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IMPULSIVE BOUNDARY-VALUE PROBLEMS FOR FIRST-ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article concerns boundary-value problems of first-order nonlinear impulsive integro-differential equations:

$$\begin{gathered} y'(t) + a(t)y(t) = f(t, y(t), (Ty)(t), (Sy)(t)), \quad t \in J_0, \\ \Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \dots, p, \\ y(0) + \lambda \int_0^c y(s)ds = -y(c), \quad \lambda \le 0, \end{gathered}$$

where $J_0 = [0, c] \setminus \{t_1, t_2, \dots, t_p\}$, $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, \mathbb{R})$ and $a(t) \le 0$ for $t \in [0, c]$. Sufficient conditions for the existence of coupled extreme quasi-solutions are established by using the method of lower and upper solutions and monotone iterative technique. Wang and Zhang [18] studied the existence of extremal solutions for a particular case of this problem, but their solution is incorrect.

1. Introduction

In recent years, many authors have paid attention to the research of differential equations with impulsive boundary conditions, because of their potential applications; see for example [4, 6, 9, 12, 13, 15, 17]. First-order and second-order impulsive differential equations with anti-periodic boundary conditions have also drawn much attention; see [1, 2, 3, 5, 7, 8, 14, 16, 19].

Recently, Wang and Zhang [18] studied the existence of extremal solutions of the following nonlinear anti-periodic boundary value problem of first-order integro-differential equation with impulse at fixed points

$$\begin{gathered} y'(t) = f(t, y(t), (Ty)(t), (Sy)(t)), \quad t \in J_0, \\ \Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \dots, p, \\ y(0) = -y(T), \end{gathered} \tag{1.1}$$

where $J = [0, T]$, $J_0 = J \setminus \{t_1, t_2, \dots, t_p\}$, $0 < t_1 < t_2 < \dots < t_p < T$, $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ denotes the jump of

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$y(t)$ at $t = t_k$; $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively.

$$(Ty)(t) = \int_0^t k(t,s)y(s)ds, \quad (Sy)(t) = \int_0^T h(t,s)y(s)ds,$$

$k \in C(D, \mathbb{R}^+)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $h \in C(J \times J, \mathbb{R}^+)$. Unfortunately, their extremal solutions $y_*(t), y^*(t)$ are wrong. In fact, by [18, Theorem 3.1] we obtain

$$\begin{gathered} y_*'(t) = f(t, y_*(t), (Ty_*)(t), (Sy_*)(t)), \quad t \in J_0, \\ \Delta y_*(t_k) = I_k(y_*(t_k)), \quad k = 1, 2, \dots, p, \\ y_*(0) = -y^*(T), \end{gathered}$$

and

$$\begin{gathered} y^{*\prime}(t) = f(t, y^*(t), (Ty^*)(t), (Sy^*)(t)), \quad t \in J_0, \\ \Delta y^*(t_k) = I_k(y^*(t_k)), \quad k = 1, 2, \dots, p, \\ y^*(0) = -y_*(T), \end{gathered}$$

which implies that $y_*(t), y^*(t)$ are not solutions of (1.1). So the conclusions of [18] are reconsidered here, for a more general equation.

In this paper, we investigate the following integral boundary value problem for first-order integro-differential equation with impulses at fixed points

$$\begin{gathered} y'(t) + a(t)y(t) = f(t, y(t), (Ty)(t), (Sy)(t)), \quad t \in J_0, \\ \Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \dots, p, \\ y(0) + \lambda \int_0^c y(s)ds = -y(c), \quad \lambda \leq 0, \end{gathered} \tag{1.2}$$

where $J = [0, c]$, $J_0 = J \setminus \{t_1, t_2, \dots, t_p\}$, $0 < t_1 < t_2 < \dots < t_p < c$, $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, \mathbb{R})$ and $a(t) \leq 0$ for $t \in J$.

$$(Ty)(t) = \int_0^t k(t,s)y(s)ds, \quad (Sy)(t) = \int_0^c h(t,s)y(s)ds,$$

$k \in C(D, \mathbb{R}^+)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $h \in C(J \times J, \mathbb{R}^+)$.

Remark 1.1. If $a(t) \equiv 0$ and $\lambda \equiv 0$, then (1.2) reduces to (1.1).

