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KRASNOSELSKII-TYPE FIXED POINT THEOREMS UNDER WEAK TOPOLOGY SETTINGS AND APPLICATIONS

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Abstract. In this article, we establish some fixed point results of Krasnoselskii type for the sum $T + S$, where S is weakly continuous and T may not be continuous. Some of the main results complement and encompass the previous ones. As an application, we study the existence of solution to one parameter operator equations. Finally, our results are used to prove the existence of solution for integral equations in reflexive Banach spaces.

1. Introduction

Recently, more and more authors are interested in the study of the existence of solutions of nonlinear abstract operator equation or the fixed point of a sum of two operators of the form

$$Sx + Tx = x, \quad x \in K, \tag{1.1}$$

where K is a closed and convex subset of a Banach space E. The reason is that, on the one hand, varieties of problems arising from the fields of natural science, when modelled under the mathematical viewpoint, involve the study of solutions of (1.1); on the other hand, several analysis and topological situations in the theory and applications of nonlinear operator lead also to the investigation of fixed point of (1.1). Especially, many problems in integral equations can be formulated in terms of (1.1). Krasnoselskii's fixed point theorem appeared as a prototyped for solving such equations. Motivated by an observation that inversion of a perturbed differential operator may yield the sum of a contraction and a compact operator, Krasnoselskii [8, 9] proved the following theorem.

Theorem 1.1. *Let K be a nonempty closed convex subset of a Banach space E. Suppose that T and S map K into E such that*

(i) *S is continuous and $S(K)$ is contained in a compact subset of E;*
(ii) *T is a contraction with constant $\alpha < 1$;*
(iii) *Any $x, y \in K$ imply $Tx + Sy \in K$.*

Then there exists $x^* \in K$ with $Sx^* + Tx^* = x^*$.

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Since the above theorem was published there have appeared a huge number of papers contributing generalizations or modifications of the Krasnoselskii's fixed point theorem and their applications, see [2, 3, 4, 6, 9, 12, 13, 14, 15, 16] and the references therein. Meanwhile, a large class of problems, for instance in integral equations and stability theory, have been adapted by the Krasnoselskii's fixed point method. Several improvements of Theorem 1.1 have been made in the literature in the course of time by modifying assumption (i), (ii) or (iii). For example, see [2, 3, 16]. It has been mentioned in [2] that the condition (iii) is too strong and hence the author, Burton, proposed the following improvement for

(iii) If $x = Tx + Sy$ with $y \in K$, then $x \in K$.

Subsequently, if T is a bounded linear operator on E, in [3], Barroso introduced the following asymptotic requirement for (iii):

If $\lambda \in (0, 1)$ and $x = \lambda Tx + Sy$ for some $y \in K$, then $x \in K$.

More recently, in [16], the authors firstly considered that the map T is expansive rather than contractive, and then relaxed the compactness of the operator S by a k-set contractive assumption.

Based on the well known fact that infinite dimensional Banach spaces are not locally compact and some practical equations of the form (1.1) may encounter the problem that the operators involved may not be continuous, inspired by papers [4, 16] and the mentioned works, in this paper, we continue to study (1.1) in the setting of locally convex (weak) topology. We should mention that other authors have already studied (1.1) in locally convex spaces [4, 6, 14, 15]. As the condition (i) involves continuity and compactness, thus, we would like to replace the continuity and the compactness by weakly continuity and weakly compactness, respectively.

The condition (ii) is also, in some sense, a little limited and artificial, since, on the one hand, this condition implies norm-continuous; on the other hand, why T can not be other type? In this article we consider a more general condition besides contraction, which includes expansive mapping and other type ones. For example, see Theorems 2.9 and 2.14, Corollaries 2.11, 2.17, 2.18 and 2.19. We also investigate some modifications on the condition (iii). The point of this paper is that we replace the contractiveness of T by the expansiveness of T in the setting of weak topology and derive some new fixed point results, some of which complement and encompass the corresponding results of [3, 4, 14]. Finally, a new existence criterion for integral equations in reflexive Banach spaces is obtained.

The remainder of this paper is organized as follows. In section 2, we state the main results and show their proofs. In section 3, we apply these fixed point results to study the existence of a solution to one parameter operator equations of the form

$$\lambda Tx + Sx = x, \quad \lambda \geq 0, \quad x \in E.$$

To illustrate the theories, our final purpose is to prove the existence of a solution to the following nonlinear integral equations of the form

$$u(t) = f(u) + \int_0^T g(s, u(s))ds, \quad t \in [0, T], \tag{1.2}$$

where u takes values in a reflexive Banach space E. By imposing some conditions on f and g (see section 4), we are able to establish the existence of a solution to (1.2).

2. Fixed point theorems for the sum of operators

At the beginning we recall several basic definitions and concepts used further on. Let $(E, \|\cdot\|)$ be a Banach space. For a sequence $\{x_n\} \subset E$ and $x \in E$ we write $x_n \to x$ whenever the sequence $\{x_n\}$ converges to x (in the norm $\|\cdot\|$). If $\{x_n\}$ converges weakly to x we will write that $x_n \rightharpoonup x$.

Next, let $X \subset E$ be a nonempty set. An operator $T : X \to E$ is said to be sequentially weakly continuous on the set X if for every sequence $\{x_n\} \subset X$ and $x \in X$ such that $x_n \rightharpoonup x$ we have that $Tx_n \rightharpoonup Tx$.

Due to the Eberlin-Šmulian's theorem ([7, Theorem 8.12.4]), it is well known that if a set K is weak compact, then each sequentially weakly continuous mapping $T : K \to E$ is weakly continuous. Therefore, it may be possible to solve equations of the form (1.1) in the weak topology setting by suitable fixed points results. As a tool to this intention, we rely on the following version of Schauder fixed point principle which was obtained by Arino, Gautier and Penot [1].

