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PARABOLIC EQUATIONS WITH ROBIN TYPE BOUNDARY CONDITIONS IN A NON-RECTANGULAR DOMAIN

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Abstract. In this article, we study the parabolic equation $\partial_t u - c^2(t)\partial_x^2 u = f$ in the non-necessarily rectangular domain

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\}.$$

The boundary conditions are of Robin type, while the right-hand side lies in the Lebesgue space $L^2(\Omega)$. Our aim is to find conditions on c and the functions $(\varphi_i)_{i=1,2}$ such that the solution belongs to the anisotropic Sobolev space $H^{1,2}(\Omega) = \{u \in L^2(\Omega) : \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\}$. For goal we use the method of approximation of domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be the triangular domain

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\},$$

where φ_1, φ_2 are the functions of parametrization with $\varphi_1(0) = \varphi_2(0)$, and T is a finite positive number. In Ω, we consider the boundary-value problem

$$\begin{gathered} \partial_t u - c^2(t)\partial_x^2 u = f \quad \text{in } L^2(\Omega) \\ b_i(t)\partial_x u + \alpha_i(t)u\big|_{x=\varphi_i(t)} = 0, \quad i = 1,2, \end{gathered} \tag{1.1}$$

where (α_i) and (b_i) are given. We look for conditions on the functions $(b_i, \alpha_i, \varphi_i)_{i=1,2}$ and the coefficient c such that (1.1) admits a unique solution u belonging to the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \{u \in L^2(\Omega) : \partial_t u, \partial_x u, \partial_x^2 u \in L^2(\Omega)\}.$$

We consider the case where $\alpha_i(t) \neq 0$ and $b_i(t) \neq 0$ for all $t \in]0,T[$. So, (1.1) may be written in the form

$$\begin{gathered} \partial_t u - c^2(t)\partial_x^2 u = f \quad \text{in } L^2(\Omega) \\ \partial_x u + \beta_i(t)u\big|_{\Gamma_i} = 0, \quad i = 1,2, \end{gathered} \tag{1.2}$$

where $\beta_i(t) = \frac{\alpha_i(t)}{b_i(t)}$, $\Gamma_i = \{(t, \varphi_i(t)), t \in]0,T[\}$, $i = 1,2$.

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In the sequel, the hypothesis

$$(-1)^i(c^2(t)\beta_i(t) - \frac{\varphi_i'(t)}{2}) \geq 0 \quad \text{a.e. } t \in]0,T[,\ i = 1,2, \tag{1.3}$$

is imposed in order to guarantee the uniqueness of the solution of (1.2). Indeed, if u is the solution of the (1.2) with a null right-hand side, the calculations show that the inner product $\langle \partial_t u - c^2(t)\partial_x^2 u, u\rangle$ in $L^2(\Omega)$ gives

$$\begin{aligned} 0 &= \sum_{i=1}^{2} \int_{\Gamma_i} (-1)^i \Big(c^2(t)\beta_i(t) - \frac{\varphi_i'(t)}{2}\Big) u^2(t,\varphi_i(t))dt \\ &\quad + \frac{1}{2}\int_{\Gamma_3} u^2 dx + \int_\Omega c^2.(\partial_x u)^2\, dt\, dx \end{aligned}$$

where $\Gamma_3 = \{(T,x) : \varphi_1(T) < x < \varphi_2(T)\}$ if $\varphi_1(T) \neq \varphi_2(T)$. The hypothesis (1.3) implies that $\partial_x u = 0$ and consequently $\partial_x^2 u = 0$. Then, (1.2) gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions and the fact that $\beta_i(t) \neq 0$ for all $t \in]0,T[$ imply $u = 0$.

We also assume that the functions $(\beta_i)_{i=1,2}$ satisfy the assumption

$$\beta_1(t) < 0, \quad \beta_2(t) > 0 \quad \text{for all } t \in [0,T]. \tag{1.4}$$

The most interesting point of the parabolic problem studied here is the fact that $\varphi_1(0) = \varphi_2(0)$ or $\varphi_1(T) = \varphi_2(T)$. In this case the domain Ω is not rectangular and cannot be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equation; see, for example Sadallah [7].

The solvability of this kind of problems with Cauchy-Dirichlet boundary conditions has been investigated in [3, 4, 8, 9]. In Sadallah [9], the same equation is studied by another approach making use of the so-called Schur's Lemma and gives the same result obtained in [8] by the *a priori* estimates technique. In [3] and [4], the authors deal with the heat equation (i.e., the case where $c(t) = 1$) set in a non-rectangular domain with a right-hand side taken in L^p, where $p \in]1,\infty[$, and have obtained optimal regularity results by the operators sum method. These results are generalized in [5] to a parabolic equation of the type

$$\partial_t u(t,x) - \partial_x^2 u(t,x) + \lambda m(t,x)u(t,x) = f(t,x)$$

where λ is a positive spectral parameter and $m(.)$ some positive weight functions. Hofmann and Lewis [2] have also considered the classical heat equation with Neumann boundary condition in noncylindrical domains satisfying some conditions of Lipschitz's type. The authors showed that the optimal L^p regularity holds for $p = 2$ and the situation gets progressively worse as p approaches 1. In Savaré [10], parabolic problems in noncylindrical domains are considered in the Hilbertian case. The author obtains some regularity results under assumption on the geometrical behavior of the boundary which cannot include our triangular domain.

