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POSITIVE SOLUTIONS FOR SECOND-ORDER SINGULAR THREE-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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Abstract. In this article, we study the existence and uniqueness of the positive solution for a second-order singular three-point boundary-value problem with sign-changing nonlinearities. Our main tool is a fixed-point theorem.

1. Introduction

In this article, we consider the second-order boundary-value problem

$$x''(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta), \quad 0 < \eta < 1, \quad 0 < \alpha < 1. \tag{1.2}$$

The singularity may appear at $t = 0$, $x = 0$ and the function f may be superlinear at $x = \infty$ and change sign.

Webb [6] employed the fixed-point index for compact maps to investigate the existence of at least one positive solution for the second-order boundary-value problem

$$\begin{gathered} x''(t) + g(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{1.3}$$

where $0 < \eta < 1$, $0 < \alpha\eta < 1$, and $f_0 = \limsup_{x\to 0} \frac{f(x)}{x}$, $f_\infty = \liminf_{x\to\infty} \frac{f(x)}{x}$ exist and $g(t) > 0$. Moreover, when $g(t)$ is a sign-changing function in $[0, 1]$ and f is nondecreasing and without any singular points, using the fixed point theorem of strict-set-contractions, Bing Liu [3] established the existence of at least two positive solutions for (1.3). When $g(t) > 0$ and f is a given sign-changing function without any singular points and any monotonicity, using the increasing operator theory and approximation process, Xian Xu [8] showed at least three solutions for the three-point boundary-value problem (1.3).

In addition, the existence of solutions of nonlinear multi-point boundary-value problems have been studied by many other authors; the readers are referred to [3, 4, 9, 10] and the references therein.

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Motivated by [2, 12], the purpose of this article is to examine the existence and the uniqueness of the positive solution of (1.1)-(1.2) under the assumption that f may be singular at $t = 0$, $x = 0$ and be superlinear at $x = \infty$ and change sign. There are only a few papers considering (1.1)-(1.2) under this assumptions. We try to fill this gap in the literature with this paper.

In this article, we use the following assumptions:

(H1) $f(t, x) \in C((0, 1] \times (0, +\infty), (-\infty, +\infty))$,
(H2) $k(t), a(t), b(t) \in C((0, 1], (0, +\infty))$, $tk(t) \in L(0, 1]$,
(H3) there exist $F(x) \in C((0, +\infty), (0, +\infty))$, $G(x) \in C([0, +\infty), [0, +\infty))$ such that $f(t, x) \leq k(t)(F(x) + G(x))$.
(S1) $f(t, x) \geq a(t)$ hold for $0 < x < b(t)$, $x \in C[0, 1]$,
(S2) $F(x)$ is decreasing in $(0, +\infty)$,
(S3) there exist $R > 1$, such that $\int_1^R \frac{dy}{F(y)} \cdot (1 + \frac{\bar{G}(R)}{F(R)})^{-1} > \int_0^1 sk(s)ds$, where $\bar{G}(R) = \max_{s\in[0,R]} G(s)$.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we obtain the existence of at least one positive solution for (1.1)-(1.2), and show an application of our results.

2. Preliminaries

Lemma 2.1 ([1]). *Let E be a Banach space, $R > 0$, $B_R = \{x \in E : \|x\| \leq R\}$, $F : B_R \to E$ be a completely continuous operator. If $x \neq \lambda F(x)$ for any $x \in E$ with $\|x\| = R$ and $0 < \lambda < 1$, then F has a fixed point in B_R.*

Let $n > [\frac{1}{\eta} + 1]$ be a natural number, $d_n = \min\{b(t) : t \in [\frac{1}{n}, 1]\}$, $b_n = \min\{d_n, \frac{1}{n}\}$, $C_n = \{x : x \in C[\frac{1}{n}, 1]\}$ with norm $\|x\| = \max\{|x(t)|, \frac{1}{n} \leq t \leq 1\}$. It is easy to see that $(C_n, \|\cdot\|)$ is a Banach space.