Lemma 2.1. *Let K be a weakly compact convex subset of a Banach space E. Then each sequentially weakly continuous map $T : K \to K$ has a fixed point in K.*

Before stating the main results we need some definitions and lemmas.

Definition 2.2. Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \to X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y), \quad \forall x, y \in M. \tag{2.1}$$

In the sequel, we shall employ the following two lemmas which have been established in [16].

Lemma 2.3. *Let M be a closed subset of a complete metric space X. Assume that the mapping $T : M \to X$ is expansive and $T(M) \supset M$, then there exists a unique point $x^* \in M$ such that $Tx^* = x^*$.*

Lemma 2.4. *Let $(X, \|.\|)$ be a linear normed space, $M \subset X$. Suppose that the mapping $T : M \to X$ is expansive with constant $h > 1$. Then the inverse of $F := I - T : M \to (I - T)(M)$ exists and*

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{h-1}\|x - y\|, x, y \in F(M). \tag{2.2}$$

Definition 2.5. Let M, K be two subsets of a linear normed space X, $T : M \to X$ and $S : K \to X$ two mappings. We denote by $\mathbb{F} = \mathbb{F}(M, K; T, S)$ the set

$$\mathbb{F} = \big\{x \in M : x = Tx + Sy \text{ for some } y \in K\big\}.$$

We are now ready to state and prove the first main result of this article.

Theorem 2.6. *Let $K \subset E$ be a nonempty closed convex subset. Suppose that T and S map K into E such that*

- (i) *S is sequentially weakly continuous;*
- (ii) *T is an expansive mapping;*
- (iii) *$z \in S(K)$ implies $T(K) + z \supset K$, where $T(K) + z = \{y + z \mid \; y \in T(K)\}$;*
- (iv) *If $\{x_n\}$ is a sequence in $\mathbb{F}(K, K; T, S)$ such that $x_n \rightharpoonup x$ and $Tx_n \rightharpoonup y$, then $y = Tx$;*
- (v) *The set $\mathbb{F}(K, K; T, S)$ is relatively weakly compact.*

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Proof. From (ii) and (iii), for each $y \in K$, we see that the mapping $T+Sy : K \to E$ satisfies the assumptions of Lemma 2.3. Therefore, the equation

$$Tx + Sy = x \tag{2.3}$$

has a unique solution $x = \tau(Sy) \in K$, so that the mapping $\tau S : K \to K$ given by $y \to \tau Sy$ is well-defined. In view of Lemma 2.4, we obtain that $\tau Sy = (I-T)^{-1}Sy$ for all $y \in K$. In addition, we observe that $\tau S(K) \subset \mathbb{F} \subset K$. We claim that τS is sequentially weakly continuous in K. To see this, let $\{x_n\}$ be a sequence in K with $x_n \rightharpoonup x$ in K. Notice that $\tau S(x_n) \in \mathbb{F}$. Thus, up to a subsequence, we may assume by (v) that $\tau S(x_n) \rightharpoonup y$ for some $y \in K$. It follows from (i) that $Sx_n \rightharpoonup Sx$. From the equality

$$\tau Sx_n = T(\tau Sx_n) + Sx_n, \tag{2.4}$$

passing the weak limit in (2.4) yields

$$T(\tau Sx_n) \rightharpoonup y - Sx.$$

The assumption (iv) now implies that $y - Sx = Ty$; i.e., $y = \tau Sx$ since $x \in K$. This proves the assertion. Let the set $C = \overline{co}(\mathbb{F})$, where $\overline{co}(\mathbb{F})$ denotes the closed convex hull of $\mathbb{F}$. Then $C \subset K$ and is a weakly compact set by the Krein-Šmulian theorem. Furthermore, it is straightforward to see that τS maps C into C. In virtue of Lemma 2.1, there exists $x^* \in C$ such that $\tau Sx^* = x^*$. From (2.3) we deduce that

$$T(\tau Sx^*) + Sx^* = \tau Sx^*;$$

that is, $Tx^* + Sx^* = x^*$. The proof is complete. □

Remark 2.7. We note that T may not be continuous since it is only expansive. If $T : K \to E$ is a contraction, then a similar result can be found in [4]. Hence Theorem 2.6 complements [4, Theorem 2.9]. The proof presented here are analogous to the arguments in [4].

Corollary 2.8. *Under the conditions of Theorem 2.6, if only the condition (iii) of Theorem 2.6 is replaced by that T maps K onto E, then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.*

It is worthy of pointing out that the condition (iii) may be a litter restrictive and the next result might be regarded as an improvement of Theorem 2.6.

Theorem 2.9. *Let $K \subset E$ be a nonempty closed convex subset. Suppose that $T : E \to E$ and $S : K \to E$ such that*

- (i) *S is sequentially weakly continuous;*
- (ii) *T is an expansive mapping;*
- (iii) *$S(K) \subset (I-T)(E)$ and $[x = Tx + Sy, y \in K] \Longrightarrow x \in K$ (or $S(K) \subset (I-T)(K)$);*
- (iv) *If $\{x_n\}$ is a sequence in $\mathbb{F}(E, K; T, S)$ such that $x_n \rightharpoonup x$ and $Tx_n \rightharpoonup y$, then $y = Tx$;*
- (v) *The set $\mathbb{F}(E, K; T, S)$ is relatively weakly compact.*

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Proof. For each $y \in K$, by (iii), there exists $x \in E$ such that $x - Tx = Sy$. By Lemma 2.4 and the second part of (iii), we have $x = (I-T)^{-1}Sy \in K$. As is shown in Theorem 2.6, one obtains that $(I-T)^{-1}S : K \to K$ is sequentially weakly continuous and there is a point $x^* \in K$ with $x^* = (I-T)^{-1}Sx^*$. This completes the proof. $\square$

Let us now state some consequences of Theorem 2.9. First, the case when E is a reflexive Banach space is considered, so that a closed, convex and bounded set is weakly compact. Rechecking the proof of Theorem 2.6, we find that it is only required $\overline{co}(\mathbb{F})$ to be weakly compact.

