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UNIQUENESS FOR N-TH ORDER DIFFERENTIAL SYSTEMS WITH STRONG SINGULARITIES

YIFEI PAN, MEI WANG

Abstract. Using a Lipschitz type condition, we obtain the uniqueness of solutions for a system of n-th order nonlinear ordinary differential equations where the coefficients are allowed to have singularities.

1. Introduction and main results

Lipschitz condition was a key part in proving classical results on the existence and uniqueness of ordinary differential equations, as extensively surveyed and summarized in [1]. In this paper, we use a Lipschitz type condition to obtain the uniqueness of solutions of n-th order nonlinear ordinary differential systems where the coefficients are allowed to have singularities.

Our results are in the spirit of the Carathéodory theorem on the existence of ordinary differential equations [5, Chapter 2], which gives a Lipschitz condition in first order differential equations. The conditions in this paper are in terms of absolute continuity, thus the uniqueness result is on solutions in weaker or more general sense. For first order differential equations, Nagumo's Theorem [8] and its generalizations [2, 3] give precise coefficients and sharp order for an isolated singularity in the Lipschitz condition. A natural generalization of the classical Carathéodory condition is to higher order linear and nonlinear differential systems (e.g. [4, 6]). Our results are on higher order differential equations with coefficients of singularities under integrability conditions.

The main results are stated below. The proofs are provided in the next section. In the last section, we provide two applicable forms of the main theorem as corollaries, and we give an example to illustrate the sharpness of the singularity order allowed in the main condition of the Lipschitz type in an n order differential equation.

Let $L^1(a,b)$ denote the set of real Lebesgue integrable functions on the interval (a,b), $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$, and $\mathcal{C}^k(a,b)$ the set of d-dimensional functions with k-th continuously differentiable components on (a,b).

The following theorem is on the uniqueness of solutions of differential systems.

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Theorem 1.1. *Consider the system of differential equations of* $y : (a,b) \to \mathbb{R}^d$,

$$\begin{gathered} y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad x \in (a,b),\ a < 0 < b, \\ y(0) = a_0 \\ y'(0) = a_1 \\ \dots \\ y^{(n-1)}(0) = a_{n-1} \end{gathered} \tag{1.1}$$

where $f : (a,b) \times \mathbb{R}^{nd} \to \mathbb{R}^d$ *satisfies the Lipschitz type condition*

$$\|f(x, s_0, \dots, s_{n-1}) - f(x, r_0, \dots, r_{n-1})\| \le \sum_{k=0}^{n-1} \lambda_k(x) \frac{\|s_k - r_k\|}{|x|^{n-k-1}}, \quad a.\ e.\ x \in (a,b) \tag{1.2}$$

for $\lambda_k(x) \ge 0$, $\lambda_k \in L^1(a,b)$ *and* $(s_0, \dots, s_{n-1})$, $(r_0, \dots, r_{n-1})$ *in a domain in* $\mathbb{R}^{nd}$ *containing* $(a_0, \dots, a_{n-1})$. *If* $u, v \in \mathcal{C}^{n-1}(a,b)$ *are solutions of* (1.1) *and* $u^{(n-1)}, v^{(n-1)}$ *are absolutely continuous with respect to the Lebesgue measure on* (a,b), *then*

$$v(x) \equiv u(x) \quad for \quad x \in (a,b).$$

The following theorem gives the condition for a function that satisfies a differential inequality to be identically zero, which can be considered as an n-th order Gronwall [7] uniqueness theorem.