Inspired by [12], we define T_n as

$$(T_n x)(t) = b_n + \int_{\frac{1}{n}}^1 G_{\frac{1}{n},1}(t, s) f(s, \max\{b_n, x(s)\})ds, \quad x \in C_n,\ t \in [\frac{1}{n}, 1],$$

where

$$G_{\frac{1}{n},1}(t, s) = \begin{cases} G_1(t, s), & \frac{1}{n} < \eta \leq s, \\ G_2(t, s), & \frac{1}{n} \leq s \leq \eta, \end{cases}$$

$$G_1(t, s) = \begin{cases} \frac{1}{1-\alpha\eta-(1-\alpha)\frac{1}{n}}(1 - s)(t - \frac{1}{n}), & \frac{1}{n} \leq t \leq s \leq 1, \\ \frac{1}{1-\alpha\eta-(1-\alpha)\frac{1}{n}}[\alpha(t - s)(\eta - \frac{1}{n}) - (t - 1)(s - \frac{1}{n})], & \eta \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t, s) = \begin{cases} \frac{(1-\alpha\eta)(t-\frac{1}{n})-s(1-\alpha)(t-\frac{1}{n})}{1-\alpha\eta-(1-\alpha)\frac{1}{n}}, & \frac{1}{n} \leq t \leq s \leq 1, \\ \frac{(1-\alpha\eta)(s-\frac{1}{n})-t(1-\alpha)(s-\frac{1}{n})}{1-\alpha\eta-(1-\alpha)\frac{1}{n}}, & \frac{1}{n} \leq s \leq t \leq 1, \end{cases}$$

and $G_{\frac{1}{n},1}(t, s)$ is Green's function to the boundary-value problem

$$x''(t) = 0, \quad \frac{1}{n} < t < 1,$$
$$x(\frac{1}{n}) = 0, \quad x(1) = \alpha x(\eta), \quad 0 < \alpha < 1, \quad 0 < \eta < 1.$$

By a standard argument we have the following result; see for example [7].

Lemma 2.2. *The operator T_n is completely continuous from C_n to C_n.*

Lemma 2.3. *There exist $x_n \in C_n$, $b_n \le x_n(t) \le R$ for $t \in [\frac{1}{n}, 1]$ such that*

$$x_n(t) = b_n + \int_{\frac{1}{n}}^{1} G_{\frac{1}{n},1}(t,s) f(s, x_n(s))ds, \quad t \in [\frac{1}{n}, 1]. \tag{2.1}$$

Proof. We prove that

$$x(t) \ne \lambda(T_n x)(t) = \lambda b_n + \lambda \int_{\frac{1}{n}}^{1} G_{\frac{1}{n},1}(t,s) f(s, \max\{b_n, x(s)\})ds, \quad t \in [\frac{1}{n}, 1], \tag{2.2}$$

for any $\|x\| = R$ and $\lambda \in (0,1)$. In fact, if (2.2) is not true, there exist $x \in C_n$ with $\|x\| = R$ and $0 < \lambda < 1$ such that

$$x(t) = \lambda(T_n x)(t) = \lambda b_n + \lambda \int_{\frac{1}{n}}^{1} G_{\frac{1}{n},1}(t,s) f(s, \max\{b_n, x(s)\})ds, \quad t \in [\frac{1}{n}, 1]. \tag{2.3}$$

It is easy to see that $x(\frac{1}{n}) = \lambda b_n$, $x(1) - \alpha x(\eta) = (1-\alpha)\lambda b_n$.

