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OSCILLATION OF SOLUTIONS FOR ODD-ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish oscillation criteria for all solutions to the neutral differential equations

$$[x(t) \pm ax(t \pm h) \pm bx(t \pm g)]^{(n)} = p \int_c^d x(t-\xi)d\xi + q \int_c^d x(t+\xi)d\xi,$$

where n is odd, h, g, a and b are nonnegative constants. We consider 10 of the 16 possible combinations of $\pm$ signs, and give some examples to illustrate our results.

1. Introduction

In this article, we study the oscillatory behavior of solutions to to n-order mixed neutral functional differential equations with distributed deviating arguments

$$[x(t) \pm ax(t \pm h) \pm bx(t \pm g)]^{(n)} = p \int_c^d x(t-\xi)d\xi + q \int_c^d x(t+\xi)d\xi, \qquad (1.1)$$

where n is odd, h, g, a and b are nonnegative constants, p and q are positive constants, and $0 < c < d$. We consider 10 of the 16 possible combinations of $\pm$ signs. The equations

$$\frac{d^2}{dt^2}(x(t) \pm x[t-\tau] \pm x[t+\sigma]) + qx[t-\alpha] + px[t+\beta] = 0$$

are encountered in the study of vibrating masses attached to an elastic bar [8], and were studied by Grace and Lalli [4]. Later Grace extended their results to n-order equations with n odd in [5], and with n even in [6]. Moreover, Grace [7] remarked that the results for the n-order equations

$$\frac{d^n}{dt^n}(x(t) + cx[t-h] + Cx[t+H]) + qx[t-g] + Qx[t+G] = 0$$

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are extendable to the equations

$$\Big(x(t)+\sum_{i=1}^{n_1}c_ix(t-h_i)+\sum_{j=1}^{n_2}C_jx(t+H_j)\Big)^{(n)}$$
$$\pm\Big(\sum_{k=1}^{n_3}q_kx(t-g_k)+\sum_{m=1}^{n_4}Q_mx(t+G_m)\Big)=0.$$

In recent years, Candan [2], and Candan and Dahiya [3] obtained some results for distributed delays, which motivate us to study (1.1). For books related to this topic, we refer the reader to [1, 8, 10].

A function x is said to be a solution of (1.1) if $x(t)\pm ax(t\pm h)\pm bx(t\pm g)$ is n times continuous differentiable and $x(t)$ satisfies (1.1) for $t\geq t_0$.

A nontrivial solution of (1.1), for all large t, is called oscillatory if it has no largest zero. Otherwise, a solution is called non-oscillatory.

The purpose of this paper is to provide sufficient conditions, only on the coefficients and on limits of the integrals, to guarantee that (1.1) is oscillatory.

2. Main Results

The following lemmas will be used in our proofs.

Lemma 2.1 ([11]). *Suppose that a and h are positive constants and $a^{1/n}\big(\frac{h}{n}\big)e>1$. Then*

(i) *the inequality*
$$x^{(n)}(t)-ax(t+h)\geq 0$$
has no eventually positive solutions when n is odd;

(ii) *the inequality*
$$x^{(n)}(t)+ax(t-h)\leq 0$$
has no eventually positive solutions when n is odd.

Lemma 2.2 ([9]). *Let $x(t)$ be a function such that it and each of its derivative up to order $(n-1)$ inclusive are absolutely continuous and of constant sign in an interval (t_0,∞). If $x^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t_1,\infty)$ for some $t_1\geq t_0$, then there exist a $t_x\geq t_0$ and an integer m, $0\leq m\leq n$ with $n+m$ even for $x^{(n)}(t)\geq 0$, or $n+m$ odd for $x^{(n)}(t)\leq 0$, and such that for every $t\geq t_x$,*
$$m>0\quad\text{implies}\quad x^{(k)}(t)>0,\quad k=0,1,\dots,m-1$$
and
$$m\leq n-1\quad\text{implies}\quad(-1)^{m+k}x^{(k)}(t)>0,\quad k=m,m+1,\dots,n-1.$$