Theorem 1.2. *Let* $y : (a,b) \to \mathbb{R}^d$, $y \in \mathcal{C}^{n-1}(a,b)$ *and* $y^{(n-1)}$ *absolute continuous with respect to the Lebesgue measure on* (a,b), $a < 0 < b$. *If*

$$y(0) = y^{(1)}(0) = \dots = y^{(n-1)}(0) = \vec{0}$$

and

$$\|y^{(n)}(x)\| \le \sum_{k=0}^{n-1} \lambda_k(x) \frac{\|y^{(k)}(x)\|}{|x|^{n-k-1}} \quad a.\ e.\ x \in (a,b) \tag{1.3}$$

with $\lambda_k \in L^1(a,b)$, $\lambda_k \ge 0$, *for* $k = 0, 1, \dots, n-1$, *then* $y(x) \equiv \vec{0}$ *for all* $x \in (a,b)$.

2. Proofs of the main results

The absolute continuity assumption in the theorems is needed in applying the Fundamental Theorem of Calculus to prove the following lemma.

Lemma 2.1. *Assume* $\phi : (a,b) \to \mathbb{R}^d$, $\phi \in \mathcal{C}^{n-1}(a,b)$, $\phi^{(n-1)}$ *absolutely continuous on* (a,b) *for* $a < 0 < b$. *If*

$$\phi^{(k)}(0) = \vec{0} \in \mathbb{R}^d, \quad k = 0, 1, \dots, n-1,$$

then for any $\lambda(x) \in L^1(a,b)$, $\lambda(x) \ge 0$ *and* $k = 0, 1, \dots, n-1$,

$$\int_0^x \lambda(t) \frac{\|\phi^{(k)}(t)\|}{|t|^{n-k-1}} dt \le \int_0^x \lambda(t) dt \int_0^x \|\phi^{(n)}(t)\| dt, \quad \forall x \in (a,b). \tag{2.1}$$

Proof. By the absolute continuity assumption of $\phi^{(n-1)}$, the Fundamental Theorem of Calculus grants the form

$$\phi^{(k)}(x) = \int_0^x \int_0^{t_{n-k-1}} \dots \int_0^{t_1} \phi^{(n)}(t) dt\ dt_1 \dots dt_{n-k-1}, \quad x \in [0,b)$$

for $k = 0, 1, \dots, n-1$. Define

$$g(x) = \int_0^x \int_0^{t_{n-1}} \dots \int_0^{t_1} \|\phi^{(n)}(t)\| dt\ dt_1 \dots dt_{n-1}, \quad x \in [0, b).$$

Then $g \in \mathcal{C}^{n-1}[0, b)$, $g^{(n)} = \|\phi^{(n)}\|$ a. e. in (a, b) and for $k = 0, 1, \dots, n-1$,

$$g^{(k)}(x) = \int_0^x \int_0^{t_{n-k-1}} \dots \int_0^{t_1} \|\phi^{(n)}(t)\| dt\, dt_1 \dots dt_{n-k-1} \quad \nearrow \quad \text{w. r. t. } x \in (0, b)$$
$$g^{(k)}(0) = 0.$$

By Taylor's formula, for any $t \in (0, b)$ and $k = 0, \dots, n-2$,

$$\begin{aligned} g^{(k)}(t) &= g^{(k)}(0) + g^{(k+1)}(c)(t-0) \quad c \in (0, t) \\ &\le g^{(k+1)}(t)\ t \le g^{(k+2)}(t)\ t^2 \le \dots \\ &\le g^{(n-1)}(t)\ t^{n-k-1} \end{aligned}$$

By $\|\phi^{(k)}(x)\| \le g^{(k)}(x)$ and the monotonicity of $g^{(k)}$ on $[0, b)$,

$$\begin{aligned} \int_0^x \lambda(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}} dt &\le \int_0^x \lambda(t) \frac{g^{(k)}(t)}{t^{n-k-1}} dt \quad x \in [0, b),\ k = 0, 1, \dots, n-1 \\ &\le \int_0^x \lambda(t) g^{(n-1)}(t) dt \le g^{(n-1)}(x) \int_0^x \lambda(t) dt \\ &= \int_0^x g^{(n)}(t) dt \int_0^x \lambda(t) dt = \int_0^x \|\phi^{(n)}(t)\| dt \int_0^x \lambda(t) dt. \end{aligned}$$