We first claim that $x(t) \ge \lambda b_n$ for any $t \in [\frac{1}{n}, 1]$. In fact if $x(\eta) < \lambda b_n$, we have $x(1) = \lambda b_n + \alpha x(\eta) - \alpha\lambda b_n < \lambda b_n$ and $x(\eta) < x(1)$. Since $x(\frac{1}{n}) = \lambda b_n > x(1)$, we can get a point $t_1 \in (\frac{1}{n}, \eta)$ such that $x(t_1) = x(1)$. Let $\gamma = \sup\{t_1 :\ t_1 \in (\frac{1}{n}, \eta), x(t_1) = x(1)\}$. It follows that $x(\gamma) = x(1)$ and $x(t) < x(\gamma) = x(1)$, $t \in (\gamma, \eta)$. Since $x(\eta) < x(1) < \lambda b_n$, we have two cases:
Case (1). There exist $t_1' \in (\eta, 1)$ such that $x(1) \le x(t_1')$. and
Case (2). $x(t) < x(1)$ for all $t \in (\eta, 1)$.

In case (1), we may get a point $t_2 \in (\eta, t_1')$ such that $x(t_2) = x(1)$. Setting $\beta = \inf\{t_2 : t_2 \in (\eta, 1), x(t_2) = x(1)\}$, we get $x(\beta) = x(1)$ and $x(t) < x(\beta) = x(1), t \in (\eta, \beta)$. In case (2), setting $\beta = 1$, we also get $x(\beta) = x(1)$ and $x(t) < x(\beta) = x(1), t \in (\eta, \beta)$. Hence, there exist an interval $[\gamma, \beta] \subseteq (\frac{1}{n}, 1](\gamma < \beta)$ such that

$$x(\gamma) = x(\beta) < \lambda b_n, x(t) < x(\gamma), x(t) < x(\beta), \quad t \in (\gamma, \beta). \tag{2.4}$$

By (2.3) and (S1), we have $x''(t) = -\lambda f(t, b_n) < 0$, $t \in [\gamma, \beta]$ and $x(t)$ is concave down on $[\gamma, \beta]$, which contradicts (2.4). Hence $x(\eta) \ge \lambda b_n$, and then $x(1) \ge \lambda b_n, x(1) \le x(\eta)$. If there exist $t_2' \in (\frac{1}{n}, \eta)$ such that $x(t_2') < \lambda b_n$, a similar argument as before yields an interval $[\gamma', \beta'] \subseteq [\frac{1}{n}, \eta](\gamma' < \beta')$, such that

$$x(t) < x(\gamma'), \quad x(t) < x(\beta'), \quad t \in (\gamma', \beta'), \quad x(\gamma') \le \lambda b_n, \quad x(\beta') \le \lambda b_n. \tag{2.5}$$

It follows from (2.3) and (S1) that $x''(t) = -\lambda f(t, b_n) < 0, t \in [\gamma', \beta']$ and $x(t)$ is concave down on $[\gamma', \beta']$, which contradicts (2.5). So we have $x(t) \ge \lambda b_n$, $t \in [\frac{1}{n}, \eta]$. By the same argument used for $t \in [\frac{1}{n}, \eta]$, we can easily show that $x(t) \ge \lambda b_n, t \in [\eta, 1]$.

Next we claim that: for any $z \in (\frac{1}{n}, 1)$, if $b_n < x(z) < R$, we have

$$\int_{b_n}^{x(z)} \frac{dx}{F(x)} \le \big(1 + \frac{\bar{G}(R)}{F(R)}\big) \int_0^z \int_t^1 k(s)\, ds\, dt. \tag{2.6}$$

Since $x(\frac{1}{n}) = \lambda b_n < R$, $x(1) \le x(\eta)$, there exist $t^* \in (\frac{1}{n}, 1)$ such that $x(t^*) = R$, $x'(t^*) = 0$. Setting $t' = \inf\{t^* : t^* \in (\frac{1}{n}, 1), x'(t^*) = 0, x(t^*) = \|x\| = R\}$, we obtain $t' \in (\frac{1}{n}, 1), x'(t') = 0, x(t') = \|x\| = R$. Obviously there exist $t'' \in (\frac{1}{n}, t')$ such that $x(t'') = b_n$. Furthermore we get a countable set $\{t_i\}$ of $(\frac{1}{n}, 1)$ such that

(1) $t'' = t_1 < t_2 \le t_3 < t_4 \le t_5 < \ldots \le t_{2m-1} < t_{2m} \le \ldots < 1$, $t_{2m} \to t'$,

(2) $x(t_1) = b_n$, $x(t_{2i}) = x(t_{2i+1})$, $x'(t_{2i}) = 0$, $i = 1, 2, 3 \dots$,
(3) $x(t)$ is strictly increasing in $[t_{2i-1}, t_{2i}]$, $i = 1, 2, 3 \dots$ (if $x(t)$ is strictly increasing in $[t'', t']$, put $m = 1$; i.e, $[t_1, t_2] = [t'', t']$).