We will give the concept of coupled quasi-solutions of BVP (1.2) in next section. It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations, for details, see [10, 11] and the references therein. The aim of this paper is to investigate the existence of coupled quasi-solutions of (1.2) by using the method of upper and lower solutions combined with a monotone iterative technique. Our result correct and generalize the main result of [18].

2. Preliminaries

In this section, we present some definitions needed for introducing the concept of quasi-solutions for (1.2). Let

$$\begin{aligned} PC(J) = \{ &y : J \to \mathbb{R} : y \text{ is continuous at } t \in J_0; \\ &y(0^+), y(T^-), y(t_k^+), y(t_k^-) \text{ exist and } y(t_k^-) = y(t_k),\ k = 1, \dots, p\}, \end{aligned}$$

$$
\begin{aligned}
PC^1(J) = \{y \in PC(J) : y \text{ is continuously differentiable for } t \in J_0;\\
y'(0^+), y'(T^-), y'(t_k^+), y'(t_k^-) \text{ exist}, \ k = 1, \dots, p\},
\end{aligned}
$$

The sets $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$
\|y\|_{PC(J)} = \sup\{|y(t)| : t \in J\}, \quad \|y\|_{PC^1(J)} = \|y\|_{PC(J)} + \|y'\|_{PC(J)}.
$$

Definition 2.1. Functions $\alpha_0, \beta_0 \in PC^1(J)$ are said to be coupled lower-upper quasi-solutions to the problem (1.2) if

$$
\begin{gathered}
\alpha_0'(t) + a(t)\alpha_0(t) \le f(t, \alpha_0(t), (T\alpha_0)(t), (S\alpha_0)(t)), \quad t \in J_0,\\
\Delta\alpha_0(t_k) \le I_k(\alpha_0(t_k)), \quad k = 1, 2, \dots, p,\\
\alpha_0(0) + \lambda \int_0^c \alpha_0(s)ds \le -\beta_0(c), \quad \lambda \le 0,\\
\beta_0'(t) + a(t)\beta_0(t) \ge f(t, \beta_0(t), (T\beta_0)(t), (S\beta_0)(t)), \quad t \in J_0,\\
\Delta\beta_0(t_k) \ge I_k(\beta_0(t_k)), \quad k = 1, 2, \dots, p,\\
\beta_0(0) + \lambda \int_0^c \beta_0(s)ds \ge -\alpha_0(c), \quad \lambda \le 0.
\end{gathered}
\tag{2.1}
$$

Note that if $\alpha_0(c) = \beta_0(c)$, then the above definition reduces to the notion of lower and upper solutions of (1.2).

Definition 2.2. Functions $v, w \in PC^1(J)$ are said to be coupled quasi-solutions to (1.2) if

$$
\begin{gathered}
v'(t) + a(t)v(t) = f(t, v(t), (Tv)(t), (Sv)(t)), \quad t \in J_0,\\
\Delta v(t_k) = I_k(v(t_k)), \quad k = 1, 2, \dots, p,\\
v(0) + \lambda \int_0^c v(s)ds = -w(c), \quad \lambda \le 0,\\
w'(t) + a(t)w(t) = f(t, w(t), (Tw)(t), (Sw)(t)), \quad t \in J_0,\\
\Delta w(t_k) = I_k(w(t_k)), \quad k = 1, 2, \dots, p,\\
w(0) + \lambda \int_0^c w(s)ds = -v(c), \quad \lambda \le 0.
\end{gathered}
\tag{2.2}
$$

Let $\alpha_0, \beta_0 \in PC^1(J)$ and $\alpha_0(t) \le \beta_0(t)$ for $t \in J_0$. In what follows we define the segment

$$
[\alpha_0, \beta_0] = \{u \in PC^1(J) : \alpha_0(t) \le u(t) \le \beta_0(t), \ t \in J\}.
$$

Definition 2.3. Let u, v be coupled quasi-solutions of (1.2) such as $u(t) \le v(t)$ for $t \in J_0$. Assume that $\alpha_0, \beta_0 \in PC^1(J)$ and $\alpha_0(t) \le \beta_0(t)$ for $t \in J_0$. Coupled quasi-solutions u, v of (1.2) are called coupled minimal-maximal quasi-solutions in segment $[\alpha_0, \beta_0]$ if $\alpha_0(t) \le u(t)$, $v(t) \le \beta_0(t)$ for $t \in J_0$ and for any U, V coupled quasi-solutions of (1.2), such as $\alpha_0(t) \le U(t)$, $V(t) \le \beta_0(t)$ for $t \in J_0$ we have $u(t) \le U(t)$ and $V(t) \le v(t)$, $t \in J_0$.