Thus the result holds for $x \in [0, b)$. For $x \in (a, 0]$, consider $x^* = -x$ and define

$$\phi^*(x^*) = \phi(-x^*), \quad g^*(x^*) = \int_0^{x^*} \int_0^{t_{n-1}} \dots \int_0^{t_1} \|{\phi^*}^{(n)}(t)\| dt\, dt_1 \dots dt_{n-1},$$

where $x^* \in [0, -a)$. Repeating the above derivation for $x \in [0, b)$ leads to

$$\int_0^{x^*} \lambda(-t) \frac{\|{\phi^*}^{(k)}(t)\|}{t^{n-k-1}} dt \le \int_0^{x^*} \|{\phi^*}^{(n)}(t)\| dt \int_0^{x^*} \lambda(-t) dt, \quad \forall x^* \in [0, -a).$$

Subsisting t by $-t$ in the integrands and replacing x^* by $x = -x^* \in (a, 0]$,

$$\int_x^0 \lambda(t) \frac{\|\phi^{(k)}(t)\|}{|t|^{n-k-1}} dt \le \int_x^0 \|\phi^{(n)}(t)\| dt \int_x^0 \lambda(t) dt, \quad \forall x \in (a, 0].$$

Therefore, (2.1) holds for all $x \in (a, b)$. This completes the proof. $\square$

The above lemma is used in the proof of Theorem 1.1 below.

Proof of Theorem 1.1. Let $\phi(x) = u(x) - v(x)$, $x \in (a, b)$. Then ϕ satisfies the conditions in Lemma 2.1: $\phi \in \mathcal{C}^{n-1}(a, b)$, $\phi^{(n-1)}$ is absolutely continuous on (a, b), and $\phi^{(k)}(0) = \vec{0}$ for $k = 0, 1, \dots, n-1$. By the Lipschitz property (1.2),

$$\begin{aligned} \|\phi^{(n)}(t)\| &= \|f(t, u(t), u'(t), \dots, u^{(n-1)}(t)) - f(t, v(t), v'(t), \dots, v^{(n-1)}(t))\| \\ &\le \sum_{k=0}^{n-1} \lambda_k(t) \frac{\|u^{(k)}(t) - v^{(k)}(t)\|}{t^{n-k-1}} = \sum_{k=0}^{n-1} \lambda_k(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}}. \end{aligned} \tag{2.2}$$

We assume f is well defined so that for solutions u, v on (a, b), $(u, u', \dots, u^{(n-1)})$, $(v, v', \dots, v^{(n-1)})$ are in the domain for which (1.2) holds. For any $\varepsilon \in (0, b)$, define

$$A(\varepsilon) = \int_0^\varepsilon \Big(\sum_{k=0}^{n-1} \lambda_k(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}} \Big) dt$$

Applying inequalities (2.1) within the sum and then using (1.2), we have

$$\begin{aligned}
A(\varepsilon) &= \sum_{k=0}^{n-1} \int_0^\varepsilon \lambda_k(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}} dt \\
&\le \sum_{k=0}^{n-1} \int_0^\varepsilon \lambda_k(t) dt \int_0^\varepsilon \|\phi^{(n)}(t)\| dt \\
&\le \sum_{k=0}^{n-1} \int_0^\varepsilon \lambda_k(t) dt \int_0^\varepsilon \Big(\sum_{k=0}^{n-1} \lambda_k(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}} \Big) dt \\
&= \Big(\int_0^\varepsilon \sum_{k=0}^{n-1} \lambda_k(t) dt \Big) A(\varepsilon).
\end{aligned}$$

If $A(\varepsilon) \neq 0$, then

$$\int_0^\varepsilon \sum_{k=0}^{n-1} \lambda_k(t) dt \ge 1.$$

However, $\lambda_k \in L^1(a, b)$ means

$$\int_0^\varepsilon \sum_{k=0}^{n-1} \lambda_k(t) dt \to 0 \quad as \quad \varepsilon \to 0.$$