Differentiating (2.3) and using the assumptions, we obtain easily

$$\begin{aligned} -x''(t) &= \lambda f(t, x(t)) \le \lambda k(t)(F(x(t)) + G(x(t))) \\ &= \lambda k(t)F(x(t))(1 + \frac{G(x(t))}{F(x(t))}) \\ &< k(t)F(x(t))(1 + \frac{\bar{G}(R)}{F(x(t))}) \\ &\le k(t)F(x(t))(1 + \frac{\bar{G}(R)}{F(R)}), \quad t \in [t_{2i-1}, t_{2i}),\ i = 1, 2, 3 \dots . \end{aligned} \tag{2.7}$$

Integrating (2.7) from t to t_{2i}, we have by the decreasing property of $F(x)$,

$$\begin{aligned} -\int_t^{t_{2i}} x''(s)ds &\le (1 + \frac{\bar{G}(R)}{F(R)}) \int_t^{t_{2i}} k(s)F(x(s))ds \\ &\le F(x(t))(1 + \frac{\bar{G}(R)}{F(R)}) \int_t^{t_{2i}} k(s)ds, \end{aligned} \tag{2.8}$$

for $t \in [t_{2i-1}, t_{2i})$, $i = 1, 2, 3 \dots$; that is to say

$$x'(t) \le F(x(t))(1 + \frac{\bar{G}(R)}{F(R)}) \int_t^{t_{2i}} k(s)ds, \quad t \in [t_{2i-1}, t_{2i}),\ i = 1, 2, 3 \dots . \tag{2.9}$$

It follows from (2.9) that

$$\frac{x'(t)}{F(x(t))} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_t^{t_{2i}} k(s)ds \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_t^1 k(s)ds, \tag{2.10}$$

for $t \in [t_{2i-1}, t_{2i})$, $i = 1, 2, 3 \dots$.

On the other hand, we can choose i_0 and $z' \in (\frac{1}{n}, 1), z' \le z$ such that $z' \in [t_{2i_0-1}, t_{2i_0})$ and $x(z') = x(z)$. Integrating (2.10) from t_{2i-1} to t_{2i}, $i = 1, 2, 3 \dots i_0 - 1$ and from t_{2i_0-1} to z', we have

$$\int_{x(t_{2i-1})}^{x(t_{2i})} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i-1}}^{t_{2i}} \int_t^1 k(s)\, ds\, dt, \quad i = 1, 2, 3 \dots i_0 - 1, \tag{2.11}$$

and

$$\int_{x(t_{2i_0-1})}^{x(z')} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i_0-1}}^{z'} \int_t^1 k(s)\, ds\, dt. \tag{2.12}$$

Summing (2.11) from 1 to $i_0 - 1$, we have by (2.12) and $x(t_{2i}) = x(t_{2i+1})$, that

$$\int_{b_n}^{x(z')} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^{z'} \int_t^1 k(s)\, ds\, dt \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^z \int_t^1 k(s)\, ds\, dt.$$

Since $x(z) = x(z')$,

$$\int_{b_n}^{x(z)} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^z \int_t^1 k(s)\, ds\, dt;$$

i.e, (2.6) holds. Letting $z \to t'$ in (2.6), we have

$$\begin{aligned}
\int_{b_n}^{R} \frac{dx}{F(x)} &\le (1+\frac{\bar{G}(R)}{F(R)}) \int_0^{t'} \int_t^1 k(s)\,ds\,dt \\
&\le (1+\frac{\bar{G}(R)}{F(R)}) \int_0^{1} \int_t^1 k(s)\,ds\,dt \\
&= (1+\frac{\bar{G}(R)}{F(R)}) \int_0^{1} sk(s)ds.
\end{aligned} \tag{2.13}$$

The inequality above contradicts $\int_1^R \frac{dx}{F(x)} > (1+\frac{\bar{G}(R)}{F(R)})\int_0^1 sk(s)ds$. Hence (2.2) holds.