The plan of this paper is as follows. In Section 2, we derive some technical lemmas which will allow us to prove an *a priori* estimate (in a sense to be defined later). In Section 3, there are two main steps. First, we prove that (1.2) admits a (unique) solution in the case of a domain which can be transformed into a rectangle. Secondly, for T small enough, we prove that the result holds true in the case of a triangular domain under some assumptions on the coefficient c and the functions $(\beta_i,\varphi_i)_{i=1,2}$ to be made more precise later on. The method used here is based on the approximation of the triangular domain by a sequence of subdomains $(\Omega_n)_n$

which can be transformed into regular domains (rectangles) and we establish an *a priori* estimate of the type

$$\|u_n\|_{H^{1,2}(\Omega_n)} \le K\|f\|_{L^2(\Omega_n)},$$

where u_n is the solution of (1.2) in Ω_n and K is a constant independent of n, which allows us to pass to the limit. Finally, in Section 4 we study (1.2) in the case where T is not necessarily small.

2. Preliminaries

Let $(\beta_i)_{i=1,2}$ be continuous real-valued functions on $]0,T[$. Assume that there exists a constant $l>0$ such that

$$\Big|\frac{(1+\beta_2(t))}{A(t)}\Big| \le l, \tag{2.1}$$

$$\Big|\frac{\beta_1(t)(1+\beta_2(t))}{A(t)}\Big| \le l, \tag{2.2}$$

where

$$A(t) = \beta_1(t)\beta_2(t) + \beta_1(t) - \beta_2(t) \ne 0, \tag{2.3}$$

for every $t \in]0,T[$.

Lemma 2.1. *Assume that β_1 and β_2 fulfil the conditions (2.1), (2.2) and (2.3). Then, for a fixed $t \in]0,1[$, there exists a positive constant K_1 independent of t, such that for each $u \in H^2_\gamma(0,1)$*

$$\|u^{(j)}\|_{L^2(0,1)} \le K_1\|u^{(2)}\|_{L^2(0,1)}, j=0,1,$$

where

$$H^2_\gamma(0,1) = \{u \in H^2(0,1) : u'(0)+\beta_1(t)u(0)=0,\ u'(1)+\beta_2(t)u(1)=0\}.$$

Proof. Let $t \in]0,1[$ and f an arbitrary fixed element of $L^2(0,1)$. Then the solution of the problem

$$\begin{gathered} u''=f \\ u'(0)+\beta_1(t)u(0)=0 \\ u'(1)+\beta_2(t)u(1)=0, \end{gathered}$$

can be written in the form

$$u(y) = \int_0^y \Big\{\int_0^x f(s)ds\Big\}dx + yu'(0) + u(0),$$

where

$$\begin{gathered} u(0) = \frac{\int_0^1 f(s)ds + \beta_2(t)\int_0^1\{\int_0^x f(s)ds\}dx}{A(t)} \\ u'(0) = -\beta_1(t)u(0). \end{gathered}$$

The uniqueness of the solution is easy to check, thanks to the boundary conditions and the condition (2.3).

Using the Cauchy-Schwarz inequality, we obtain the following two estimates

$$|u(0)| \leq C|\frac{(1+\beta_2(t))}{A(t)}|\|f\|_{L^2(0,1)}$$
$$|u'(0)| \leq C|\frac{\beta_1(t)(1+\beta_2(t))}{A(t)}|\|f\|_{L^2(0,1)},$$

which will allow us to obtain the desired estimates, thanks to the conditions (2.1), (2.2). □

Lemma 2.2. *Under the assumptions* (2.1), (2.2) *and* (2.3) *on* $(\beta_i)_{i=1,2}$ *and for a fixed* $t \in]0,1[$, *there exists a constant* C_1 *(independent of* a *and* b*) such that*

$$\|v^{(j)}\|^2_{L^2(a,b)} \leq C_1(b-a)^{2(2-j)}\|v^{(2)}\|^2_{L^2(a,b)}, \quad j=0,1,$$

for each $v \in H^2_\gamma(a,b)$, *with*

$$H^2_\gamma(a,b) = \{v \in H^2(a,b) : v'(a) + \frac{\beta_1(t)}{b-a}v(a) = 0,\ v'(b) + \frac{\beta_2(t)}{b-a}v(b) = 0\}.$$

Proof. It is a direct consequence of Lemma 2.1 by using the affine change of variable $[0,1] \to [a,b]$, $x \to (1-x)a + xb = y$. □

3. Solution of the problem (1.2)

3.1. **A domain that can be transformed into a rectangle.** Let

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\}$$

where T is a finite positive number, while φ_1 and φ_2 are Lipschitz continuous in $[0,T]$, such that $\varphi_1(t) < \varphi_2(t)$ for all $t \in [0,T]$. Consider c a continuous function on $[0,T]$, such that

$$0 < d_1 \leq c \leq d_2, \tag{3.1}$$

where d_1, d_2 are two constants.

Theorem 3.1. *Under assumptions* (1.3), (2.1), (2.2) *and* (2.3) *on* $(\beta_i)_{i=1,2}$, *the problem*

$$\begin{gathered}\partial_t u - c^2(t)\partial_x^2 u = f \quad \text{in } L^2(\Omega),\\ u\big|_{t=0} = 0,\\ \partial_x u + \beta_i(t)u\big|_{x=\varphi_i(t)} = 0, \quad i=1,2,\end{gathered} \tag{3.2}$$

admits a (unique) solution $u \in H^{1,2}(\Omega)$.