For convenience, we assume the following conditions are satisfied

- (H1) Functions $\alpha_0(t), \beta_0(t)$ are coupled lower-upper quasi-solutions of (1.2) such that $\alpha_0(t) \le \beta_0(t)$ for $t \in J_0$.
- (H2) There exist $M > 0, N, N_1 \ge 0$ such that

$$
f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \ge -M(x_1 - x_2) - N(y_1 - y_2) - N_1(z_1 - z_2),
$$

for $\alpha_0 \le x_2 \le x_1 \le \beta_0$, $T\alpha_0 \le y_2 \le y_1 \le T\beta_0$, $S\alpha_0 \le z_2 \le z_1 \le S\beta_0$, $t \in J$.

(H3) There exist $0 \le L_k < 1$, $k = 1, 2, \dots, p$, satisfy

$$I_k(x) - I_k(y) \ge -L_k(x - y),$$

for $\alpha_0 \le y \le x \le \beta_0$, $t \in J$.

Now we consider the problem

$$\begin{gathered} y'(t) + My(t) + N(Ty)(t) + N_1(Sy)(t) = \sigma(t), \quad t \in J_0, \\ \Delta y(t_k) = -L_k y(t_k) + b_k, \quad k = 1, 2, \dots, p, \\ y(0) = b, \end{gathered} \tag{2.3}$$

where $M > 0$, $N, N_1 \ge 0$, $L_k < 1$, $k = 1, 2, \dots, p$.

Lemma 2.4. *If $y \in PC^1(J)$, $M > 0$, $N, N_1 \ge 0$, $L_k < 1$, $k = 1, 2, \dots, p$, and*

$$\bar{k} + \bar{h} + \sum_{i=1}^{p} L_i < 1, \tag{2.4}$$

where

$$\bar{k} = \begin{cases} k_0 c M^{-1}(1 - e^{-Mc}), & \text{if } M > 1, \\ k_0 c M^{-1}(1 - M e^{-Mc}), & \text{if } 0 < M \le 1, \\ \frac{1}{2} k_0 c^2, & \text{if } M = 0. \end{cases}$$

$$\bar{h} = \begin{cases} h_0 c M^{-1}(1 - e^{-Mc}), & \text{if } M > 0, \\ h_0 c^2, & \text{if } M = 0, \end{cases}$$

where $k_0 = \max_{0 \le s \le t \le c} k(t,s)$ and $h_0 = \max_{0 \le t,s \le c} h(t,s)$. Then (2.3) has a unique solution.

Proof. If $y \in PC^1(J)$ is a solution of (2.3), then, by integrating, we obtain

$$\begin{aligned} y(t) = be^{-Mt} &+ \int_0^t e^{-M(t-s)}[\sigma(s) - N(Ty)(s) - N_1(Sy)(s)]ds \\ &+ \sum_{0<t_i<t} e^{-M(t-t_i)}(-L_i y(t_i) + b_i). \end{aligned} \tag{2.5}$$

Conversely, if $y(t) \in PC(J)$ is solution of the above-mentioned integral equation (2.5), then it is easy to check that $y'(t) = -My(t) - N(Ty)(t) - N_1(Sy)(t) + \sigma(t)$, $t \ne t_k$, $\Delta y(t_k) = -L_k y(t_k) + b_k$, $k = 1, 2, \dots, p$, and $y(0) = b$. So (2.3) is equivalent to the integral equation (2.5). Now, we define operator $B : PC(J) \to PC(J)$ as

$$\begin{aligned} (By)(t) = be^{-Mt} &+ \int_0^t e^{-M(t-s)}[\sigma(s) - N(Ty)(s) - N_1(Sy)(s)]ds \\ &+ \sum_{0<t_i<t} e^{-M(t-t_i)}(-L_i y(t_i) + b_i). \end{aligned} \tag{2.6}$$