Corollary 2.10. *Suppose that the conditions (i)-(iv) of Theorem 2.9 for T and S are fulfilled. If $\mathbb{F}(E,K;T,S)$ is a bounded subset of a reflexive Banach space E, then $T+S$ has at least one fixed point in K.*

The second consequence of Theorem 2.9 is concerned the case when T is non-contractive on $M \subset E$, i.e., $\|Tx - Ty\| \geq \|x - y\|$ for all $x, y \in M$.

Corollary 2.11. *Let $K \subset E$ be a nonempty convex and weakly compact subset. Suppose that $T : E \to E$ and $S : K \to E$ are sequentially weakly continuous such that*

(i) *T is non-contractive on E (or K);*
(ii) *There is a sequence $\lambda_n > 1$ with $\lambda_n \to 1$ such that $S(K) \subset (I - \lambda_n T)(E)$ and $[x = \lambda_n Tx + Sy, y \in K] \Longrightarrow x \in K$ (or $S(K) \subset (I - \lambda_n T)(K)$).*

Then $T+S$ has a fixed point in K.

Proof. Notice that $\lambda_n T : E \to E$ is expansive with constant $\lambda_n > 1$. By Theorem 2.9, there exists $x_n^* \in K$ such that

$$Sx_n^* + \lambda_n Tx_n^* = x_n^*. \tag{2.5}$$

Up to a subsequence we may assume that $x_n^* \rightharpoonup x^*$ in K since K is convex and weakly compact. Passing the weak limit in (2.5) we complete the proof. $\square$

Given by Lemma 2.4, Theorem 2.9 and [4, Theorem 2.9], the following weak type Krasnoselskii fixed point theorem may be easily formulated, which clearly contains, but not limited to (see Remarks 2.13 and 2.20), Theorem 2.9 and [4, Theorem 2.9].

Theorem 2.12. *Let $K \subset E$ be a nonempty closed convex subset. Suppose that $T : E \to E$ and $S : K \to E$ such that*

(i) *S is sequentially weakly continuous;*
(ii) *$(I - T)$ is one-to-one;*
(iii) *$S(K) \subset (I - T)(E)$ and $[x = Tx + Sy, y \in K] \Longrightarrow x \in K$ (or $S(K) \subset (I - T)(K)$);*
(iv) *If $\{x_n\}$ is a sequence in $\mathbb{F}(E,K;T,S)$ such that $x_n \rightharpoonup x$ and $Tx_n \rightharpoonup y$, then $y = Tx$;*
(v) *The set $\mathbb{F}(E,K;T,S)$ is relatively weakly compact.*

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Remark 2.13. If $T : E \to E$ is a contraction mapping, then $(I - T)(E) = E$ and hence $S(K) \subset (I-T)(E)$. It can be easily seen by (ii) and (iii) that $\mathbb{F}(E,K;T,S) = (I-T)^{-1}S(K)$. It has been shown under the assumptions of [14] that $\mathbb{F}$ is relatively weakly compact. Therefore, Theorem 2.12 also encompasses the main result of [14, Theorem 2.1]. Moreover, the condition (iv) is weaker than the condition that T is sequentially weakly continuous.

For a given $r > 0$, let B_r denote the set $\{x \in E : \|x\| \leq r\}$. Taking advantage of the linearity of the operator T, we derive the following result.

Theorem 2.14. *Let E be a reflexive Banach space, $T : E \to E$ a linear operator and $S : E \to E$ a sequentially weakly continuous map. Assume that the following conditions are satisfied.*

- (i) *$(I - T)$ is continuously invertible;*
- (ii) *There exists $R > 0$ such that $S(B_R) \subset B_{\beta R}$, where $\beta \leq \|(I - T)^{-1}\|^{-1}$;*
- (iii) *$S(B_R) \subset (I - T)(E)$.*

Then $T + S$ possesses a fixed point in B_R.

Proof. Let $F = I - T : E \to (I - T)(E)$. By (i), one can easily see from the fact that T is linear and $\beta \leq \|(I - T)^{-1}\|^{-1}$ that

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{\beta}\|x - y\|, \quad \forall x, y \in F(E). \tag{2.6}$$

It follows from (2.6) that $F^{-1} : F(E) \to E$ is continuous. Recall that F^{-1} being linear implies that F^{-1} is weakly continuous. Consequently, one knows from (iii) that $F^{-1}S : B_R \to E$ is sequentially weakly continuous. For any $x \in B_R$, one easily derive from (2.6) and (ii) that $\|F^{-1}Sx\| \leq R$. Hence, $F^{-1}S$ maps B_R into itself. Applying Lemma 2.1, we obtain that $F^{-1}S$ has a fixed point in B_R. This completes the proof. $\square$

Next, we shall present some concrete mappings which fulfil the condition (i) of Theorem 2.14. Before stating the consequences, we introduce the following two lemmas. The first one is known, its proof can be directly shown or founded in [16].

Lemma 2.15. *Let $(X, \|.\|)$ be a linear normed space, $M \subset X$. Assume that the mapping $T : M \to X$ is contractive with constant $\alpha < 1$, then the inverse of $F := I - T : M \to (I - T)(M)$ exists and*

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{1 - \alpha}\|x - y\|, x, y \in F(M). \tag{2.7}$$

The second one is as follows, we shall provide all the details for the sake of convenience.