Proof. The uniqueness of the solution is easy to check, thanks to (1.3). Let us prove the existence. The change of variables

$$(t,x) \mapsto (t,y) = \Big(t, \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\Big)$$

transforms Ω into the rectangle $R =]0,T[\times]0,1[$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, then Problem (3.2) becomes

$$\begin{gathered}\partial_t v(t,y) + a(t,y)\partial_y v(t,y) - \frac{1}{b^2(t)}\partial_y^2 v(t,y) = g(t,y)\\ v\big|_{t=0} = 0\\ \frac{1}{\varphi(t)}\partial_y v + \beta_1(t)v\big|_{y=0} = 0,\\ \frac{1}{\varphi(t)}\partial_y v + \beta_2(t)v\big|_{y=1} = 0,\end{gathered} \tag{3.3}$$

where

$$\begin{gathered}\varphi(t) = \varphi_2(t) - \varphi_1(t)\\ b(t) = \frac{\varphi(t)}{c(t)}\\ a(t,y) = -\frac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}.\end{gathered}$$

This change of variables conserves the spaces $H^{1,2}$ and L^2. In other words

$$\begin{gathered}f \in L^2(\Omega) \Leftrightarrow g \in L^2(R)\\ u \in H^{1,2}(\Omega) \Leftrightarrow v \in H^{1,2}(R).\end{gathered}$$

□

Lemma 3.2. *The operator*

$$\begin{aligned}B: \quad H_\gamma^{1,2}(R) &\to L^2(R)\\ v &\mapsto Bv = a(t,y)\partial_y v\end{aligned}$$

is compact, where for a fixed $t \in]0,T[$,

$$H_\gamma^{1,2}(R) = \{v \in H^{1,2}(R) : v\big|_{\Gamma_0} = 0, \partial_y v + \varphi(t)\beta_i(t)v\big|_{\Gamma_{i,R}} = 0,\ i = 1,2\},$$

with $\Gamma_0 = \{0\}\times]0,1[$, $\Gamma_{1,R} =]0,T[\times\{0\}$ *and* $\Gamma_{2,R} =]0,T[\times\{1\}$.

Proof. R has the "horn property" of Besov [1], so

$$\begin{aligned}\partial_y: \quad H_\gamma^{1,2}(R) &\to H^{\frac{1}{2},1}(R)\\ v &\mapsto \partial_y v\end{aligned}$$

is continuous. Since R is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(R)$ into $L^2(R)$, see for instance [1]. Here

$$H^{\frac{1}{2},1}(R) = L^2(0,T;H^1]0,1[) \cap H^{\frac{1}{2}}(0,T;L^2]0,1[).$$

See [6] for the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces.

Then ∂_y is a compact operator from $H_\gamma^{1,2}(R)$ to $L^2(R)$. Furthermore, since $a(.,.)$ is a bounded function, the operator $B = a\partial_y$ is then compact from $H_\gamma^{1,2}(R)$ into $L^2(R)$. □

So, it is sufficient to show that the operator

$$\partial_t - \frac{c^2}{\varphi^2}\partial_y^2 : H_\gamma^{1,2}(R) \to L^2(R)$$

is an isomorphism. A simple change of variable $t = h(s)$ with $h'(s) = \frac{\varphi^2}{c^2}(t)$, transforms the problem

$$\begin{gathered}\partial_t v(t,y) - \frac{c^2}{\varphi^2}(t)\partial_y^2 v(t,y) = g(t,y) \in L^2(R),\\ v\big|_{t=0} = 0,\\ \frac{1}{\varphi(t)}\partial_y v + \beta_1(t)v\big|_{y=0} = 0,\\ \frac{1}{\varphi(t)}\partial_y v + \beta_2(t)v\big|_{y=1} = 0,\end{gathered}$$

into

$$\begin{gathered}\partial_s w(s,y) - \partial_y^2 w(s,y) = \zeta(s,y)\\ w\big|_{s=h^{-1}(0)} = 0\\ \frac{1}{\varphi(h(s))}\partial_y w + \beta_1(h(s))w\big|_{y=0} = 0,\\ \frac{1}{\varphi(h(s))}\partial_y w + \beta_2(h(s))w\big|_{y=1} = 0,\end{gathered} \tag{3.4}$$

with $\zeta(s,y) = \frac{g(t,y)}{h'(s)}$ and $w(s,y) = v(t,y)$. Note that this change of variables preserves the spaces L^2 and $H^{1,2}$. It follows from (1.4) that there exists a unique $w \in H^{1,2}$ solution of the problem (3.4). This implies that Problem (3.2) admits a unique solution $u \in H^{1,2}(\Omega)$. We obtain the function u by setting $u(t,x) = v(t,y) = w(h^{-1}(t),y)$. This completes the proof of Theorem 3.1.

We shall need the following result in order to justify the calculus of the next section.