For each $u, v \in PC(J)$, we have

$$
\begin{aligned}
|(Bu)(t)-(Bv)(t)| \le & N\Big|\int_0^t e^{-M(t-s)}(Tu-Tv)(s)ds\Big| \\
& + N_1\Big|\int_0^t e^{-M(t-s)}(Su-Sv)(s)ds\Big| \\
& + \sum_{0<t_i<t} L_i|e^{-M(t-t_i)}(u(t_i)-v(t_i))|.
\end{aligned} \tag{2.7}
$$

We easily check that

$$
\begin{aligned}
&|\int_0^t e^{-M(t-s)}(Tu-Tv)(s)ds| \\
&\le \begin{cases} k_0tM^{-1}(1-e^{-Mt})\|u-v\|_{PC}, & \text{if } M>1, \\ k_0tM^{-1}(1-Me^{-Mt})\|u-v\|_{PC}, & \text{if } 0<M\le 1, \\ k_0\frac{1}{2}t^2\|u-v\|_{PC}, & \text{if } M=0, \end{cases}
\end{aligned} \tag{2.8}
$$

and

$$
|\int_0^t e^{-M(t-s)}(Su-Sv)(s)ds| \le \begin{cases} h_0cM^{-1}(1-e^{-Mt})\|u-v\|_{PC}, & \text{if } M>0, \\ h_0ct\|u-v\|_{PC}, & \text{if } M=0. \end{cases} \tag{2.9}
$$

Substituting (2.8) and (2.9) into (2.7), we obtain

$$
\|Bu-Bv\|_{PC} \le (\bar{k}+\bar{h}+\sum_{i=1}^{p} L_i)\|u-v\|_{PC}.
$$

This indicates that B is a contraction mapping (by (2.4)). Then there is one unique $y \in PC(J)$ such that $By=y$, that is, (2.3) has a unique solution. □

Lemma 2.5 ([18]). *Assume that $y \in PC^1(J)$ satisfies*

$$
\begin{gathered}
y'(t)+My(t)+N(Ty)(t)+N_1(Sy)(t) \le 0, \quad t\in J_0, \\
\Delta y(t_k) \le -L_k y(t_k), \quad k=1,2,\dots,p, \\
y(0)\le 0,
\end{gathered} \tag{2.10}
$$

where $M>0$, $N, N_1 \ge 0$, $L_k<1$, $k=1,2,\dots,p$, and

$$
\int_0^c q(s)ds \le \prod_{j=1}^{p}(1-\bar{L}_j) \tag{2.11}
$$

with $\bar{L}_k=\max\{L_k,0\}$, $k=1,2,\dots,p$,

$$
q(t)=N\int_0^t k(t,s)e^{M(t-s)}\prod_{s<t_k<c}(1-L_k)ds+N_1\int_0^c h(t,s)e^{M(t-s)}\prod_{s<t_k<c}(1-L_k)ds,
$$

then $y\le 0$.

3. Main result

Theorem 3.1. *If* (H1),(H2),(H3) *are satisfied, and, in addition, if there exist $M>0$, $N, N_1\ge 0$, $L_k<1$, $k=1,2,\dots,p$, such that* (2.4) *and* (2.11) *hold, then* (1.2) *has, in segment $[\alpha_0,\beta_0]$ the coupled minimal-maximal quasi-solutions.*

Proof. For convenience, let $(K\phi)(t) = N(T\phi)(t) + N_1(S\phi)(t)$. We now construct two sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ that satisfy the following problems

$$\begin{gathered} \alpha_i'(t) + a(t)\alpha_{i-1}(t) + M\alpha_i(t) + (K\alpha_i)(t) \\ = f(t, \alpha_{i-1}(t), (T\alpha_{i-1})(t), (S\alpha_{i-1})(t)) + M\alpha_{i-1}(t) + (K\alpha_{i-1})(t), \quad t \in J_0, \\ \Delta\alpha_i(t_k) = I_k(\alpha_{i-1}(t_k)) - L_k(\alpha_i(t_k) - \alpha_{i-1}(t_k)), \quad k = 1, 2, \dots, p, \\ \alpha_i(0) + \lambda \int_0^c \alpha_{i-1}(s)ds = -\beta_{i-1}(c), \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} \beta_i'(t) + a(t)\beta_{i-1}(t) + M\beta_i(t) + (K\beta_i)(t) \\ = f(t, \beta_{i-1}(t), (T\beta_{i-1})(t), (S\beta_{i-1})(t)) + M\beta_{i-1}(t) + (K\beta_{i-1})(t), \quad t \in J_0, \\ \Delta\beta_i(t_k) = I_k(\beta_{i-1}(t_k)) - L_k(\beta_i(t_k) - \beta_{i-1}(t_k)), \quad k = 1, 2, \dots, p, \\ \beta_i(0) + \lambda \int_0^c \beta_{i-1}(s)ds = -\alpha_{i-1}(c). \end{gathered} \tag{3.2}$$