Theorem 2.3. *Suppose that $b>0$, either*
$$\Big(\frac{p(d-c)}{b}\Big)^{1/n}\Big(\frac{g+c}{n}\Big)e>1, \tag{2.1}$$
or
$$\Big(\frac{(p+q)(d-c)}{b}\Big)^{1/n}\Big(\frac{g-d}{n}\Big)e>1,\quad g>d, \tag{2.2}$$
and
$$\Big(\frac{q(d-c)}{1+a}\Big)^{1/n}\big(\frac{c}{n}\big)e>1. \tag{2.3}$$

Then

$$[x(t) + ax(t-h) - bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi, \tag{2.4}$$

is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (2.4). We may assume that $x(t)$ is eventually positive; that is, there exists a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$. If $x(t)$ is an eventually negative solution, the proof follows the same arguments. Let

$$z(t) = x(t) + ax(t-h) - bx(t+g), \quad t \geq t_0 + h.$$

From (2.4), we have

$$z^{(n)}(t) = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi \tag{2.5}$$

for $t \geq t_1 \geq t_0 + h$, which implies that $z^{(n)}(t) > 0$. Then $z^{(i)}(t)$, $i = 0, 1, \dots, n$ are of constant sign on $[t_1, \infty)$. We have two possible cases to consider: $z(t) < 0$ for $t \geq t_1$, and $z(t) > 0$ for $t \geq t_1$.

Case 1: $z(t) < 0$ for $t \geq t_1$. Let $v(t) = -z(t)$. Then from (2.5), we obtain

$$v^{(n)}(t) + p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi = 0. \tag{2.6}$$

On the other hand, since

$$0 < v(t) = -z(t) = -x(t) - ax(t-h) + bx(t+g) \leq bx(t+g) \quad \text{for } t \geq t_1,$$

there is a $t_2 \geq t_1$ such that

$$x(t) \geq \frac{v(t-g)}{b} \quad \text{for } t \geq t_2. \tag{2.7}$$

In view of (2.7) it follows from (2.6) that

$$v^{(n)}(t) + \frac{p}{b}\int_c^d v(t-g-\xi)d\xi + \frac{q}{b}\int_c^d v(t-g+\xi)d\xi \leq 0 \quad \text{for } t \geq t_3 > t_2. \tag{2.8}$$

It is clear that from either (2.6) or (2.8), $v^{(n)}(t) < 0$ for $t \geq t_3$. Therefore, by Lemma 2.2 $v^{(n-1)}(t) > 0$ for $t \geq t_3$. Now, we want to show that $v'(t) < 0$ for $t \geq t_3$. Suppose on the contrary $v'(t) > 0$ for $t \geq t_3$, then there exists a constant $k > 0$ and $t_4 \geq t_3$ such that

$$v(t-g-\xi) \geq k, \qquad v(t-g+\xi) \geq k$$

for $t \geq t_4$ and $\xi \in [c, d]$. Thus,

$$v^{(n)}(t) \leq -\frac{k(p+q)(d-c)}{b} \quad \text{for } t \geq t_4$$

and

$$v^{(n-1)}(t) \leq v^{(n-1)}(t_4) - \frac{k(p+q)(d-c)(t-t_4)}{b} \to -\infty \quad \text{as } t \to \infty,$$

which is a contradiction. Thus, $v'(t) < 0$ and therefore $(-1)^i v^{(i)}(t) > 0$ for $t \geq t_4$ and $i = 0, 1, \dots, n$. Then from (2.8), we have

$$v^{(n)}(t) + \frac{p(d-c)}{b}v(t-(g+c)) \leq 0, \tag{2.9}$$

and

$$v^{(n)}(t)+\frac{(p+q)(d-c)}{b}v(t-(g-d))\leq 0,\quad t\geq t_4. \tag{2.10}$$

Thus, from Lemma 2.1 (*ii*) and condition (2.1), (2.9) has no eventually positive solutions or from Lemma 2.1 (ii) and condition (2.2), (2.10) has no eventually positive solutions, which is a contradiction.