Lemma 3.3. *The space*

$$W = \{u \in D([0,T];H^2(0,1)) : \partial_x u + \beta_i(t)u\big|_{\Gamma_i} = 0,\ i = 1,2\}$$

is dense in

$$H_\gamma^{1,2}(]0,T[\times]0,1[) = \{u \in H^{1,2}(]0,T[\times]0,1[) : \partial_x u + \beta_i(t)u\big|_{\Gamma_i} = 0,\ i = 1,2\}$$

where $\Gamma_1 =]0,T[\times\{0\}$ *and* $\Gamma_2 =]0,T[\times\{1\}$*.*

The above lemma is a particular case of [6, Theorem 2.1].

Remark 3.4. We can replace in Lemma 3.3 $R =]0,T[\times]0,1[$ by Ω with the help of the change of variables defined above.

3.2. **Case of a triangular domain.** In this case, we define Ω by

$$\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\}$$

with

$$\begin{gathered}\varphi_1(0) = \varphi_2(0)\\ \varphi_1(T) < \varphi_2(T).\end{gathered} \tag{3.5}$$

We assume that the functions $(\varphi_i)_{i=1,2}$ satisfy

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \to 0 \quad \text{as } t \to 0,\ i = 1,2. \tag{3.6}$$

For each $n \in \mathbb{N}$, we define Ω_n by

$$\Omega_n = \{(t,x) \in \mathbb{R}^2 : a_n < t < T, \varphi_1(t) < x < \varphi_2(t)\}$$

where $(a_n)_n$ is a decreasing sequence to zero. Thus, we have

$$\varphi_1(a_n) < \varphi_2(a_n),$$
$$\varphi_1(T) < \varphi_2(T).$$

Setting $f_n = f\big|_{\Omega_n}$, where $f \in L^2(\Omega)$, we denote $u_n \in H^{1,2}(\Omega_n)$ the solution of (3.2) in Ω_n

$$\begin{gathered} \partial_t u_n - c^2(t)\partial_x^2 u_n = f_n \quad \text{in } L^2(\Omega_n) \\ u_{n/t=a_n} = 0 \\ \partial_x u_n + \beta_i(t) u_{n/\Gamma_{n,i}} = 0, \quad i = 1, 2, \end{gathered} \tag{3.7}$$

here $\Gamma_{n,i} = \{(t, \varphi_i(t)), a_n < t < T\}$, $i = 1, 2$, and c is a bounded differentiable coefficient depending on time such that

$$0 < \alpha \le c(t)c'(t) \le \beta \tag{3.8}$$

for every $t \in]0, T[$, where α and β are two constants. We also assume that

$$(\beta_1 c^2) \text{ is an increasing function on }]0, T[\tag{3.9}$$
$$(\beta_2 c^2) \text{ is a decreasing function on }]0, T[. \tag{3.10}$$

Such a solution u_n exists by Theorem 3.1.

Theorem 3.5. *There exists a constant $K > 0$ independent of n such that*

$$\|u_n\|^2_{H^{1,2}(\Omega_n)} \le K\|f_n\|^2_{L^2(\Omega_n)} \le K\|f\|^2_{L^2(\Omega)}.$$

To prove Theorem 3.5, we need some preliminary results.

Lemma 3.6. *For every $\epsilon > 0$ satisfying $(\varphi_2(t) - \varphi_1(t)) \le \epsilon$, there exists a constant $C > 0$ independent of n, such that*

$$\|\partial_x^j u_n\|^2_{L^2(\Omega_n)} \le C\epsilon^{2(2-j)}\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}, \quad j = 0, 1.$$

Proof. Replacing in Lemma 2.2 v by u_n and $]a, b[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^j u_n)^2 dx &\le C(\varphi_2(t) - \varphi_1(t))^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2 dx \\ &\le C\epsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2 dx \end{aligned}$$

where C is the constant of Lemma 2.2. Integrating with respect to t, we obtain the desired estimates. □

Proposition 3.7. *There exists a constant $C > 0$ independent of n such that*

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \le C\|f\|^2_{L^2(\Omega)}.$$

Then Theorem 3.5 is a direct consequence of Lemma 3.6 and Proposition 3.7, since ϵ is independent of n.

Proposition 3.7. Thanks to the density results, Lemma 2.2 and Remark 3.4, it is sufficient to prove the first part of the proposition (Relationship (3.11) below) in the case when $u_n \in \{v \in H^2(\Omega_n), \partial_x v + \beta_i(t)v\big|_{\Gamma_{n,i}} = 0, i = 1, 2\}$ without assuming the Cauchy condition $u_{n/t=a_n} = 0$.

For this end, we develop the inner product in $L^2(\Omega_n)$

$$\begin{aligned}\|f_n\|^2_{L^2(\Omega_n)} &= \langle \partial_t u_n - c^2\partial_x^2 u_n, \partial_t u_n - c^2\partial_x^2 u_n\rangle \\ &= \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|c^2.\partial_x^2 u_n\|^2_{L^2(\Omega_n)} - 2\langle \partial_t u_n, c^2\partial_x^2 u_n\rangle.\end{aligned}$$

Calculating the last term of the previous relation, we obtain

$$\begin{aligned}\langle \partial_t u_n, c^2\partial_x^2 u_n\rangle &= \int_{\Omega_n} \partial_t u_n.c^2\partial_x^2 u_n \,dt\,dx \\ &= -\int_{\Omega_n} c^2\partial_x\partial_t u_n.\partial_x u_n \,dt\,dx + \int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n \nu_x d\sigma.\end{aligned}$$