For each $\phi, \psi \in [\alpha_0, \beta_0]$, we consider the equation

$$\begin{gathered} y'(t) + My(t) + (Ky)(t) \\ = f(t, \phi(t), (T\phi)(t), (S\phi)(t)) - a(t)\phi(t) + M\phi(t) + (K\phi)(t), \quad t \in J_0, \\ \Delta y(t_k) = I_k(\phi(t_k)) - L_k(y(t_k) - \phi(t_k)), \quad k = 1, 2, \dots, p, \\ y(0) + \lambda \int_0^c \phi(s)ds = -\psi(c). \end{gathered} \tag{3.3}$$

By condition (2.4) and Lemma 2.4, we know that (3.3) has a unique solution $y(t) \in PC^1(J)$. Define the operator $A : PC^1(J) \times PC^1(J) \to PC^1(J)$ as $A(\phi, \psi) = y$. Let $\alpha_n(t) = A(\alpha_{n-1}, \beta_{n-1})(t)$ and $\beta_n(t) = A(\beta_{n-1}, \alpha_{n-1})(t)$, $n = 1, 2, \dots$, we will prove that $\{\alpha_n\}$, $\{\beta_n\}$ have the following properties.

(i) $\alpha_{i-1} \le \alpha_i$, $\beta_i \le \beta_{i-1}$;
(ii) $\alpha_i \le \beta_i$, $i = 1, 2, \dots$.

Firstly, we prove that $\alpha_0 \le \alpha_1$. Set $p(t) = \alpha_0(t) - \alpha_1(t)$, it follows that

$$\begin{gathered} p'(t) + Mp(t) + N(Tp)(t) + N_1(Sp)(t) = p'(t) + Mp(t) + (Kp)(t) \le 0, \\ \Delta p(t_k) \le -L_k p(t_k), \quad k = 1, 2, \dots, p, \\ p(0) \le 0. \end{gathered} \tag{3.4}$$

Then by condition (2.11) and Lemma 2.5, we get $p(t) \le 0$, which implies that $\alpha_0(t) \le \alpha_1(t)$, for all $t \in J_0$. In a similar way, it can be proved that $\beta_1(t) \le \beta_0(t)$, for all $t \in J_0$. Now we prove that $\alpha_1(t) \le \beta_1(t)$, for all $t \in J_0$. In fact, setting $p(t) = \alpha_1(t) - \beta_1(t)$ and using assumption, we obtain

$$\begin{aligned} &p'(t) + Mp(t) + N(Tp)(t) + N_1(Sp)(t) \\ &= \alpha_1'(t) - \beta_1'(t) + M(\alpha_1(t) - \beta_1(t)) + N(T\alpha_1(t) - T\beta_1(t)) + N_1(S\alpha_1(t) - S\beta_1(t)) \\ &= f(t, \alpha_0(t), (T\alpha_0)(t), (S\alpha_0)(t)) - a(t)\alpha_0(t) + M\alpha_0(t) + N(T\alpha_0)(t) + N_1(S\alpha_0)(t) \\ &\quad - f(t, \beta_0(t), (T\beta_0)(t), (S\beta_0)(t)) + a(t)\beta_0(t) - M\beta_0(t) - N(T\beta_0)(t) - N_1(S\beta_0)(t) \\ &\le a(t)(\beta_0(t) - \alpha_0(t)) \le 0, \quad t \in J_0, \end{aligned}$$

and

$$\Delta p(t_k) = -L_k p(t_k) + I_k(\alpha_0(t_k)) - I_k(\beta_0(t_k)) + L_k\alpha_0(t_k) - L_k\beta_0(t_k) \le -L_k p(t_k),$$

$$p(0)=\alpha_1(0)-\beta_1(0)=\lambda\int_0^c(\beta_0(s)-\alpha_0(s))ds+\alpha_0(c)-\beta_0(c)\le 0.$$