Case 2: $z(t)>0$ for $t\geq t_1$. Let

$$w(t)=z(t)+az(t-h)-bz(t+g),\quad t\geq t_1+h.$$

Thus, one can show that

$$w^{(n)}(t)=p\int_c^d z(t-\xi)d\xi+q\int_c^d z(t+\xi)d\xi, \tag{2.11}$$

then

$$[w(t)+aw(t-h)-bw(t+g)]^{(n)}=p\int_c^d w(t-\xi)d\xi+q\int_c^d w(t+\xi)d\xi. \tag{2.12}$$

Since n is odd, by Lemma 2.2 $z'(t)>0$ for $t\geq t_2^*\geq t_1+h$. From equation (2.11), $w^{(n)}(t)>0$ and $w^{(n+1)}(t)>0$ for $t\geq t_3^*\geq t_2^*$. Therefore, $w^{(i)}(t)>0$ for $i=0,1\dots,n+1$ and $t\geq t_3^*$. Using this results and (2.12) we obtain

$$(1+a)w^{(n)}(t)\geq p\int_c^d w(t-\xi)d\xi+q\int_c^d w(t+\xi)d\xi\geq q\int_c^d w(t+\xi)d\xi$$

and then

$$w^{(n)}(t)\geq\frac{q(d-c)}{1+a}w(t+c),\quad t\geq t_3^*.$$

This last equation does not have a positive solution by Lemma 2.1 (i) and condition (2.3). Therefore, it is a contradiction, and the proof is complete. □

Example 2.4. Consider the neutral differential equation

$$[x(t)+x(t-\pi)-x(t+\frac{9\pi}{2})]'''=\frac{1}{2}\int_{\pi/2}^{3\pi}x(t-\xi)d\xi+\frac{1}{2}\int_{\pi/2}^{3\pi}x(t+\xi)d\xi,$$

so that $n=3$, $a=b=1$, $c=\frac{\pi}{2}$, $d=3\pi$, $p=q=\frac{1}{2}$, $h=\pi$, $g=\frac{9\pi}{2}$. One can verify that the conditions of Theorem 2.3 are satisfied. We shall note that $x(t)=\cos t$ is a solution of this problem.

Theorem 2.5. *Suppose $c>h$, $c>g$, $a>0$,*

$$\Big(\frac{p(d-c)}{a}\Big)^{1/n}\Big(\frac{c-h}{n}\Big)e>1, \tag{2.13}$$

$$\Big(\frac{q(d-c)}{1+b}\Big)^{1/n}\Big(\frac{c-g}{n}\Big)e>1. \tag{2.14}$$

Then

$$[x(t)-ax(t-h)+bx(t+g)]^{(n)}=p\int_c^d x(t-\xi)d\xi+q\int_c^d x(t+\xi)d\xi, \tag{2.15}$$

is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (2.15). Without loss of generality we may assume that $x(t)$ is eventually positive; that is, there exists a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$. If $x(t)$ is eventually negative solution, the proof follows the same arguments. Let

$$z(t) = x(t) - ax(t-h) + bx(t+g), \quad t \geq t_0 + h.$$

As in the proof of the Theorem 2.3 the function $z^{(i)}(t)$ are of constant sign for $t \geq t_1 \geq t_0 + h$ and $i = 0, 1, \dots, n$, hence we have two possible cases to consider for $z(t)$: $z(t) < 0$ for $t \geq t_1$, and $z(t) > 0$ for $t \geq t_1$.

Case 1: $z(t) < 0$ **for** $t \geq t_1$**.** Let $v(t) = -z(t)$. Then we obtain

$$v^{(n)}(t) + p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi = 0. \tag{2.16}$$

On the other hand, since

$$0 < v(t) = -z(t) = -x(t) + ax(t-h) - bx(t+g) \leq ax(t-h) \quad \text{for } t \geq t_1,$$

there is a $t_2 \geq t_1$ such that

$$x(t) \geq \frac{v(t+h)}{a} \quad \text{for } t \geq t_2. \tag{2.17}$$

In view of (2.17) it follows from (2.16) that

$$v^{(n)}(t) + \frac{p}{a}\int_c^d v(t+h-\xi)d\xi + \frac{q}{a}\int_c^d v(t+h+\xi)d\xi \leq 0 \quad \text{for } t \geq t_3 \geq t_2. \tag{2.18}$$

As in the proof of the Theorem 2.3 (case 1) we show that $(-1)^i v^{(i)}(t) > 0$ for $t \geq t_4 \geq t_3$ and $i = 0, 1, \dots, n$, and using this in (2.18) we see that

$$v^{(n)}(t) + \frac{p(d-c)}{a} v(t-(c-h)) \leq 0 \quad \text{for } t \geq t_4. \tag{2.19}$$

Thus, from Lemma 2.1 (ii) and condition (2.13), (2.19) has no eventually positive solutions, which is a contradiction.