So,

$$\begin{aligned}&-2\langle \partial_t u_n, c^2\partial_x^2 u_n\rangle \\ &= \int_{\Omega_n} c^2\partial_t(\partial_x u_n)^2\,dt\,dx - 2\int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n\nu_x d\sigma \\ &= -\int_{\Omega_n} 2cc'(\partial_x u_n)^2\,dt\,dx + \int_{\partial\Omega_n} c^2(\partial_x u_n)^2\nu_t d\sigma - 2\int_{\partial\Omega_n} c^2\partial_t u_n.\partial_x u_n\nu_x d\sigma \\ &= \int_{\partial\Omega_n} c^2[(\partial_x u_n)^2\nu_t - 2\partial_t u_n.\partial_x u_n\nu_x]d\sigma - \int_{\Omega_n} 2cc'(\partial_x u_n)^2\,dt\,dx,\end{aligned}$$

where ν_t, ν_x are the components of the outward normal vector at the boundary of Ω_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω_n where $t = a_n$, we have $\nu_x = 0$ and $\nu_t = -1$. The corresponding boundary integral

$$A_1 = -\int_{\varphi_2(a_n)}^{\varphi_1(a_n)} c^2(\partial_x u_n)^2 dx = \int_{\varphi_1(a_n)}^{\varphi_2(a_n)} c^2(\partial_x u_n)^2 dx \geq 0.$$

On the part of the boundary of Ω_n where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$A_2 = \int_{\varphi_1(T)}^{\varphi_2(T)} c^2(\partial_x u_n)^2 dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1+(\varphi_i')^2(t)}}, \quad \nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}}$$

and $\partial_x u_n(t, \varphi_i(t)) + \beta_i(t)u_n(t, \varphi_i(t)) = 0$, $i = 1, 2$. Consequently, the corresponding integral is

$$\begin{aligned}&\int_{a_n}^T c^2\varphi_1'(t)[\partial_x u_n(t,\varphi_1(t))]^2 dt - 2\int_{a_n}^T (\beta_1 c^2)(t)\partial_t u_n(t,\varphi_1(t)).u_n(t,\varphi_1(t))dt \\ &-\int_{a_n}^T c^2\varphi_2'(t)[\partial_x u_n(t,\varphi_2(t))]^2 dt + 2\int_{a_n}^T (\beta_2 c^2)(t)\partial_t u_n(t,\varphi_2(t)).u_n(t,\varphi_2(t))dt.\end{aligned}$$

By setting

$$I_{n,k}=(-1)^{k+1}\int_{a_n}^{T}c^2\varphi_k'(t)[\partial_x u_n(t,\varphi_k(t))]^2dt, k=1,2,$$

$$J_{n,k}=(-1)^{k}2\int_{a_n}^{T}(\beta_k c^2)(t)\partial_t u_n(t,\varphi_k(t)).u_n(t,\varphi_k(t))dt, k=1,2,$$

we have

$$-2\langle\partial_t u_n,c^2\partial_x^2u_n\rangle\geq-|I_{n,1}|-|I_{n,2}|-|J_{n,1}|-|J_{n,2}|-\int_{\Omega_n}2cc'(\partial_x u_n)^2\,dt\,dx. \quad (3.11)$$

□

1. Estimation of $I_{n,k}, k=1,2$.

Lemma 3.8. *There exists a constant $K>0$ independent of n such that*

$$|I_{n,k}|\leq K\epsilon\|\partial_x^2u_n\|^2_{L^2(\Omega_n)},\quad k=1,2.$$

Proof. We convert the boundary integral $I_{n,1}$ into a surface integral by setting

$$\begin{aligned}[\partial_x u_n(t,\varphi_1(t))]^2&=-\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}[\partial_x u_n(t,x)]^2\Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)}\\&=-\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\partial}{\partial x}\{\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}[\partial_x u_n(t,x)]^2\}dx\\&=-2\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}\partial_x u_n(t,x)\partial_x^2u_n(t,x)dx\\&\quad+\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi_2(t)-\varphi_1(t)}[\partial_x u_n(t,x)]^2dx.\end{aligned}$$

Then

$$\begin{aligned}I_{n,1}&=\int_{a_n}^{T}c^2(t)\varphi_1'(t)[\partial_x u_n(t,\varphi_1(t))]^2dt\\&=\int_{\Omega_n}c^2(t)\frac{\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)}(\partial_x u_n)^2\,dt\,dx\\&\quad-2\int_{\Omega_n}c^2(t)\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}\varphi_1'(t)(\partial_x u_n)(\partial_x^2u_n)\,dt\,dx.\end{aligned}$$

Thanks to Lemma 3.6, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)}[\partial_x u_n(t,x)]^2dx\leq C[\varphi_2(t)-\varphi_1(t)]^2\int_{\varphi_1(t)}^{\varphi_2(t)}[\partial_x^2u_n(t,x)]^2dx.$$