This contradiction implies that there exists $\varepsilon \in (0, b)$ such that $A(\tilde{\varepsilon}) = 0$, $\forall \tilde{\varepsilon} \le \varepsilon$. Integrating (2.2) on both sides,

$$\int_0^\varepsilon \|\phi^{(n)}(t)\| dt \le \int_0^\varepsilon \Big(\sum_{k=0}^{n-1} \lambda_k(t) \frac{\|\phi^{(k)}(t)\|}{t^{n-k-1}} \Big) dt = A(\varepsilon) = 0$$

Hence

$$\|\phi^{(n)}(t)\| = 0 \text{ a. e. } t \in (0, \varepsilon) \quad \Longrightarrow \quad \phi^{(n)}(t) = \vec{0} \text{ a. e. } t \in (0, \varepsilon).$$

Recall $\phi^{(n-1)}(0) = u^{(n-1)}(0) - v^{(n-1)}(0) = \vec{0}$. Thus

$$\phi^{(n-1)}(\varepsilon) = \int_0^\varepsilon \phi^{(n)}(t) dt = \vec{0} \quad \Longrightarrow \quad \phi^{(n-1)}(t) = \vec{0} \text{ a. e. } t \in (0, \varepsilon).$$

Repeating the argument results in

$$\phi(t) = u(t) - v(t) = \vec{0} \quad a.\ e.\ t \in (0, \varepsilon).$$

Since $u, v \in \mathcal{C}^{n-1}(a, b)$ and $u(0) = v(0)$, we obtain

$$\phi(t) = u(t) - v(t) \equiv \vec{0}, \quad \forall t \in [0, \varepsilon).$$

Let $\varepsilon' = \max\{\varepsilon : \phi(t) = \vec{0},\ \forall t \in (0, \varepsilon)\}$. If $\varepsilon' < b$, then $\phi(t) = \vec{0}$ for $t \in (0, \varepsilon']$ by the continuity of u and v. Then applying the above derivation to functions $u(x - \varepsilon')$,

$v(x-\varepsilon')$ on the interval $[\varepsilon', b)$ would yield $u(t)-v(t)=\vec{0}$ for $t\in(\varepsilon', \varepsilon'+\varepsilon'')$ for some $\varepsilon''>0$, which contradicts the definition of ε'. Therefore we must have

$$u(t)-v(t)=\vec{0}, \quad \forall t\in[0,b).$$

To obtain the results on $(a,0]$, replacing $u(x)$, $v(x)$, $x\in(a,b)$ by $u^*(x)=u(-x)$, $v^*(x)=v(-x)$, $x\in(-b,-a)$ to obtain

$$u^*(t)-v^*(t)=\vec{0} \quad \forall t\in[0,-a).$$

Combining the results we arrived at

$$u(t)-v(t)=\vec{0} \quad \forall t\in(a,b).$$

This concludes the proof for Theorem 1.1. □

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1.

Proof of Theorem 1.2. Notice that y satisfies the assumptions in Lemma 2.1. Define

$$B(\varepsilon)=\int_0^{\varepsilon}\Big(\sum_{k=0}^{n-1}\lambda_k(t)\frac{\|y^{(k)}(t)\|}{t^{n-k-1}}\Big)dt$$

for any $\varepsilon\in(0,b)$. Applying (2.1) in Lemma 2.1 and then (1.3),

$$\begin{aligned}
B(\varepsilon)&=\sum_{k=0}^{n-1}\int_0^{\varepsilon}\lambda_k(t)\frac{\|y^{(k)}(t)\|}{t^{n-k-1}}dt\le\sum_{k=0}^{n-1}\int_0^{\varepsilon}\lambda_k(t)dt\int_0^{\varepsilon}\|y^{(n)}(t)\|dt\\
&\le\sum_{k=0}^{n-1}\int_0^{\varepsilon}\lambda_k(t)dt\int_0^{\varepsilon}\Big(\sum_{k=0}^{n-1}\lambda_k(t)\frac{\|y^{(k)}(t)\|}{t^{n-k-1}}\Big)dt\\
&=\Big(\int_0^{\varepsilon}\sum_{k=0}^{n-1}\lambda_k(t)dt\Big)B(\varepsilon).
\end{aligned}$$