Lemma 2.16. *Let E be a Banach space. Assume that $T : E \to E$ is linear and bounded and T^p is a contraction for some $p \in \mathbb{N}$. Then $(I - T)$ maps E onto E, the inverse of $F := I - T : E \to E$ exists and*

$$\|F^{-1}x - F^{-1}y\| \leq \gamma_p\|x - y\|, x, y \in E, \tag{2.8}$$

where

$$\gamma_p = \begin{cases} \frac{p}{1 - \|T^p\|}, & \text{if } \|T\| = 1, \\ \frac{1}{1 - \|T\|}, & \text{if } \|T\| < 1, \\ \frac{\|T\|^p - 1}{(1 - \|T^p\|)(\|T\| - 1)}, & \text{if } \|T\| > 1. \end{cases}$$

Proof. Let $y \in E$ be fixed and define the map $T_y : E \to E$ by

$$T_y x = Tx + y.$$

We first show that T_y^p is a contraction. To this end, let $x_1, x_2 \in E$. Notice that T is linear. One has

$$\|T_y x_1 - T_y x_2\| = \|Tx_1 - Tx_2\|.$$

Again

$$\|T_y^2 x_1 - T_y^2 x_2\| = \|T^2 x_1 - T^2 x_2\|.$$

By induction,

$$\|T_y^p x_1 - T_y^p x_2\| = \|T^p x_1 - T^p x_2\| \le \|T^p\|\|x_1 - x_2\|.$$

So T_y^p is a contraction on E. Next, we claim that both $(I-T)$ and $(I-T^p)$ map E onto E. Indeed, by Banach contraction mapping principle, there is a unique $x^* \in E$ such that $T_y^p x^* = x^*$. It then follows that $T_y x^*$ is also a fixed point of T_y^p. In view of uniqueness, we obtain that $T_y x^* = x^*$ and x^* is the unique fixed point of T_y. Hence, we have

$$(I-T)x^* = y,$$

which implies that $(I-T)$ maps E onto E. It is clear that $(I-T^p)$ maps E onto E. The claim is proved. Next, for each $x, y \in E$ and $x \neq y$, one easily obtain that

$$\|(I-T^p)x - (I-T^p)y\| \ge (1 - \|T^p\|)\|x-y\| > 0,$$

which shows that $(I-T^p)$ is one-to-one. Summing the above arguments, we derive that $(I-T^p)^{-1}$ exists on E. Therefore, we infer that $(I-T)^{-1}$ exists on E due to the fact that

$$(I-T)^{-1} = (I-T^p)^{-1} \sum_{k=0}^{p-1} T^k. \tag{2.9}$$

Since T^p is a contraction, we know from (2.7) that

$$\|(I-T^p)^{-1}\| \le \frac{1}{1-\|T^p\|}. \tag{2.10}$$

We conclude from Lemma 2.15, (2.9) and (2.10) that

$$\|(I-T)^{-1}\| \le \begin{cases} \frac{p}{1-\|T^p\|}, & \text{if } \|T\| = 1, \\ \frac{1}{1-\|T\|}, & \text{if } \|T\| < 1, \\ \frac{\|T\|^p - 1}{(1-\|T^p\|)(\|T\|-1)}, & \text{if } \|T\| > 1. \end{cases} \tag{2.11}$$

This proves the lemma. □

Together Lemmas 2.4, 2.15, 2.16 and Theorem 2.3 immediately yield the following results.

Corollary 2.17. *Let E, S be the same as Theorem 2.14. Assume that $T : E \to E$ is a linear expansion with constant $h > 1$ such that $S(B_R) \subset B_{(h-1)R}$ for some $R > 0$ and $S(B_R) \subset (I-T)(E)$. Then fixed point for $T + S$ is achieved in B_R.*

Corollary 2.18. *Let E, S be the same as Theorem 2.14. Assume that $T : E \to E$ is a linear contraction with constant $\alpha < 1$ such that $S(B_R) \subset B_{(1-\alpha)R}$ for some $R > 0$. Then the equation $Tx + Sx = x$ has at least one solution in B_R.*

Corollary 2.19. *Let E, S be the same as Theorem 2.14. Assume that $T : E \to E$ is linear and bounded and T^p is a contraction for some $p \in \mathbb{N}$ such that $S(B_R) \subset B_{\gamma_p^{-1} R}$ for some $R > 0$, where γ_p is given in Lemma 2.16. Then the equation $Tx + Sx = x$ has at least one solution in B_R.*

Remark 2.20. Given by Lemma 2.16, it is easily verified that, under the conditions in [3, Theorem 2.1], all the assumptions of Theorem 2.12 are fulfilled. Furthermore, when $T \in \mathcal{L}(E)$ and $\|T^p\| \leq 1$ for some $p \geq 1$, instead of requiring $[x = Tx+Sy, y \in K] \Longrightarrow x \in K$, we assume the following condition holds in Theorem 2.12.

$$[\lambda \in (0,1) \text{ and } x = \lambda Tx + Sy, y \in K] \Longrightarrow x \in K.$$

Then Theorem 2.12 also covers the main result [3, Theorem 2.2]. However, it does not necessarily require that T is linear in Theorem 2.12.

Finally, inspired by the work of Barroso [5], we give the following asymptotic version of the Krasnoselskii fixed point theorem.

Theorem 2.21. *Let K, E, S, T and the conditions (ii), (iii) and (v) for S and T be the same as Theorem 2.12. In addition, assume that the following hypotheses are fulfilled.*

(a) *S is demicontinuous, that is, if $\{x_n\} \subset K$ and $x_n \to x$ then $Sx_n \rightharpoonup Sx$;*
(b) *T is sequentially weakly continuous and $T\theta = \theta$;*

Then there exists a sequence $\{u_n\}$ in K so that $(u_n - (S+T)u_n)_n$ converges weakly to zero.