Therefore,

$$\int_{\varphi_1(t)}^{\varphi_2(t)}[\partial_x u_n(t,x)]^2\frac{|\varphi_1'|}{\varphi_2-\varphi_1}dx\leq C|\varphi_1'|[\varphi_2-\varphi_1]\int_{\varphi_1(t)}^{\varphi_2(t)}[\partial_x^2u_n(t,x)]^2dx,$$

consequently,

$$|I_{n,1}|\leq C\int_{\Omega_n}c^2(t)|\varphi_1'|[\varphi_2-\varphi_1](\partial_x^2u_n)^2\,dt\,dx+2\int_{\Omega_n}c^2(t)|\varphi_1'||\partial_x u_n||\partial_x^2u_n|\,dt\,dx,$$

since $|\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}| \leq 1$. So, for all $\epsilon > 0$, we have

$$\begin{aligned} |I_{n,1}| &\leq C \int_{\Omega_n} c^2(t)|\varphi_1'|[\varphi_2-\varphi_1](\partial_x^2 u_n)^2\,dt\,dx \\ &\quad + \int_{\Omega_n} \epsilon c^2(t)(\partial_x^2 u_n)^2\,dt\,dx + \frac{1}{\epsilon}\int_{\Omega_n} c^2(t)(\varphi_1')^2(\partial_x u_n)^2\,dt\,dx. \end{aligned}$$

Lemma 3.6 yields

$$\frac{1}{\epsilon}\int_{\Omega_n} c^2(t)(\varphi_1')^2(\partial_x u_n)^2\,dt\,dx \leq C\frac{1}{\epsilon}\int_{\Omega_n} c^2(t)(\varphi_1')^2[\varphi_2-\varphi_1]^2(\partial_x^2 u_n)^2\,dt\,dx.$$

Thus, there exists a constant $M > 0$ independent of n such that

$$\begin{aligned} |I_{n,1}| &\leq C \int_{\Omega_n} c^2(t)[|\varphi_1'||\varphi_2-\varphi_1| + \frac{1}{\epsilon}(\varphi_1')^2|\varphi_2-\varphi_1|^2](\partial_x^2 u_n)^2\,dt\,dx \\ &\quad + \int_{\Omega_n} c^2(t)\epsilon(\partial_x^2 u_n)^2\,dt\,dx \\ &\leq M\epsilon \int_{\Omega_n} (\partial_x^2 u_n)^2\,dt\,dx, \end{aligned}$$

because $|\varphi_1'(\varphi_2-\varphi_1)| \leq \epsilon$ and $c^2(t)$ is bounded. The inequality

$$|I_{n,2}| \leq K\epsilon\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}$$

can be proved by a similar argument.

Estimation of $J_{n,k}$, $k = 1, 2$. We have

$$\begin{aligned} J_{n,1} &= -2\int_{a_n}^T (\beta_1 c^2)(t)\partial_t u_n(t,\varphi_1(t)).u_n(t,\varphi_1(t))dt \\ &= -\int_{a_n}^T (\beta_1 c^2)(t)[\partial_t u_n^2(t,\varphi_1(t))]dt. \end{aligned}$$

By setting $h(t) = u_n^2(t,\varphi_1(t))$, we obtain

$$\begin{aligned} J_{n,1} &= -\int_{a_n}^T \beta_1 c^2.\big[h'(t) - \varphi_1'(t)\partial_x u_n^2(t,\varphi_1(t))\big]dt \\ &= -\beta_1 c^2.h(t)\big]_{a_n}^T + \int_{a_n}^T (\beta_1 c^2)'.h(t)dt + \int_{a_n}^T \beta_1 c^2.\varphi_1'(t)\partial_x u_n^2(t,\varphi_1(t))dt. \end{aligned}$$

Thanks to (1.4), (3.9) and the fact that $u_n^2(a_n,\varphi_1(a_n)) = 0$, we have

$$-\beta_1 c^2.h(t)\big]_{a_n}^T + \int_{a_n}^T (\beta_1 c^2)'.h(t)dt \geq 0.$$

The last boundary integral in the expression of $J_{n,1}$ can be treated by a similar argument used in Lemma 3.8. So, we obtain the existence of a positive constant K independent of n, such that

$$\Big|\int_{a_n}^T \beta_1 c^2.\varphi_1'(t)\partial_x u_n^2(t,\varphi_1(t))dt\Big| \leq K\epsilon\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}. \tag{3.12}$$

By a similar method, we obtain

$$J_{n,2} = \beta_2(t)c^2(t)u_n^2(t,\varphi_2(t))\big]_{a_n}^T - \int_{a_n}^T (\beta_2 c^2)'.u_n^2(t,\varphi_2(t))dt$$
$$- \int_{a_n}^T \beta_2 c^2.\varphi_2'(t)\partial_x u_n^2(t,\varphi_2(t))dt.$$

Thanks to (1.4), (3.10) and the fact that $u_n^2(a_n,\varphi_2(a_n)) = 0$, we have

$$\beta_2(t)c^2(t)u_n^2(t,\varphi_2(t))\big]_{a_n}^T - \int_{a_n}^T (\beta_2 c^2)'.u_n^2(t,\varphi_2(t))dt \geq 0.$$

Then

$$| - \int_{a_n}^T \beta_2 c^2.\varphi_2'(t)\partial_x u_n^2(t,\varphi_2(t))dt| \leq K\epsilon \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \tag{3.13}$$

where K is a positive constant independent of n.

