 317-322.
[5] Z. X. Chen and C. C. Yang; *Some further results on the zeros and growth of entire solutions of second order differential equations.* Kodai Math. J., 22 (1999), 273-285.
[6] Z. X. Chen and C. C. Yang; *Quantitative estimations on the zeros and growths of entire solutions of linear differential equations.* Complex Var., 42 (2000), 119-133.
[7] W. Fuchs; *Proof of a conjecture of G. Pólya concerning gap series.* Illinois J. Math., 7 (1963), 661-667.

[8] G. Gundersen; *Estimates for the logarithmic derivative of a meromorphic funtion. plus similar estimates*. J. London Math. Soc., 305(2), (1998), 88-104.
[9] G. Gundersen; *Finite order solutions of second order linear differential equations*. Trans. Amer. Math. Soc., 305 (1988), 415-429.
[10] W. Hayman; *Meromorphic Functions*, Clarendon press, Oxford, 1964.
[11] W. Hayman; *The local growth of power series: a survey of the Wiman-Valiron method*. Canad. Math. Bull., 17 (1974), 317-358.
[12] J. Heittokangas, R. Korhonen, and Jouni Rättyä; *Growth estimates for solutions of nonhomogeneous linear complex differential equations*. Ann. Acad. Sci. Fenn. 34 (2009), 145-156.
[13] S. Hellerstein, J. Miles and J. Rossi; *On the growth of solutions of* $f'' + gf' + hf = 0$. Trans. Amer. Math. Soc., 324 (1991), 693-706.
[14] S. Hellerstein, J. Miles and J. Rossi; *On the growth of solutions of certain linear differential equations*. Ann. Acad. Sci. Fenn. Ser A. I. Math., 17 (1992), 343-365.
[15] G. Jank and L. Volkmann; *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel-Boston, 1985.
[16] L. Kinnunen; *Linear differential equations with solutions of finite iterated order*. Southeast Asian Bull. Math. (4) 22 (1998) 385-405.
[17] T. Kobayashi; *On the deficiency of an entire function of finite genus*. Kodai Math. Sem. Rep., 27 (1976), 320-328.
[18] I. Laine; *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Math., Vol. 15, Walter de Gruyter, Berlin, New York, 1993.
[19] J. Tu and G. T. Deng; *Growth of solutions of certain higher order linear differential equations*. Complex Var. Elliptic Equ., (7), 53 (2008), 2693-2703.
[20] J. Tu and Z. X. Chen; *Growth of solution of complex differential equations with meromorphic coefficients of finite iterated order*. Southeast Asian Bull. Math., 33 (2009), 153-164.
[21] H. X. Yi and C. C. Yang; *The Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 2003.

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