Proof. Keeping the conditions (a) and (b) in mind, using the essentially same reasoning as in Theorem 2.6, one can show easily that $(I-T)^{-1}S : C \to C$ is demicontinuous, where $C = \overline{co}(\mathbb{F})$. Due to [5, Theorem 3.3] there is a sequence $\{u_n\}$ in C such that $u_n - (I-T)^{-1}Su_n \rightharpoonup \theta$, i.e., $(I-T)^{-1}[u_n - (S+T)u_n] \rightharpoonup \theta$. Invoking again the item (b), one can readily deduce that $u_n - (S+T)u_n \rightharpoonup \theta$. This ends the proof. $\square$

3. Fixed point results to one parameter operator equation

Throughout this section, E will denote a reflexive Banach space. The main purpose of this section is to present some existence results for the following nonlinear abstract operator equation in Banach spaces.

$$\lambda Tx + Sx = x, \tag{3.1}$$

where $T, S : E \to E$ and $\lambda \geq 0$ is a parameter. The first result concerning about (3.1) is as follows.

Theorem 3.1. *Let T and S map E into E being sequentially weakly continuous operators. Suppose that there exists $\lambda_0 > 0$ such that*

(i) *T is expansive with constant $h > 1$ and $S(B_R) \subset (I-\lambda T)(E)$ for all $\lambda \geq \lambda_0$;*
(ii) *$S(B_R) \subset \{x \in E : \|x+\lambda T\theta\| \leq (\lambda h-1)R\}$ for some $R > 0$ and all $\lambda \geq \lambda_0$.*

Then (3.1) is solvable for all $\lambda \geq \lambda_0$.

Proof. For each $\lambda \geq \lambda_0$, it follows from the first part of (i) that

$$\|(I-\lambda T)x - (I-\lambda T)y\| \geq (\lambda h - 1)\|x-y\|,$$

which implies that

$$\|(I-\lambda T)x + \lambda T\theta\| \geq (\lambda h - 1)\|x\|. \tag{3.2}$$

Assume now that $x = \lambda Tx + Sy$ with $y \in B_R$, then it follows from (3.2) and (ii) that

$$(\lambda h - 1)\|x\| \leq \|(I-\lambda T)x + \lambda T\theta\| = \|Sy + \lambda T\theta\| \leq (\lambda h - 1)R,$$

Thus $x \in B_R$, and hence, since B_R is weakly compact, it follows from Corollary 2.10 that $\lambda T + S$ has a fixed point in B_R. This completes the proof. $\square$

Remark 3.2. Particularly, in Theorem 3.1, if $T\theta = \theta$, then condition (ii) can be replaced by that $S(B_R) \subset \{x \in E : \|x\| \le (\lambda_0 h - 1)R\}$ for some $R > 0$. And the result of Theorem 3.1 also holds.

Corollary 3.3. *Assume that the condition (ii) of Theorem 3.1 holds. In addition, if T is expansive and onto, then* (3.1) *is solvable for all $\lambda \ge \lambda_0$.*

Next, we can modify some assumptions to study (3.1). Before proceeding to the theorem, we shall give a needed definition.

Definition 3.4. Let (X, d) be a metric space and M be a subset of X. A mapping $T : M \to X$ is said to be weakly expansive, if there exists a constant $\beta > 0$ such that

$$d(Tx, Ty) \ge \beta d(x, y), \quad \forall x, y \in M. \tag{3.3}$$

Remark 3.5. Clearly, if $\beta > 1$, then weakly expansive map is just an expansive one. If T is weakly expansive and satisfies similar assumptions as (i), (ii) in Theorem 3.1, then there exists $\lambda_1 \ge \lambda_0$ such that (3.1) has a solution for $\lambda \ge \lambda_1$.

Our second result of this section is as follows.

Theorem 3.6. *Suppose that T and $S : E \to E$ are sequentially weakly continuous operators such that*

(i) *T is weakly expansive with constant $h > 0$ and onto;*
(ii) *$S(B_R) \subset B_R$ and $\|T\theta\| < hR$ for some $R > 0$.*

Then there exists $\lambda_0 > 0$ such that (3.1) *is solvable for all $\lambda \ge \lambda_0$.*

Proof. We first choose $\lambda_1, \epsilon > 0$ such that $\lambda_1 h > 1$ and

$$\frac{\lambda h}{1+\epsilon} > 1, \quad \text{for all } \lambda \ge \lambda_1. \tag{3.4}$$

In view of $\|T\theta\| < hR$, for such a small ϵ there exists $\lambda_2 > 0$ such that

$$\lambda(hR - \|T\theta\|) \ge 2(1+\epsilon)R, \quad \text{for all} \quad \lambda \ge \lambda_2. \tag{3.5}$$

We define $T', S' : E \to E$ by

$$T'x = \frac{\lambda Tx}{1+\epsilon} \quad \text{and} \quad S'y = \frac{Sy + \epsilon y}{1+\epsilon}.$$

Then, T', S' are sequentially weakly continuous, S' maps B_R into itself, and it is easy to see from (3.3) and (3.4) that T' is expansive with constant $\lambda h/(1+\epsilon) > 1$ for $\lambda \ge \lambda_0$, where $\lambda_0 = \max\{\lambda_1, \lambda_2\}$. Together with the expression of T' and (i), Lemma 2.3 tells us that $I - T'$ maps E onto E. Therefore $S'(B_R) \subset (I - T')(E)$. Now, if $x = T'x + S'y$ with $y \in B_R$, then

$$(I - \lambda T)x = Sy + \epsilon y - \epsilon x. \tag{3.6}$$

From (3.2), (3.5) and (3.6), we deduce that

$$[\lambda h - (1+\epsilon)]\|x\| \le R + \lambda\|T\theta\| + \epsilon R \le [\lambda h - (1+\epsilon)]R, \quad \text{for } \lambda \ge \lambda_0.$$