Now, we can complete the proof of Proposition 3.7. Summing up the estimates (3.8), (3.11), (3.12) and (3.13), and making use of Lemma 3.6, we then obtain

$$\begin{aligned}
&\|f_n\|^2_{L^2(\Omega_n)} \\
&\geq \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|c^2\partial_x^2 u_n\|^2_{L^2(\Omega_n)} - K\epsilon\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} - K_2\epsilon\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \\
&\geq \|\partial_t u_n\|^2_{L^2(\Omega_n)} + (d_1^2 - K\epsilon - K_2\epsilon)\|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}
\end{aligned}$$

where K_2 is a positive number. Then, it is sufficient to choose ϵ such that $(d_1^2 - K\epsilon - K_2\epsilon) > 0$, to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|^2_{L^2(\Omega_n)} \geq K_0(\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)}).$$

However,

$$\|f_n\|_{L^2(\Omega_n)} \leq \|f\|_{L^2(\Omega)},$$

then, there exists a constant $C > 0$, independent of n satisfying

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\partial_x^2 u_n\|^2_{L^2(\Omega_n)} \leq C\|f_n\|^2_{L^2(\Omega_n)} \leq C\|f\|^2_{L^2(\Omega)}.$$

This completes the proof of Proposition 3.7. □

Passage to the limit. We are now in position to prove the first main result of this work.

Theorem 3.9. *Assume that the following conditions are satisfied*

(1) $(\varphi_i)_{i=1,2}$ *fulfil the assumptions* (3.5) *and* (3.6)*.*
(2) *the coefficient* c *verifies the conditions* (3.1) *and* (3.8)*.*
(3) $(\beta_i)_{i=1,2}$ *fulfil the conditions* (1.4)*,* (2.1)*,* (2.2) *and* (2.3)*.*
(4) $(\varphi_i, \beta_i, c)_{i=1,2}$ *fulfil the conditions* (1.3)*,* (3.9) *and* (3.10)*.*

Then, for T *small enough,* (1.2) *admits a (unique) solution* u *belonging to*

$$H_\gamma^{1,2}(\Omega) = \{u \in H^{1,2}(\Omega); (\partial_x u + \beta_i(t)u)\big|_{\Gamma_i} = 0, i = 1,2\},$$

where Γ_i*,* $i = 1,2$ *are the parts of the boundary of* Ω *where* $x = \varphi_i(t)$*.*

Proof. Choose a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of the domains defined above, such that $\Omega_n \subseteq \Omega$ with (a_n) a decreasing sequence to 0, as $n \to \infty$. Then, we have $\Omega_n \to \Omega$, as $n \to \infty$.

Consider the solution $u_n \in H^{1,2}(\Omega_n)$ of the Robin boundary value problem

$$\begin{gathered}\partial_t u_n - c^2(t)\partial_x^2 u_n = f_n \quad \text{in } \Omega_n\\ u_{n/t=a_n} = 0\\ \partial_x u_n + \beta_i(t) u_n\big|_{\Gamma_{n,i}} = 0, \quad i=1,2,\end{gathered}$$

where $\Gamma_{n,i}$ are the parts of the boundary of Ω_n where $x=\varphi_i(t)$, $i=1,2$. Such a solution u_n exists by Theorem 3.1. Let $\widetilde{u_n}$ the 0−extension of u_n to Ω. In virtue of Theorem 3.5, we know that there exists a constant $K>0$ such that

$$\|\widetilde{u_n}\|^2_{L^2(\Omega)} + \|\widetilde{\partial_t u_n}\|^2_{L^2(\Omega)} + \|\widetilde{\partial_x u_n}\|^2_{L^2(\Omega)} + \|\widetilde{\partial_x^2 u_n}\|^2_{L^2(\Omega)} \le K\|f\|^2_{L^2(\Omega)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_x^j u_n}$, for $j=1,2$ are bounded functions in $L^2(\Omega)$. So, for a suitable increasing sequence of integers n_k, $k=1,2,\dots$, there exist functions u, v and v_j, $j=1,2$ in $L^2(\Omega)$ such that

$$\begin{gathered}\widetilde{u_{n_k}} \rightharpoonup u \quad \text{weakly in } L^2(\Omega),\ k\to\infty\\ \widetilde{\partial_t u_{n_k}} \rightharpoonup v \quad \text{weakly in } L^2(\Omega),\ k\to\infty\\ \widetilde{\partial_x^j u_{n_k}} \rightharpoonup v_j \quad \text{weakly in } L^2(\Omega),\ k\to\infty,\ j=1,2.\end{gathered}$$

Clearly $v=\partial_t u$, $v_1=\partial_x u$ and $v_2=\partial_x^2 u$ in the sense of distributions in Ω. So, $u\in H^{1,2}(\Omega)$ and

$$\partial_t u - c^2(t)\partial_x^2 u = f \quad \text{in } \Omega.$$

On the other hand, the solution u satisfies the boundary conditions

$$\partial_x u + \beta_i(t) u\big|_{\Gamma_i} = 0, \quad i=1,2,$$

since for all $n\in\mathbb{N}$, $u\big|_{\Omega_n} = u_n$. This proves the existence of solution to (1.2).

The uniqueness of the solution is easy to check, thanks to the hypothesis (1.3). □

4. The case of an arbitrary T

Assume that Ω satisfies (3.5). In the case where T is not in the neighborhood of zero, we set $\Omega = D_1 \cup D_2 \cup \Gamma_{T_1}$ where

$$\begin{gathered}D_1 = \{(t,x)\in\mathbb{R}^2 : 0<t<T_1, \varphi_1(t)<x<\varphi_2(t)\}\\ D_2 = \{(t,x)\in\mathbb{R}^2 : T_1<t<T, \varphi_1(t)<x<\varphi_2(t)\}\\ \Gamma_{T_1} = \{(T_1,x)\in\mathbb{R}^2 : \varphi_1(T_1)<x<\varphi_2(T_1)\}\end{gathered}$$

with T_1 small enough.