$B(\varepsilon)\neq 0$ would imply

$$\int_0^{\varepsilon}\sum_{k=0}^{n-1}\lambda_k(t)dt\ge 1.$$

On the other hand, $\lambda_k\in L^1(a,b)$ implies

$$\int_0^{\varepsilon}\sum_{k=0}^{n-1}\lambda_k(t)dt\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus there must exist $\varepsilon\in(0,b)$ such that $B(\tilde{\varepsilon})=0, \forall\tilde{\varepsilon}\le\varepsilon$. Integrating (1.3) on both sides,

$$\int_0^{\varepsilon}\|y^{(n)}(t)\|dt\le\int_0^{\varepsilon}\Big(\sum_{k=0}^{n-1}\lambda_k(t)\frac{\|y^{(k)}(t)\|}{t^{n-k-1}}\Big)dt=B(\varepsilon)=0$$

which leads to

$$\|y^{(n)}(t)\|=0 \text{ a. e. } t\in(0,\varepsilon) \quad\Longrightarrow\quad y^{(n)}(t)=\vec{0} \quad \text{a. e. } t\in(0,\varepsilon).$$

Consequently,

$$y^{(n-1)}(\varepsilon)=\int_0^{\varepsilon}y^{(n)}(t)dt=\vec{0} \quad\Longrightarrow\quad y^{(n-1)}(t)=\vec{0} \quad \text{a. e. } t\in(0,\varepsilon).$$

This argument leads to

$$y(t) = \vec{0} \quad \text{a. e. } t \in (0, \varepsilon).$$

where "a. e." can be removed by the continuity of y. The interval $(0, \varepsilon)$ on which $y \equiv \vec{0}$ can be extended to $(0, b)$ and $(a, 0)$ using arguments analogous to the ones used in the proof of Theorem 1.1. This concludes the proof of Theorem 1.2. □

3. Corollaries and an example

In Corollary 3.1 below, we give an explicit form of the L^1 functions in the Lipschitz condition (1.2) in Theorem 1.1 in terms of the Jacobians, under stronger differentiability conditions on the function f in the differential system (1.1) in Theorem 1.1. Recall that

$$f(x, s_0, s_1, \dots, s_{n-1}) : (a, b) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \quad \to \quad \mathbb{R}^d$$

where $f = (f_1, \dots, f_d) \in \mathbb{R}^d$, $s_k = (s_{k1}, \dots, s_{kd}) \in \mathbb{R}^d$, $k = 0, \dots, n-1$. For each $x \in (a, b)$, denote the Jacobian

$$J_k = J_k(x) = \det\Big(\frac{\partial f(x, s_0, \dots, s_{n-1})}{\partial s_k}\Big) = \begin{vmatrix} \frac{\partial f_1}{\partial s_{k1}} & \dots & \frac{\partial f_1}{\partial s_{kd}} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial s_{k1}} & \dots & \frac{\partial f_d}{\partial s_{kd}} \end{vmatrix},$$

for $k = 0, 1, \dots, n-1$.