Hence, $x \in B_R$. Applying Theorem 2.9, we know that $T' + S'$ has a fixed point in B_R. This completes the proof. $\square$

Theorem 3.7. *Let $T : E \to E$ be a linear weakly expansive mapping with constant $h > 0$ and $S : E \to E$ a bounded and sequentially weakly continuous operator. If there exist $R > 0$ and $\lambda_0 \geq 0$ such that $S(B_R) \subset (I - \lambda T)(E)$ for all $\lambda \geq \lambda_0$, then there exists $\lambda_1 \geq \lambda_0$ such that the equation $Sx + \lambda Tx = x$ is solvable in B_R for all $\lambda \geq \lambda_1$.*

Proof. Choose $\lambda_1' \geq \lambda_0$ so that $\lambda_1' h > 1$. Thus $\lambda T : E \to E$ is expansive with constant $\lambda h > 1$ for all $\lambda \geq \lambda_1'$. Let $F_\lambda = I - \lambda T$. By Lemma 2.4, we know that the inverse of $F_\lambda : E \to F_\lambda(E)$ exists and

$$\|F_\lambda^{-1}(x) - F_\lambda^{-1}(y)\| \leq \frac{1}{\lambda h - 1}\|x - y\|, \quad \forall x, y \in F_\lambda(E). \tag{3.7}$$

It follows from (3.7) that F_λ^{-1} is linearly bounded. So F_λ^{-1} is weakly continuous. Consequently, one knows from the assumptions that $F_\lambda^{-1}S : B_R \to E$ is sequentially weakly continuous. There is $\lambda_1 \geq \lambda_1'$ such that $\|Sx\| \leq (\lambda h - 1)R$ for all $x \in B_R$ and $\lambda \geq \lambda_1$ since S is bounded. Thus, we deduce from (3.7) that

$$\|F_\lambda^{-1}Sx\| \leq R, \quad \text{for all } x \in B_R \quad \text{and } \lambda \geq \lambda_1. \tag{3.8}$$

It follows from (3.8) that $F_\lambda^{-1}S$ maps B_R into itself. Using Lemma 2.1, we obtain that $F_\lambda^{-1}S$ has a fixed point in B_R for all $\lambda \geq \lambda_1$. The proof is complete. □

As for the Lipschitzian mapping, by analogous argument, we derive the following result.

Theorem 3.8. *Let $T : E \to E$ be a bounded linear operator and $S : E \to E$ a sequentially weakly continuous operator. Suppose that there exist $R > 0$ and $\lambda_0 \geq 0$ such that $S(B_R) \subset B_{(1-\lambda_0\|T\|)R}$. Then there exists $\lambda_1 \in [0, \lambda_0]$ such that* (3.1) *has at least one solution in B_R for all $\lambda \in [0, \lambda_1]$.*

Proof. For each $\lambda \in [0, \lambda_0]$, we have $\lambda\|T\| < 1$ and hence $\lambda T : E \to E$ is a contraction with constant $\lambda\|T\| < 1$. Let $F_\lambda = I - \lambda T$. One easily know from Lemma 2.15 that the inverse of $F_\lambda : E \to E$ exists and

$$\|F_\lambda^{-1}(x) - F_\lambda^{-1}(y)\| \leq \frac{1}{1 - \lambda\|T\|}\|x - y\|, \quad \forall x, y \in E. \tag{3.9}$$

One readily sees from (3.9) that F_λ^{-1} is weakly continuous. Therefore, $F_\lambda^{-1}S : E \to E$ is sequentially weakly continuous. One can obtain from the hypothesis that $\|Sx\| \leq (1 - \lambda\|T\|)R$ for all $x \in B_R$ and $\lambda \in [0, \lambda_0]$. We derive from (3.9) that

$$\|F_\lambda^{-1}Sx\| \leq R, \quad \text{for all } x \in B_R \quad \text{and } \lambda \in [0, \lambda_0]. \tag{3.10}$$

It follows from (3.10) that $F_\lambda^{-1}S$ maps B_R into itself. Invoking Lemma 2.1, we infer that $F_\lambda^{-1}S$ has a fixed point in B_R for $\lambda \in [0, \lambda_1]$. The proof is complete. □

Corollary 3.9. *Let T, S be the same as Theorem 3.4. Suppose that for each $x \in E$, we have $\|Sx\| \leq a\|x\|^p + b$, where $b \geq 0$, $0 < p \leq 1$, $a \in [0, 1)$ if $p = 1$; $a \geq 0$ if $0 < p < 1$. Then there exists $\lambda_1 \in [0, \lambda_0]$ such that* (3.1) *is solvable in B_R for all $\lambda \in [0, \lambda_1]$.*

Proof. For the case that $p = 1$, since $0 \leq a < 1$, there is $\lambda_0 \geq 0$ such that $a < 1 - \lambda_0\|T\|$. Obviously, there exists sufficiently large $R > 0$ such that

$$\frac{b}{R} \leq (1 - \lambda_0\|T\| - a). \tag{3.11}$$

It follows from (3.11) and the hypothesis that the conditions of Theorem 3.8 is satisfied.

Next, for the case that $p \in (0, 1)$, it suffices to choose $\lambda_0 \geq 0$ with $\lambda_0 \|T\| < 1$ and $R > 0$ such that $aR^p + b \leq (1 - \lambda_0 \|T\|)R$. This is obvious. □

Remark 3.10. The fixed point results of section 2 can be applied to study the eigenvalue problems of Krasnosel'skii-type in the critical case, that is, the map $T : M \subset E \to E$ is non-expansive. Their arguments are fully analogous to the discission presented in this section. Hence we omit it.