In the sequel, f stands for an arbitrary fixed element of $L^2(\Omega)$ and $f_i = f\big|_{D_i}$, $i=1,2$.

Theorem 3.9 applied to the triangular domain D_1, shows that there exists a unique solution $u_1\in H^{1,2}(D_1)$ of the problem

$$\begin{gathered}\partial_t u_1 - c^2(t)\partial_x^2 u_1 = f_1 \quad \text{in } L^2(D_1)\\ \partial_x u_1 + \beta_i(t) u_{1/\Gamma_{i,1}} = 0, \quad i=1,2,\end{gathered} \tag{4.1}$$

where $\Gamma_{i,1}$ are the parts of the boundary of D_1, and $x=\varphi_i(t)$, $i=1,2$.

Lemma 4.1. *If $u \in H^{1,2}(]0,T[\times]0,1[)$, then $u\big|_{t=0} \in H^1(\gamma_0)$, $u\big|_{x=0} \in H^{\frac{3}{4}}(\gamma_1)$ and $u\big|_{x=1} \in H^{\frac{3}{4}}(\gamma_2)$, where $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ and $\gamma_2 =]0,T[\times\{1\}$.*

The above lemma is a particular case of [6, Theorem 2.1, Vol.2]. The transformation

$$(t,x) \longmapsto (t',x') = (t, (\varphi_2(t) - \varphi_1(t))x + \varphi_1(t))$$

leads to the following lemma.

Lemma 4.2. *If $u \in H^{1,2}(D_2)$, then $u\big|_{\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$, $u\big|_{x=\varphi_1(t)} \in H^{\frac{3}{4}}(\Gamma_{1,2})$ and $u\big|_{x=\varphi_2(t)} \in H^{\frac{3}{4}}(\Gamma_{2,2})$, where $\Gamma_{i,2}$ are the parts of the boundary of D_2 where $x = \varphi_i(t)$, $i = 1,2$.*

Hereafter, we denote the trace $u_1\big|_{\Gamma_{T_1}}$ by ψ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $u_1 \in H^{1,2}(D_1)$ (see Lemma 4.2).

Now, consider the following problem in D_2

$$\begin{gathered}\partial_t u_2 - c^2(t)\partial_x^2 u_2 = f_2 \quad \text{in } L^2(D_2)\\ u_2\big|_{\Gamma_{T_1}} = \psi \\ \partial_x u_2 + \beta_i(t) u_{2/\Gamma_{i,2}} = 0, \quad i = 1,2,\end{gathered} \tag{4.2}$$

where $\Gamma_{i,2}$ are the parts of the boundary of D_2, and $x = \varphi_i(t)$, $i = 1,2$.

We use the following result, which is a consequence of [6, Theorem 4.3, Vol.2] to solve (4.2).

Proposition 4.3. *Let Q be the rectangle $]0,T[\times]0,1[$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$. Then the problem*

$$\begin{gathered}\partial_t u - c^2(t)\partial_x^2 u = f \quad \text{in } Q\\ u\big|_{\gamma_0} = \psi\\ \partial_x u + \beta_i(t) u\big|_{\gamma_i} = 0, \quad i = 1,2,\end{gathered}$$

where $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ and $\gamma_2 =]0,T[\times\{1\}$, admits a (unique) solution $u \in H^{1,2}(Q)$.

Remark 4.4. In the application of [6, Theorem 4.3, Vol 2], we can observe that there are no compatibility conditions to satisfy because $\partial_x\psi$ is only in $L^2(\gamma_0)$.

Thanks to the transformation

$$(t,x) \longmapsto (t,y) = (t, (\varphi_2(t) - \varphi_1(t))x + \varphi_1(t)),$$

we deduce the following result.

Proposition 4.5. *Problem (4.2) admits a (unique) solution $u_2 \in H^{1,2}(D_2)$.*

So, the function

$$u = \begin{cases} u_1 & \text{in } D_1\\ u_2 & \text{in } D_2\end{cases}$$

is the (unique) solution of (1.2) for an arbitrary T. Our second main result is as follows.

Theorem 4.6. *Assume that the following conditions are satisfied*

(1) $(\varphi_i)_{i=1,2}$ *satisfies assumptions* (3.5) *and* (3.6).

(2) *the coefficient c satisfies conditions* (3.1) *and* (3.8).
(3) $(\beta_i)_{i=1,2}$ *fulfil the conditions* (1.4), (2.1), (2.2) *and* (2.3).
(4) $(\varphi_i, \beta_i, c)_{i=1,2}$ *fulfil the conditions* (1.3), (3.9) *and* (3.10).

Then, (1.2) *admits a (unique) solution u belonging to*

$$H^{1,2}_{\gamma}(\Omega) = \{u \in H^{1,2}(\Omega); (\partial_x u + \beta_i(t)u)\big|_{\Gamma_i} = 0, i = 1, 2\},$$

where Γ_i, $i = 1, 2$ *are the parts of the boundary of* Ω *where* $x = \varphi_i(t)$.

Remark 4.7. Using the same method in the case where $\varphi_1(T) = \varphi_2(T)$ we can obtain a result similar to Theorem 4.6.

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