It follows from Lemma 2.1 and (2.2) that T_n has a fixed point x_n in C_n. Using x_n and 1 in the place of x and λ in (2.2), we obtain easily $b_n \le x_n(t) \le R, t \in [\frac{1}{n}, 1]$. The proof is complete. □

Lemma 2.4. *For a fixed $h \in (0, \min\{\frac{1}{2}, \eta\})$, suppose $m_{n,h} = \min\{x_n(t), t \in [h,1]\}$. Then $m_h = \inf\{m_{n,h}\} > 0$.*

Proof. Since $x_n(t) \ge b_n > 0$, we get $m_h \ge 0$. For any fixed natural numbers n ($n > [\frac{1}{\eta}] + 1$), let $t_n \in [h,1]$ such that $x_n(t_n) = \min\{x_n(t), t \in [h,1]\}$. If $m_h = 0$, there exist a countable set $\{n_i\}$ such that

$$\lim_{n_i \to +\infty} x_{n_i}(t_{n_i}) = 0. \tag{2.14}$$

So there exist N such that $x_{n_i}(t_{n_i}) \le \min\{b(t), t \in [\frac{h}{2}, 1]\}$, $n_i > N$. Then we have two cases.

Case 1. There exist $n_k \in \{n_i\}, n_k > N$ and $t^*_{n_k} \in [\frac{h}{2}, h]$ such that $x_{n_k}(t^*_{n_k}) \ge x_{n_k}(t_{n_k})$. By the same argument in Lemma 2.3, we can get $t'_{n_k}, t''_{n_k} \in [\frac{h}{2}, 1], t'_{n_k} < t''_{n_k}$ such that

$$x_{n_k}(t) \le \min\{b(t), t \in [\frac{h}{2}, 1]\}, \quad t \in [t'_{n_k}, t''_{n_k}],$$

$$x_{n_k}(t) \le x_{n_k}(t'_{n_k}), x_{n_k}(t) \le x_{n_k}(t''_{n_k}), \quad t \in (t'_{n_k}, t''_{n_k}), \tag{2.15}$$

$$x''_{n_k}(t) = -f(t, x_{n_k}(t)) < 0, \quad t \in (t'_{n_k}, t''_{n_k}). \tag{2.16}$$

Inequality (2.15) shows that $x_{n_k}(t)$ is concave down in $[t'_{n_k}, t''_{n_k}]$, which contradicts (2.16).

Case 2. $x_{n_i}(t) < x_{n_i}(t_{n_i})$, $t \in [\frac{h}{2}, h]$ for any $n_i \in \{n_i\}$, $n_i > N$. And so we have

$$\lim_{n_i \to +\infty} x_{n_i}(t) = 0, \quad t \in [\frac{h}{2}, h]. \tag{2.17}$$

On the other hand for any $t \in [\frac{h}{2}, h]$,

$$\begin{aligned}
x_{n_i}(t) &= \frac{2}{h}\int_{\frac{h}{2}}^{t} (t-\frac{h}{2})(h-s)f(s, x_{n_i}(s))ds \\
&\quad + \frac{2}{h}\int_t^h (s-\frac{h}{2})(h-t)f(s, x_{n_i}(s))ds + x_{n_i}(\frac{h}{2}) + x_{n_i}(h) \\
&\ge \frac{2}{h}[\int_{\frac{h}{2}}^{t} (t-\frac{h}{2})(h-s)a(s)ds + \int_t^h (s-\frac{h}{2})(h-t)a(s)ds] > 0,
\end{aligned} \tag{2.18}$$

which contradicts (2.17). The proof is complete. $\square$

3. Main Result

Theorem 3.1. *If* (S1)–(S3) *hold, the three-point boundary-value problem* (1.1)-(1.2) *has at least one positive solution.*

Proof. For any natural numbers $n \geq [\frac{1}{\eta}+1]$, it follows from Lemma 2.3 that there exist $x_n \in C_n, b_n \leq x_n \leq R$ satisfying (2.1). Now we divide the proof into three steps.