Again by Lemma 2.5, we obtain $p(t)\le 0$, that is, $\alpha_1(t)\le\beta_1(t)$ for all $t\in J_0$. Thus we have $\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)$ for all $t\in J_0$. Continuing this process, by induction, one can obtain monotone sequence $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ such that

$$\alpha_0(t)\le\alpha_1(t)\le\dots\le\alpha_n(t)\le\dots\le\beta_n(t)\le\dots\beta_1(t)\le\beta_0(t),\quad t\in J_0,$$

where each $\alpha_i(t),\beta_i(t)\in PC^1(J)$ satisfies (3.1) and (3.2). As the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are uniformly bounded and equi-continuous, by employing the standard arguments Ascoli-Arzela criterion [12], we conclude that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly on J_0 with

$$\lim_{n\to\infty}\alpha_n(t)=y_*(t),\quad \lim_{n\to\infty}\beta_n(t)=y^*(t).$$

Obviously, $y_*(t),y^*(t)$ are coupled lower-upper quasi-solutions of (1.2). Now we have to prove that (y_*,y^*) are coupled minimal-maximal quasi-solutions of problem (1.2) in segment $[\alpha_0,\beta_0]$. Let x,z be coupled quasi-solutions of (1.2) such that

$$\alpha_n(t)\le x(t),\quad z(t)\le\beta_n(t),\quad t\in J_0$$

for some $n\in\mathbf{N}$. Put $q(t)=\alpha_{n+1}(t)-x(t)$, for $t\in J_0$. Form definition of α_{n+1} and properties of quasi-solution $x(t)$, we obtain

$$\begin{aligned}
&q'(t)+Mq(t)+N(Tq)(t)+N_1(Sq)(t)\\
&=f\big(t,\alpha_n(t),(T\alpha_n)(t),(S\alpha_n)(t)\big)-a(t)\alpha_n(t)+M\alpha_n(t)+N(T\alpha_n)(t)\\
&\quad+N_1(S\alpha_n)(t)-f\big(t,x(t),(Tx)(t),(Sx)(t)\big)+a(t)x(t)-Mx(t)\\
&\quad-N(Tx)(t)-N_1(Sx)(t)\\
&\le a(t)(x(t)-\alpha_n(t))\le 0,\quad t\in J_0,
\end{aligned}$$

and

$$\Delta q(t_k)=-L_kq(t_k)+I_k(\alpha_n(t_k))-I_k(x(t_k))+L_k\alpha_n(t_k)-L_kx(t_k)\le -L_kq(t_k),$$
$$q(0)=\alpha_{n+1}(0)-x(0)=\lambda\int_0^c(x(s)-\alpha_n(s))ds+z(c)-\beta_n(c)\le 0.$$

By Lemma 2.5, we have $q(t)\le 0$ for all $t\in J_0$, that is $\alpha_{n+1}(t)\le x(t)$. Similarly, we can prove that $z(t)\le\beta_{n+1}(t)$ for all $t\in J_0$.

By induction, we obtain

$$\alpha_m(t)\le x(t),\quad z(t)\le\beta_m(t),\quad t\in J_0,\quad \text{for } m\in\mathbf{N}.$$

If $m\to\infty$, it yields

$$y_*(t)\le x(t),\quad z(t)\le y^*(t),\quad t\in J_0.$$

It shows that (y_*,y^*) are coupled minimal-maximal quasi-solutions of problem (1.2) in segment $[\alpha_0,\beta_0]$. □