4. Application to integral equation

In this section, our aim is to present some existence results for the nonlinear integral equation

$$u(t) = f(u) + \int_0^T g(s, u(s))ds, \quad u \in C(J, E), \tag{4.1}$$

where E is a reflexive Banach space and $J = [0, T]$. The integral in (4.1) is understood to be the Pettis integral. To study (4.1), we assume for the remained of this section the following hypotheses are satisfied:

- (H1) $f : E \to E$ is sequentially weakly continuous and onto;
- (H2) $\|f(x) - f(y)\| \geq h\|x - y\|$, $(h \geq 2)$ for all $x, y \in E$; and f maps relatively weakly compact sets into bounded sets and is uniformly continuous on weakly compact sets;
- (H3) for any $t \in J$, the map $g_t = g(t, \cdot) : E \to E$ is sequentially weakly continuous;
- (H4) for each $x \in C(J, E)$, $g(\cdot, x(\cdot))$ is Pettis integrable on $[0, T]$;
- (H5) there exist $\alpha \in L^1[0, T]$ and a nondecreasing continuous function ϕ from $[0, \infty)$ to $(0, \infty)$ such that $\|g(t, x)\| \leq \alpha(t)\phi(\|x\|)$ for a.e. $t \in [0, T]$ and all $x \in E$. Further, assume that $\int_0^T \alpha(s)ds < \int_{\|f(\theta)\|}^\infty \frac{dr}{\phi(r)}$.

We now state and prove an existence principle for (4.1).

Theorem 4.1. *Suppose that the conditions (H1)-(H5) are fulfilled. Then* (4.1) *has at least one solution $u \in C(J, E)$.*

Proof. Put

$$\beta(t) = \int_{\|f(\theta)\|}^t \frac{dr}{\phi(r)} \quad \text{and} \quad b(t) = (h - 1)^{-1}\beta^{-1}\big(\int_0^t \alpha(s)ds\big).$$

Then

$$\int_{\|f(\theta)\|}^{(h-1)b(t)} \frac{dr}{\phi(r)} = \int_0^t \alpha(s)ds. \tag{4.2}$$

It follows from (4.2) and the final part of (H5) that $b(T) < \infty$. We define the set

$$K = \big\{x \in C(J, E) : \|x(t)\| \leq (h - 1)b(t) \text{ for all } t \in J\big\}.$$

Then K is a closed, convex and bounded subset of $C(J, E)$. Let us now introduce the nonlinear operators T and S as follows:

$$(Tx)(t) = f(x(t)) - f(\theta),$$

$$(Sy)(t) = f(\theta) + \int_0^t g(s, y(s))ds.$$

The conditions (H1) and (H4) imply that T and S are well defined on $C(J,E)$, respectively.

Our idea is to use Theorem 2.9 to find the fixed point for the sum $T+S$ in K. The proof will be shown in several steps.

Step 1: Prove that S maps K into K, $S(K)$ is equicontinuous and relatively weakly compact.

For any $y \in K$, we shall show that $Sy \in K$. Let $t \in J$ be fixed. Without loss of generality, we may assume that $(Sy)(t) \neq 0$. In view of the Hahn-Banach theorem there exists $y_t^* \in E^*$ with $\|y_t^*\| = 1$ such that $\langle y_t^*, (Sy)(t)\rangle = \|(Sy)(t)\|$. Thus, one can deduce from (H5) and (4.2) that

$$\begin{aligned} \|(Sy)(t)\| &= \langle y_t^*, f(\theta)\rangle + \int_0^t \langle y_t^*, g(s, y(s))\rangle ds \\ &\leq \|f(\theta)\| + \int_0^t \alpha(s)\phi(\|y(s)\|)ds \\ &\leq \|f(\theta)\| + \int_0^t \alpha(s)\phi((h-1)b(s))ds \\ &= \|f(\theta)\| + (h-1)\int_0^t b'(s)ds = (h-1)b(t). \end{aligned} \tag{4.3}$$

It shows from (4.3) that $S(K) \subset K$ and hence is bounded. This proves the first claim of Step 1. Next, let $t, s \in J$ with $s \neq t$. We may assume that $(Sy)(t) - (Sy)(s) \neq 0$. Then there exists $x_t^* \in E^*$ with $\|x_t^*\| = 1$ and $\langle x_t^*, (Sy)(t) - (Sy)(s)\rangle = \|(Sy)(t) - (Sy)(s)\|$. Consequently,

$$\begin{aligned} \|(Sy)(t) - (Sy)(s)\| &\leq \int_s^t \alpha(\tau)\phi(\|y(\tau)\|)d\tau \\ &\leq \int_s^t \alpha(\tau)\phi((h-1)b(\tau))d\tau \\ &\leq (h-1)\Big|\int_s^t b'(\tau)d\tau\Big| = (h-1)|b(t) - b(s)|. \end{aligned} \tag{4.4}$$

It follows from (4.4) that $S(K)$ is equicontinuous. The reflexiveness of E implies that $S(K)(t)$ is relatively weakly compact for each $t \in J$, where $S(K)(t) = \{z(t) : z \in S(K)\}$. By a known result (see [10, 11]), one can easily get that $S(K)$ is relatively weakly compact in $C(J,E)$. This completes Step 1.

Step 2: Prove that $S : K \to K$ is sequentially weakly continuous. Let $\{x_n\}$ be a sequence in K with $x_n \rightharpoonup x$ in $C(J,E)$, for some $x \in K$. Then $x_n(t) \rightharpoonup x(t)$ in E for all $t \in J$. Fix $t \in (0,T]$. From the item (H3) one sees that $g(t, x_n(t)) \rightharpoonup g(t, x(t))$ in E. Together with (H5) and the Lebesgue dominated convergence theorem for the Pettis integral yield for each $\varphi \in E^*$ that

$$\langle \varphi, (Sx_n)(t)\rangle \to \langle \varphi, (Sx)(t)\rangle;$$

i.e., $(Sx_n)(t) \rightharpoonup (Sx)(t)$ in E. We can do this for each $t \in J$ and notice that $S(K)$ is equicontinuous, and accordingly $Sx_n \rightharpoonup Sx$ by [11]. The Step 2 is proved.