Corollary 3.1. *If $f(x, s_0, \dots, s_{n-1})$ is differentiable on $(s_0, \dots, s_{n-1}) \in \mathbb{R}^{nd}$, a. e. $x \in (a, b)$, and*

$$J_k(x)\ x^{n-k-1} \in L^1(a, b), \quad k = 0, \dots, n-1,$$

then there are $\lambda_k(x) \in L^1(a, b)$ such that the Lipschitz condition (1.2) *holds in Theorem 1.1.*

Proof. By the differentiability of f and the mean value theorem,

$$f(x, s_0, \dots, s_{n-1}) - f(x, r_0, \dots, r_{n-1}) = Df(x, s')(s_0 - r_0, \dots, s_{n-1} - r_{n-1}),$$

a. e. $x \in (a, b)$, where

$$Df(x, s') = \Big(\frac{\partial f(x, s_0, \dots, s_{n-1})}{\partial s_0}, \dots, \frac{\partial f(x, s_0, \dots, s_{n-1})}{\partial s_{n-1}}\Big)\Big|_{(x, s')}$$

and $(x, s') = (x, s'_0, \dots, s'_{n-1}) \in (a, b) \times \mathbb{R}^{nd}$ is on the line connecting the two points $(x, s_0, \dots, s_{n-1})$ and $(x, r_0, \dots, r_{n-1})$. By matrix multiplication,

$$f(x, s_0, \dots, s_{n-1}) - f(x, r_0, \dots, r_{n-1}) = \sum_{k=0}^{n-1} \frac{\partial f(x, s_0, \dots, s_{n-1})}{\partial s_k}(x, s')\ (s_k - r_k),$$

a. e. $x \in (a, b)$. Taking the norm, we have

$$\|f(x, s_0, \dots, s_{n-1}) - f(x, r_0, \dots, r_{n-1})\| \le \sum_{k=0}^{n-1} |J_k(x, s'_k)|\, \|s_k - r_k\|,$$

a. e. $x \in (a, b)$. Therefore, the functions

$$\lambda_k(x) = J_k(x, s')\ x^{n-k-1} \in L^1(a, b), \quad k = 0, 1, \dots, n-1$$

satisfy the Lipschitz condition (1.2). □

When f in Theorem 1.1 is linear, the results can be stated as the corollary below.

Corollary 3.2. *Let $a < 0 < b$ and $y : (a, b) \to \mathbb{R}^d$ be a solution of*

$$y^{(n)} + a_{n-1}(x, y)y^{(n-1)} + \cdots + a_o(x, y)y = 0, \quad x \in (a, b),$$

where $y^{(n-1)}$ is absolutely continuous on (a, b), and the coefficient functions

$$|a_k(x, y)| \leq \frac{|\lambda_k(x)|}{|x|^{n-k-1}}, \quad \lambda_k \in L^1(a, b),\ k = 0, 1, \ldots, n-1.$$

If

$$y(0) = y^{(1)}(0) = \cdots = y^{(n-1)}(0) = \vec{0},$$

then $y \equiv \vec{0}$ on (a, b).

Theorem 1.1 is on uniqueness of n-th order differential systems where the nth derivative of the solution only needs to exist almost everywhere, which is in the spirit of the classical Carathéodory theorem on the existence of ordinary differential equations. Theorem 1.1 for $n = 1$ can be stated as follows.

Consider the differential equation

$$y' = f(x, y(x)), \quad x \in (a, b), \quad y(x_o) = y_o.$$

If $f : (a, b) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\|f(x, y_1) - f(x, y_2)\| \leq \lambda(x)\|y_1 - y_2\|, \quad a.\ e.\ x \in (a, b)$$

where $\lambda \in L^1(a, b)$, $\lambda \geq 0$, then the solution of the differential equation is unique.

We conclude by an example to show the sharpness of the orders in (1.2) and (1.3).

Example. For $p \in (0, 1/2)$, let

$$y = e^{-1/|x|^p}, \quad x \in (-1, 1), \quad y^{(k)}(0) = 0, \quad k = 0, \ldots, n-1.$$

Then y and its derivatives are even functions. For $x \in (0, 1)$, we have

$$\begin{aligned}
y' &= p\frac{y}{x^{p+1}} \\
y'' &= p\frac{y'}{x^{p+1}} - p(p+1)\frac{y}{x^{p+2}} \\
y''' &= p\frac{y''}{x^{p+1}} - 2p(p+1)\frac{y'}{x^{p+2}} + p(p+1)(p+2)\frac{y'}{x^{p+3}}
\end{aligned}$$