Step 1. There exist a convergent subsequence of $\{x_n\}$ in (0,1]. For a natural number $k \geq \max\{3, [\frac{1}{\eta}]+1\}$, it follows from Lemma 2.4 that $0 < m_{\frac{1}{k}} \leq x_n(t) \leq R$, $t \in [\frac{1}{k}, 1]$ for any natural numbers $n \geq [\frac{1}{\eta}+1]$; i.e., $\{x_n\}$ is uniformly bounded in $[\frac{1}{k}, 1]$. Since x_n also satisfies

$$\begin{aligned}
x_n(t) = &-\int_{\frac{1}{k}}^{t}(t-s)f(s,x_n(s))ds \\
&+ \frac{t-\frac{1}{k}}{1-\alpha\eta-\frac{1}{k}(1-\alpha)}[\int_{\frac{1}{k}}^{1}(1-s)f(s,x_n(s))ds - \alpha\int_{\frac{1}{k}}^{\eta}(\eta-s)f(s,x_n(s))ds] \\
&+ x_n(\frac{1}{k}) + \frac{(t-\frac{1}{k})(1-\alpha)}{1-\alpha\eta-\frac{1}{k}(1-\alpha)}(b_n - x_n(\frac{1}{k})), \quad t \in [\frac{1}{k}, 1],
\end{aligned}$$

we have

$$\begin{aligned}
x_n'(t) = &-\int_{\frac{1}{k}}^{t} f(s,x_n(s))ds + \frac{\int_{\frac{1}{k}}^{1}(1-s)f(s,x_n(s))ds - \alpha\int_{\frac{1}{k}}^{\eta}(\eta-s)f(s,x_n(s))ds}{1-\alpha\eta-\frac{1}{k}(1-\alpha)} \\
&+ \frac{(1-\alpha)(b_n - x_n(t))}{1-\alpha\eta-\frac{1}{k}(1-\alpha)}, \quad t \in [\frac{1}{k}, 1].
\end{aligned}$$

Obviously

$$|x_n'(t)| \leq \frac{3-\eta}{1-\eta}\max\{|f(t,x(t))| : (t,x) \in [\frac{1}{k}, 1] \times [m_{\frac{1}{k}}, R]\} + \frac{2R}{1-\eta}, \tag{3.1}$$

for $t \in [\frac{1}{k}, 1]$. It follows from inequality (3.1) that $\{x_n\}$ is equicontinuous in $[\frac{1}{k}, 1]$. The Ascoli-Arzela theorem guarantees that there exists a subsequence of $\{x_n(t)\}$ which converges uniformly on $[\frac{1}{k}, 1]$. We may choose the diagonal sequence $\{x_k^{(k)}(t)\}$ (see more details in [13]) which converges everywhere in $(0,1]$ and it is easy to verify that $\{x_k^{(k)}(t)\}$ converges uniformly on any interval $[c,d] \subseteq (0,1]$. Without loss of generality, let $\{x_k^{(k)}(t)\}$ be $\{x_n(t)\}$ in the rest. Putting $x(t) = \lim_{n\to+\infty} x_n(t), t \in (0,1]$, we have $x(t)$ is continuous in $(0,1]$ and $x(t) \geq m_h > 0, t \in (0,1]$ by Lemma 2.4.