Example 3.2. Consider the problem

$$
\begin{gathered}
y'(t) - \frac{t}{4}(1-e^{-t})y(t) = -y(t) - \frac{1}{8}\int_0^t te^{-(t-s)}y(s)ds - \frac{5}{6}\int_0^1 y(s)ds, \\
t \in [0,t_1) \cup (t_1,1], \\
\Delta y(t_1) = -\frac{1}{9}y(t_1), \quad t_1 = \frac{1}{3} \\
y(0) - \frac{1}{6}\int_0^1 y(s)ds = -y(1).
\end{gathered} \tag{3.5}
$$

where $a(t) = -\frac{t}{4}(1-e^{-t}) \le 0$, $I_1(x) = -\frac{1}{9}x$, $L_1 = \frac{1}{9}$ and $\lambda = -\frac{1}{6} < 0$. Let $f(t,x,y,z) = -Mx - Ny - N_1 z$, $M = 1, N = \frac{3}{8}$, $N_1 = \frac{5}{6}$, $J = [0,1]$, $c = 1$, $k(t,s) = \frac{t}{3}e^{-(t-s)}$, $h(t,s) = 1$, then for $t \in J$, $x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2$, $x_1 \ge x_2$, $y_1 \ge y_2$, $z_1 \ge z_2$,

$$
f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) = -(x_1 - x_2) - \frac{3}{8}(y_1 - y_2) - \frac{5}{6}(z_1 - z_2).
$$

Thus the condition (H2) holds. It is easy to see that $k_0 = \frac{1}{3}$, $h_0 = 1$, $\bar{k} = \frac{1}{3}\bar{h} = \frac{1}{3}(1-e^{-1})$ and

$$
\bar{h} + \bar{k} + L_1 = 0.9359 < 1.
$$

Hence the condition (2.4) holds. Moreover, we have

$$
\begin{aligned}
\int_0^1 q(s)ds &\le \int_0^1 \Big(\frac{3}{8}\int_0^t \frac{t}{3}e^{-(t-s)}e^{(t-s)}(1-L_1)ds + \frac{5}{6}\int_0^1 e^{(t-s)}(1-L_1)ds\Big)dt \\
&= \int_0^1 \Big(\frac{t^2}{18} + \frac{20}{27}(1-e^{-1})e^t\Big)dt \\
&= \frac{1}{54} + \frac{20}{27}(e + e^{-1} - 2) = 0.8231 < 0.8889 = 1 - L_1,
\end{aligned}
$$

which implies that the condition (2.11) holds. Let

$$
\alpha_0(t) = -\frac{5}{4}, \quad \beta_0(t) = 2 - t, \quad t \in [0,1].
$$

Then $\alpha_0(t)$ and $\beta_0(t)$ are coupled lower-upper quasi-solutions of problem (**??**). In fact,

$$
\begin{aligned}
\alpha_0'(t) + a(t)\alpha_0(t) &= \frac{5}{16}t(1-e^{-t}) \le 2 + \frac{5}{32}t(1-e^{-t}) \\
&< \frac{5}{4} + \frac{5}{32}\int_0^t te^{-(t-s)}ds + \frac{25}{24}\int_0^1 ds \\
&= f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t)),
\end{aligned}
$$

$$
\Delta\alpha_0(1/3) = 0 < \frac{5}{36} = -L_1\alpha_0(1/3)
$$

$$
\alpha_0(0) - \frac{1}{6}\int_0^1 \alpha_0(s)ds = -\frac{25}{24} < -1 = -\beta_0(1),
$$

and

$$\begin{aligned}
\beta_0'(t) + a(t)\beta_0(t) &= -1 - \frac{1}{4}t(1-e^{-t})(2-t) \\
&\geq -1 - \frac{1}{4}(1-e^{-1}) \\
&> -\frac{27}{12} + \frac{3}{8}e^{-1} \\
&\geq t - 2 - \frac{1}{8}t(3-t) + \frac{3}{8}te^{-t} - \frac{15}{12} \\
&= t - 2 - \frac{1}{8}\int_0^t te^{-(t-s)}(2-s)ds - \frac{5}{6}\int_0^1 (2-s)ds \\
&= f(t, \beta_0(t), (T\beta_0)(t), (S\beta_0)(t)),
\end{aligned}$$

$$\Delta\beta_0(1/3) = 0 > -\frac{5}{27} = -L_1\beta_0(1/3)$$

$$\beta_0(0) - \frac{1}{6}\int_0^1 \beta_0(s)ds = \frac{7}{4} > \frac{5}{4} = -\alpha_0(1).$$

Obviously, $\alpha_0(t) \leq \beta_0(t)$. Thus, all the conditions of Theorem 3.1 are satisfied, so problem (3.5) has the coupled minimal-maximal quasi-solutions in the segment $[\alpha_0(t), \beta_0(t)]$.

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