Step 3: Prove that the conditions (ii) and (iii) of Theorem 2.9 hold. Since E is reflexive and f is continuous on weakly compact sets, it shows that T transforms $C(J,E)$ into itself. This, in conjunction with the first part of (H2), one easily gets that $T : C(J,E) \to C(J,E)$ is expansive with constant $h \geq 2$. For all $x, y \in C(J,E)$, one can see from the first part of (H2) that

$$\|(I-T)x(t)-(I-T)y(t)\| \geq (h-1)\|x(t)-y(t)\| \geq \|x(t)-y(t)\|,$$

where I is identity map. Thus, one has

$$\|(I-T)x(t)\| \geq (h-1)\|x(t)\| \geq \|x(t)\|, \quad \forall x \in C(J,E). \tag{4.5}$$

Assume now that $x = Tx + Sy$ for some $y \in K$. We conclude from (4.3) and (4.5) that

$$\|x(t)\| \leq \|(I-T)x(t)\| = \|(Sy)(t)\| \leq (h-1)b(t),$$

which shows that $x \in K$. Therefore, the second part of (iii) in Theorem 2.9 is fulfilled. Next, for each $y \in C(J,E)$, we define $T_y : C(J,E) \to C(J,E)$ by

$$(T_y x)(t) = (Tx)(t) + y(t).$$

Then T_y is expansive with constant $h \geq 2$ and onto since f maps E onto E. By Lemma 2.3, we know there exists $x^* \in C(J,E)$ such that $T_y x^* = x^*$, that is $(I-T)x^* = y$. Hence $S(K) \subset (I-T)(E)$. This completes Step 3.

Step 4: Prove that the condition (v) of Theorem 2.9 is satisfied. For each $x \in \mathbb{F}(E,K;T,S)$, then by the definition of $\mathbb{F}$ and Lemma 2.4 there exists $y \in K$ such that

$$x = (I-T)^{-1}Sy. \tag{4.6}$$

Hence, for $t, s \in J$, we obtain from Lemma 2.4, (4.6) and (4.4) that

$$\|x(t)-x(s)\| \leq |b(t)-b(s)|,$$

which illustrates that $\mathbb{F}(E,K;T,S)$ is equicontinuous in $C(J,E)$. Let $\{x_n\}$ be a sequence in $\mathbb{F}$. Then $\{x_n\}$ is equicontinuous in $C(J,E)$ and there exists $\{y_n\}$ in K with $x_n = Tx_n + Sy_n$. Thus, one has from (4.3) and (4.5) that

$$\|x_n(t)\| \leq \frac{1}{h-1}\|(Sy_n)(t)\| \leq b(t), \forall t \in J.$$

It follows that, for each $t \in J$, the set $\{x_n(t)\}$ is relatively weakly compact in E. The above discussion tells us that $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact. The Eberlein-Šmulian theorem implies that $\mathbb{F}$ is relatively weakly compact. This achieves Step 4

Step 5: Prove that T fulfils the condition (iv) of Theorem 2.2. By the second part of (H2) and the fact that $\mathbb{F}$ is relatively weakly compact we obtain that $T(\mathbb{F})$ is bounded. Again by the second part of (H2) and the fact that $\mathbb{F}$ is equicontinuous, one can readily deduce that $T(\mathbb{F})$ is also equicontinuous. Now, let $\{x_n\} \subset \mathbb{F}$ with $x_n \rightharpoonup x$ in $C(J,E)$ for some $x \in K$. It follows from (H1) that $(Tx_n)(t) \rightharpoonup (Tx)(t)$. Since $\{Tx_n : n \in \mathbb{N}\}$ is equicontinuous in $C(J,E)$, as before, we conclude that $Tx_n \rightharpoonup Tx$ in $C(J,E)$. The Step 5 is proved.

Now, invoking Theorem 2.9 we obtain that there is $x^* \in K$ with $Tx^* + Sx^* = x^*$; i.e., x^* is a solution to (4.1). This accomplishes the proof. $\square$

Remark 4.2. It is clearly seen that the following locally "Lipscitizan" type condition fulfills the second part of (H2): For each bounded subset U of E, there exists a continuous function $\psi_U : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_U(0) = 0$, such that $\|f(x)-f(y)\| \leq \psi_U(\|x-y\|)$ for all $x, y \in U$. Although the proof of Theorem 4.1 is analogous to that of [4, Theorem 5.1], it clarifies some vague points made in [4]. Moreover, it can be easily known that Theorem 4.1 does not contain the corresponding result of [4, Theorem 5.1], vice versa. Therefore, Theorem 4.1 and [4, Theorem 5.1] are complementary.

We conclude this artcile by presenting a class of maps which fulfil the assumptions (H1) and (H2) in Theorem 4.1. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and is coercive, i.e., f satisfies the inequality

$$(f(x) - f(y), x - y) \geq \alpha(\|x - y\|)\|x - y\|, \quad \forall x, y \in \mathbb{R}^n,$$

where $\alpha(0) = 0, \alpha(t) > 0$ for all $t > 0$; $\lim_{t\to\infty} \alpha(t) = \infty$. Then it is well known that f is surjective. Particularly, if $\alpha(t) = ht$, $h > 0$, then $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. Specifically, let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^k + hx + x_0$, where $h \geq 2$, k is a positive odd number, and x_0 is a given constant. Then f satisfies the assumptions (H1) and (H2) in Theorem 4.1 for $E = \mathbb{R}$.

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