Case 2: $z(t) > 0$ **for** $t \geq t_1$**.** Let

$$w(t) = z(t) - az(t-h) + bz(t+g)\,.$$

Then one sees that

$$w^{(n)}(t) = p\int_c^d z(t-\xi)d\xi + q\int_c^d z(t+\xi)d\xi,$$

$$[w(t) - aw(t-h) + bw(t+g)]^{(n)} = p\int_c^d w(t-\xi)d\xi + q\int_c^d w(t+\xi)d\xi.$$

As in the proof of the Theorem 2.3 (case 2), we have $w^{(i)}(t) > 0$ for $t \geq t_2^* \geq t_1$ and $i = 0, 1, \dots, n+1$. Then, we obtain

$$(1+b)w^{(n)}(t+g) \geq p\int_c^d w(t-\xi)d\xi + q\int_c^d w(t+\xi)d\xi \geq q\int_c^d w(t+\xi)d\xi.$$

Since $w'(t) > 0$ for $t \geq t_2^*$,

$$w^{(n)}(t) \geq \frac{q(d-c)}{1+b} w(t+(c-g)).$$

The above equation does not have a positive solution by Lemma 2.1 (i) and condition (2.14). Thus, the proof is complete. □

Example 2.6. Consider the neutral differential equation

$$[x(t) - x(t-\pi) + 2x(t+\pi)]^{(5)} = \int_{2\pi}^{4\pi} x(t-\xi)d\xi + \frac{1}{2}\int_{2\pi}^{4\pi} x(t+\xi)d\xi,$$

so that $n = 5$, $a = 1$, $b = 2$, $c = 2\pi$, $d = 4\pi$, $p = 1$, $q = \frac{1}{2}$, $g = h = \pi$. One can check that the conditions of Theorem 2.5 are satisfied. By direct substitution it is easy to see that $x(t) = t\cos t$ is a solution of this problem.

Example 2.7. Consider the neutral differential equation

$$[x(t) - x(t-\pi) + 2x(t+\pi)]^{(9)} = \frac{3}{4}\int_{6\pi}^{8\pi} x(t-\xi)d\xi + \frac{3}{4}\int_{6\pi}^{8\pi} x(t+\xi)d\xi.$$

We see that $n = 9$, $a = 1$, $b = 2$, $c = 6\pi$, $d = 8\pi$, $p = q = \frac{3}{4}$, $g = h = \pi$. One can verify that the conditions of Theorem 2.5 are satisfied. It is easy to show that $x(t) = t\sin t$ is a solution of this problem.

Since the proofs of the following two theorems are similar to that of Theorems 2.3 and 2.5, they are omitted.

Theorem 2.8. *Suppose that $c > g$, $b > 0$, (2.3) holds, and*

$$\Big(\frac{p(d-c)}{b}\Big)^{1/n}\Big(\frac{c-g}{n}\Big)e > 1\,.$$

Then

$$[x(t) + ax(t-h) - bx(t-g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

Theorem 2.9. *Suppose that $c > h$, $b > 0$, (2.1) or (2.2) hold, and*

$$\Big(\frac{q(d-c)}{1+a}\Big)^{1/n}\Big(\frac{c-h}{n}\Big)e > 1\,.$$

Then

$$[x(t) + ax(t+h) - bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

Theorem 2.10. *Suppose $c > g$, and*

$$\Big(\frac{q(d-c)}{1+a+b}\Big)^{1/n}\Big(\frac{c-g}{n}\Big)e > 1. \tag{2.20}$$

Then

$$[x(t) + ax(t-h) + bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi, \tag{2.21}$$

is oscillatory.