The general form can be written as

$$y^{(m)}(x) = \sum_{k=0}^{m-1} C_k^n \frac{y^{(k)}}{|x|^{m-k+p}}, \quad x \in (-1, 1) \setminus \{0\},\ m = 1, \ldots, n. \tag{3.1}$$

where C_k^m are constants depending on n, k and p. Formula (3.1) can be verified by induction. Taking derivative of (3.1) for $x \in (0,1)$,

$$\begin{aligned} y^{(m+1)}(x) &= \sum_{k=0}^{m-1} C_k^m \frac{y^{(k+1)}}{x^{m-k+p}} + \sum_{k=0}^{m-1} C_k^m(-m+k-p)\frac{y^{(k)}}{x^{m-k+p+1}} \quad (k'=k+1) \\ &= \sum_{k'=1}^{m} C_{k'-1}^m \frac{y^{(k')}}{x^{m+1-k'+p}} + \sum_{k=0}^{m-1} C_k^m(-m+k-p)\frac{y^{(k)}}{x^{m+1-k+p}} \\ &= \sum_{k=0}^{m} C_k^{m+1} \frac{y^{(k)}}{x^{m+1-k+p}} \end{aligned}$$

where C_k^{m+1} are constants depending on n, k and p. Therefore (3.1) holds for all $m = 1, \dots, n$. We may write the case of $m = n$ as

$$|y^{(n)}(x)| \le \sum_{k=0}^{n-1} \Big(\frac{C_k^n}{|x|^{1-p}}\Big) \frac{y^{(k)}(x)}{|x|^{n-k-1+2p}} = \sum_{k=0}^{n-1} \lambda_k(x) \frac{y^{(k)}(x)}{|x|^{n-k-1+2p}}, \quad \forall x \in (-1,1)\setminus\{0\}$$

where $\lambda_k \in L^1(-1,1)$ for $p \in (0,1/2)$. In (1.2) and (1.3), the order of singularity corresponding to $y^{(k)}$ is $n-k-1$. In this example, the corresponding order is $n-k-1+2p$, where $p \in (0,1/2)$ can be arbitrarily small. However it is enough to make $y \not\equiv 0$ on $(-1,1)$. Alternatively, we may write

$$|y^{(n)}(x)| \le \sum_{k=0}^{n-1} \Big(\frac{C_k^n}{|x|^{1+p}}\Big) \frac{y^{(k)}(x)}{|x|^{n-k-1}} = \sum_{k=0}^{n-1} \lambda_k^*(x) \frac{y^{(k)}(x)}{|x|^{n-k-1}}, \quad \forall x \in (-1,1)\setminus\{0\}$$

Now the order of singularity corresponding to $y^{(k)}$ is $n-k-1$ as in (1.2) and (1.3), however

$$\lambda_k^*(x) = \frac{C_k^n}{|x|^{1+p}} \quad \in L^q(-1,1), \quad \forall q < \frac{1}{1+p}, \quad p \in (0,1/2).$$

Thus we do not have $y \equiv 0$ as in the conclusions of the theorems.

We are interested in the uniqueness of solutions ordinary differential systems when the coefficients are allowed to have singularities [9, 10]. In this paper, We give a Lipschitz type condition for the uniqueness of weak solutions in the style of the Carathéodory theorem for nonlinear nth order nonlinear ordinary differential systems. We also give a unique continuation condition for functions satisfying an nth order Gronwall differential inequality. We use an example to show the sharpness of the order of singularity required in the conditions.

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Yifei Pan

Department of Mathematical Sciences, Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.

School of Mathematics and Informatics, Jiangxi Normal University, Nanchang, China

E-mail address: pan@ipfw.edu

Mei Wang

Department of Statistics, University of Chicago, Chicago, IL 60637, USA

E-mail address: meiwang@galton.uchicago.edu