Step 2. $x(t)$ satisfies (1.1). Fixed $t \in (0,1]$, we may choose $h \in (0, \min\{\frac{1}{2}, \eta\})$ such that $t \in (h,1]$ and

$$\begin{aligned} x_n(t) = &- \int_h^t (t-s) f(s, x_n(s)) ds \\ &+ \frac{t-h}{1-\alpha\eta - h(1-\alpha)} [\int_h^1 (1-s) f(s, x_n(s)) ds - \alpha \int_h^\eta (\eta - s) f(s, x_n(s)) ds] \\ &+ x_n(h) + \frac{(t-h)(1-\alpha)}{1-\alpha\eta - h(1-\alpha)} (b_n - x_n(h)), \quad t \in (h,1]. \end{aligned} \tag{3.2}$$

Letting $n \to +\infty$ in (3.2), we have

$$\begin{aligned} x(t) = &- \int_h^t (t-s) f(s, x(s)) ds + \frac{t-h}{1-\alpha\eta - h(1-\alpha)} \\ &\times [\int_h^1 (1-s) f(s, x(s)) ds - \alpha \int_h^\eta (\eta - s) f(s, x(s)) ds] \\ &+ x(h) + \frac{(t-h)(1-\alpha)}{1-\alpha\eta - h(1-\alpha)} (-x(h)), \quad t \in (h,1]. \end{aligned} \tag{3.3}$$

Differentiating (3.3), we get the desired result.

Step 3. $x(t)$ satisfies (1.2). Let

$$t_n = \inf\{t : x_n(t) = \|x_n\|, x_n'(t) = 0, t \in [\frac{1}{n}, 1]\},$$

where $\|x_n\| = \max_{\frac{1}{n} \le t \le 1} x_n(t) \le R$. Then

$$t_n \in [\frac{1}{n}, 1], \quad x_n(t_n) = \|x_n\|, \quad x_n'(t_n) = 0.$$

Using $x_n(t), 1$ and t_n in place of $x(t), \lambda$ and t' in Lemma 2.3, we obtain easily by (2.13)

$$\int_{b_n}^{\|x_n\|} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^{t_n} \int_t^1 k(s)\, ds\, dt. \tag{3.4}$$

It follows from (3.4) and Lemma 2.4 that $0 < a = \inf\{t_n\} \le 1$. Fixed $z \in (0,a)$, then $b_n < x_n(z) < \|x_n\| \le R$. By Lemma 2.3 we easily get

$$\int_{b_n}^{x_n(z)} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^{z} \int_t^1 k(s)\, ds\, dt, \quad z \in (0,a). \tag{3.5}$$

Letting $n \to +\infty$ in (3.5) and noticing $b_n \to 0$, we have

$$\int_{0}^{x(z)} \frac{dx}{F(x)} \le (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^{z} \int_t^1 k(s)\, ds\, dt, \quad z \in (0,a). \tag{3.6}$$

It follows from (3.6) that $x(0) = \lim_{z \to 0^+} x(z) = 0$. Using 1 in place of λ in (2.3), we obtain easily

$$x_n(1) = \alpha x_n(\eta) + (1-\alpha) b_n. \tag{3.7}$$

Letting $n \to +\infty$, we have $x(1) = \alpha x(\eta)$. This complete the proof. □

When $G(x) \equiv 0$ in (H3), it is easy to see that the assumption (S3) is satisfied by the decreasing property of $F(x)$. Then under the assumption $G(x) \equiv 0$ we get the following corollaries to Theorem 3.1.

Corollary 3.2. *Suppose* (S1), (S2) *hold. Then* (1.1)-(1.2) *has at least one positive solution*

Corollary 3.3. *Suppose the assumptions of Corollary 3.2 hold. If further $f(t,\cdot)$ is non-increasing in $(0,+\infty)$ for each $t \in (0,1)$, the solution of* (1.1)-(1.2) *is unique.*