Proof. Suppose there exist a nonoscillatory solution $x(t)$ of (2.21). Without loss of generality we may say that $x(t) > 0$ for $t \geq t_0$. Let

$$z(t) = x(t) + ax(t-h) + bx(t+g), \quad t \geq t_0 + h.$$

Clearly $z(t) > 0$ for $t \geq t_0 + h$. Thus, using (2.21), we get

$$z^{(n)}(t) = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi$$

for $t \geq t_1$ for some $t_1 \geq t_0 + h$. Therefore, we conclude that $z^{(i)}(t)$, $i = 0, 1, \dots, n$ are of constant sign, by Lemma 2.2 $z(t) > 0$ and $z'(t) > 0$ on $[t_1, \infty)$. Let

$$w(t) = z(t) + az(t-h) + bz(t+g),$$

then we show that

$$w^{(n)}(t) = p\int_c^d z(t-\xi)d\xi + q\int_c^d z(t+\xi)d\xi \tag{2.22}$$

and then

$$[w(t) + aw(t-h) + bw(t+g)]^{(n)} = p\int_c^d w(t-\xi)d\xi + q\int_c^d w(t+\xi)d\xi. \tag{2.23}$$

Since $z(t) > 0$ and $z'(t) > 0$ are eventually increasing, from (2.22) we see that $w^{(n)}(t) > 0$ and $w^{(n+1)}(t) > 0$ for $t \geq t_2 \geq t_1$. As a result of this $w^{(i)}(t) > 0$ for $i = 0, 1, \dots, n+1$ and $t \geq t_2$. Thus from (2.23), we have

$$(1 + a + b)w^{(n)}(t+g) \geq q\int_c^d w(t+\xi)d\xi,$$

and then using the eventually increasing nature of $w(t)$, we obtain

$$w^{(n)}(t+g) \geq \frac{q(d-c)}{1+a+b}w(t+c)$$

or

$$w^{(n)}(t) \geq \frac{q(d-c)}{1+a+b}w(t+(c-g)), \quad t \geq t_3 \geq t_2. \tag{2.24}$$

In view of Lemma 2.1(i) and (2.20), the inequality (2.24) has no eventually positive solutions, which leads to a contradiction. Thus, the proof is complete. $\square$

Example 2.11. Consider the neutral differential equation

$$[x(t) + x(t-\pi) + x(t+\frac{3\pi}{2})]''' = \frac{1}{4}\int_{5\pi/2}^{7\pi/2} x(t-\xi)d\xi + \frac{1}{4}\int_{5\pi/2}^{7\pi/2} x(t+\xi)d\xi,$$

so that $n = 3$, $a = b = 1$, $c = \frac{5\pi}{2}$, $d = \frac{7\pi}{2}$, $p = q = \frac{1}{4}$, $h = \pi$, $g = \frac{3\pi}{2}$. One can see that the conditions of Theorem 2.10 are satisfied. In fact $x(t) = \sin t + \cos t$ is an oscillatory solution of this problem.

The proofs of the following two theorems are similar to that of Theorem 2.10 and therefore omitted.

Theorem 2.12. *Suppose that $c > g > h$, and (2.20) holds. Then the equation*

$$[x(t) + ax(t+h) + bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

Theorem 2.13. *Suppose that*

$$\Big(\frac{q(d-c)}{1+a+b}\Big)^{1/n}\big(\frac{c}{n}\big)e > 1.$$

Then

$$[x(t)+ax(t-h)+bx(t-g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

Theorem 2.14. *Suppose $a>0$, $c>h$,*

$$\Big(\frac{p(d-c)}{a+b}\Big)^{1/n}\Big(\frac{c-h}{n}\Big)e > 1, \tag{2.25}$$

$$\Big(q(d-c)\Big)^{1/n}\big(\frac{c}{n}\big)e > 1. \tag{2.26}$$

Then

$$[x(t)-ax(t-h)-bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi, \tag{2.27}$$

is oscillatory.

Proof. Suppose that $x(t)$ is a non-oscillatory solution of (2.27). We may assume that $x(t)$ is eventually positive, say $x(t)>0$ for $t\ge t_0$. Let

$$z(t) = x(t) - ax(t-h) - bx(t+g), \quad t\ge t_0+h. \tag{2.28}$$

From (2.27), we have

$$z^{(n)}(t) = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi \tag{2.29}$$

for $t\ge t_1$ for some $t_1\ge t_0+h$, implies that $z^{(i)}(t)$, $i=0,1,\dots,n$ are of constant sign on $[t_1,\infty)$. We have two cases: $z(t)>0$ for $t\ge t_1$, and $z(t)<0$ for $t\ge t_1$.

Case 1: $z(t)>0$ for $t\ge t_1$. From (2.28),

$$x(t)\ge z(t). \tag{2.30}$$

In view of (2.29) and (2.30), we have

$$z^{(n)}(t)\ge q\int_c^d z(t+\xi)d\xi \quad \text{for } t\ge t_1.$$

As in the proof of Theorem 2.3, $z'(t)$ is eventually positive. Thus

$$z^{(n)}(t)\ge q(d-c)z(t+c),$$

which contradicts to Lemma 2.1 (i) and condition (2.26).

Case 2: $z(t)<0$ for $t\ge t_1$. Let

$$0<v(t) = -z(t) = -x(t)+ax(t-h)+bx(t+g),$$

then

$$v^{(n)}(t) + p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi = 0.$$

Set

$$w(t) = -v(t)+av(t-h)+bv(t+g)\,.$$

Then

$$w^{(n)}(t) + p\int_c^d v(t-\xi)d\xi + q\int_c^d v(t+\xi)d\xi = 0 \tag{2.31}$$

and since the function satisfies (2.27), we obtain

$$[-w(t) + aw(t-h) + bw(t+g)]^{(n)} + p\int_c^d w(t-\xi)d\xi + q\int_c^d w(t+\xi)d\xi = 0.$$

If $w(t) < 0$ for $t \geq t_1$, we can handle as in case 1. Now suppose $w(t) > 0$ for $t \geq t_1$. On the other hand, $v'(t) < 0$ for $t \geq t_2 \geq t_1$, otherwise from (2.31) we see that $w^{(n)}(t) < 0$ and $w^{(n+1)}(t) < 0$ for $t \geq t_2$ which is a contradiction. As a result of this,

$$(-1)^i w^{(i)}(t) > 0 \quad \text{for } i = 0, 1, \dots, n+1 \quad \text{and} \quad t \geq t_2,$$

and then

$$(a+b)w^{(n)}(t-h) + p\int_c^d w(t-\xi)d\xi \leq 0,$$
$$w^{(n)}(t) + \frac{p(d-c)}{a+b} w(t-(c-h)) \leq 0,$$

which leads to a contradiction by condition (2.25) and Lemma 2.1 (*ii*). This completes the proof. $\square$

Example 2.15. Consider the equation

$$[x(t) - \frac{3}{2}x(t - \frac{3\pi}{2}) - \frac{4}{3}x(t+2\pi)]''' = \frac{7}{12}\int_{2\pi}^{7\pi/2} x(t-\xi)d\xi + \frac{11}{12}\int_{2\pi}^{7\pi/2} x(t+\xi)d\xi.$$

We see that $n = 3$, $a = \frac{3}{2}$, $b = \frac{4}{3}$, $c = 2\pi$, $d = \frac{7\pi}{2}$, $p = \frac{7}{12}$, $q = \frac{11}{12}$, $h = \frac{3\pi}{2}$, $g = 2\pi$. Clearly the conditions of Theorem 2.14 are satisfied. In fact, $x(t) = \sin t$ is a solution of this problem.

The proofs of the following two theorems are similar to that of Theorem 2.14, hence the proofs are omitted.

Theorem 2.16. *Suppose $a > 0$, $h > g$, and (2.25) and (2.26) hold. Then*

$$[x(t) - ax(t-h) - bx(t-g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

Theorem 2.17. *Suppose $b > 0$, $h > g$, $\lambda = \mu = -1$, $\alpha = \beta = 1$. In addition, if (2.26) and*

$$\Big(\frac{p(d-c)}{a+b}\Big)^{1/n}\Big(\frac{c+g}{n}\Big)e > 1,$$

Then

$$[x(t) - ax(t+h) - bx(t+g)]^{(n)} = p\int_c^d x(t-\xi)d\xi + q\int_c^d x(t+\xi)d\xi,$$

is oscillatory.

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