Proof. Suppose $x_1(t)$ and $x_2(t)$ are two solutions of (1.1)-(1.2). We need to prove that $x_1(t) \equiv x_2(t), t \in [0,1]$. Let $z(t) = x_1(t) - x_2(t), t \in [0,1]$. It follows that $z(0) = 0, z(1) = \alpha z(\eta)$. We first show that $x_1(\eta) = x_2(\eta)$, which implies that $x_1(1) = x_2(1)$. In fact, if it is not true, without loss of generality, we can suppose $x_1(\eta) > x_2(\eta)$. That is to say $z(\eta) > 0, 0 < z(1) = \alpha z(\eta) < z(\eta)$. Setting $t_1 = \max\{t \in (0,\eta), z(t) = z(1)\}$ and $t_2 = \min\{t \in (\eta,1), z(t) = z(1)\}$, we get

$$z(t_1) = z(t_2) = z(1), \quad z(t) = x_1(t) - x_2(t) > z(1) > 0, \quad t \in (t_1,t_2).$$

Letting $s(t) = z(t) - z(1)$, we have that $s(t_1) = s(t_2) = 0$ and $s(t) > 0, t \in (t_1,t_2)$. It follows from (1.1) and the monotonicity of $f(t,\cdot)$ that $s''(t) = z''(t) \geq 0, t \in (t_1,t_2)$. An elementary form of the maximum principle implies $s(t) \leq 0$ for all $t \in (t_1,t_2)$ and hence a contradiction. Then, $x_1(\eta) = x_2(\eta)$, which also yields that $x_1(1) = x_2(1)$. That is to say $z(0) = z(\eta) = z(1) = 0$.

We next claim that $x_1(t) = x_2(t), t \in (0,\eta)$. In fact, if it is not true, without loss of generality, we can get $x_1(t_0) > x_2(t_0)$ for some $t_0 \in (0,\eta)$. Let $t_3 = \max\{t \in (0,t_0), z(t) = 0\}, t_4 = \min\{t \in (t_0,\eta), z(t) = 0\}$(note $z(\eta) = 0$)). Then $z(t_3) = z(t_4) = 0$ and $z(t) > 0, t \in (t_3,t_4)$. Let $s_1(t) = z_1(t) - z_2(t)$, $t \in [t_3,t_4]$. Then $s_1(t) > 0$ for all $t \in [t_3,t_4]$. On the other hand, the monotonicity of $f(t,\cdot)$ implies that $s_1''(t) \geq 0, t \in (t_3,t_4)$. An elementary form of the maximum principle implies $s_1(t) \leq 0$ for all $t \in (t_3,t_4)$ and hence a contradiction.

The same argument yields that $x_1(t) = x_2(t), t \in (\eta,1)$. Hence we get $x_1(t) = x_2(t), t \in [0,1]$. Thus the result is proved. □

Example. Consider the second order singular three-point boundary-value problem

$$x''(t) + \frac{1}{4}(x^2(t) + \frac{1}{x^2(t)} - \frac{x^3(t)}{t^5} - \frac{1}{t^2}) = 0, \quad 0 < t < 1, \tag{3.8}$$

$$x(0) = 0, \quad x(1) = \frac{1}{3}x(\frac{1}{4}). \tag{3.9}$$

Set $\alpha = \frac{1}{3}$, $\eta = \frac{1}{4}$,

$$f(t,x) = \frac{1}{4}(x^2 + \frac{1}{x^2} - \frac{x^3}{t^5} - \frac{1}{t^2}), \quad k(t) = \frac{1}{4}, \quad F(x) = \frac{1}{x^2},$$
$$G(x) = x^2, \quad a(t) = \frac{1}{4t^2}, \quad b(t) = \frac{t}{2}.$$

It is easy to prove that $f(t,x) \leq k(t)(F(x)+G(x))$ and (S1)–(S3) hold. By Theorem 3.1, the three-point boundary-value problem (3.8)-(3.9) has at least one positive solution. Moreover, if $f(t,x) = \frac{1}{x^2(t)} - \frac{x^3(t)}{t^5}$ in (3.8), the three-point boundary-value problem (3.8)-(3.9) has only one positive solution by Corollaries 3.2 and 3.3.

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