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CONTINUABILITY AND BOUNDEDNESS OF SOLUTIONS TO NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Continuability, boundedness, and monotonicity of solutions for a class of second-order nonlinear differential equations are discussed. It is proved that all solutions are eventually monotonic and can be extended to infinity under some natural assumptions. Moreover, necessary and sufficient conditions for boundedness of all solutions are established. The results obtained have extended and improved some analogous existing ones.

1. Introduction

In this article we consider the continuability, boundedness, and monotonicity of solutions for the second-order nonlinear differential equation

$$[p(t)h(x(t))f(x'(t))]' = q(t)g(x(t)), \quad t \geq a. \tag{1.1}$$

The behavior, such as continuability, boundedness, monotonicity, osciallation, and asymptoticity, of solutions to second-order differential equations

$$[p(t)x'(t)]' = q(t)g(x(t)), \quad t \geq a, \tag{1.2}$$

$$[p(t)h(x(t))x'(t)]' = q(t)g(x(t)), \quad t \geq a, \tag{1.3}$$

both are special cases of (1.1) with $f(r) = r$, has been extensively discussed by many authors; see, e.g., [7, 8, 9, 15, 18] and references therein. In the case of $f(r) = \Phi_p(r) = |r|^{p-2}r$, $p > 1$, the so-called p-Laplacian operator, that are for half-linear equations

$$[p(t)\Phi_p(x'(t))]' = q(t)\Phi_p(x(t)), \quad t \geq a, \tag{1.4}$$

Emden-Fowler type equations

$$[p(t)\Phi_p(x'(t))]' = q(t)\Phi_\beta(x(t)), \quad t \geq a, \tag{1.5}$$

and more general equations

$$[p(t)\Phi_p(x'(t))]' = q(t)g(x(t)), \quad t \geq a. \tag{1.6}$$

A considerable effort has been devoted to the study of continuability, boundedness, monotonicity, and asymptoticity of solutions due to their various applications; see for example [1, 3, 4, 5, 6, 10, 11, 12, 14, 16, 17, 19].

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Wang [20] discussed properties of solutions for (1.1) with general increasing functions f in the case $h \equiv 1$; the results in [20] extended and improved many results obtained for (1.4), (1.5), and (1.6). However, we discovered that nontrivial functions h play an important role to the continuability, boundedness, and monotonicity for solutions of (1.1). The main contribution of this article is to address the role of the functions h in the discussion of the continuability and boundedness for solutions of (1.1); see Theorems 2.3, 2.4, and 4.1. For example, from [13] we know that the assumption (H3) (see Section 2) can not be omitted for the continuability and boundedness of solutions to Emden-Fowler equation (1.5), but (H3) does not hold in the case $\beta > p$. This situation can be improved by introducing a nontrivial function h in the differential operator; consider a simple differential equation

$$[t^2 h(x)x']' = \frac{1}{t^2}\Phi_3(x), \quad t \geq 1. \tag{1.7}$$

we can not use the results in [13] to decide the continuability and boundedness of all solutions for (1.7) in the case $h \equiv 1$ since (H3) is obviously invalid. However, with $h(r) = r^2 + 1$, (H3) holds since

$$\int_1^\infty \frac{dr}{f^{-1}(z(r))} = \int_1^\infty \frac{(r^2+1)dr}{r^3} = \infty$$

and

$$\int_{-\infty}^{-1} \frac{dr}{f^{-1}(z(r))} = -\int_{-\infty}^{-1} \frac{(r^2+1)dr}{r^3} = -\infty.$$

By Theorem 2.3 all solutions of (1.7) can be extended to $[1, \infty)$. Moreover, it is easy to verify that

$$J_1 = \int_1^\infty \Big(\frac{1}{t^2}\int_1^t \frac{1}{s^2}ds\Big)\,dt < \infty, \quad J_2 = -\int_1^\infty \Big(\frac{1}{t^2}\int_1^t \frac{1}{s^2}ds\Big)\,dt > -\infty.$$

It follows from Theorem 4.1 that all solutions of (1.7) are bounded on $[1, \infty)$.

The results obtained in this article generalize, complement, or improve some analogous ones existing in the literature. By solution of (1.1) we mean a differentiable function x such that $p(t)h(x(t))f(x'(t))$ is differentiable and satisfies (1.1) on $[a, \alpha_x)$, $\alpha_x \leq \infty$, the maximum existence interval of x. A solution x of (1.1) is said to be eventually monotonic if there exists a $t_1 \geq a$ such that it is monotonic on $[t_1, \alpha_x)$.

In this article we consider only nontrivial solutions of (1.1), in other words, solutions that are not identically equal to zero on their existence interval.

Throughout this article, we assume that

(H)
- $p(t), q(t) : [a, \infty) \to \mathbb{R}$ are continuous and $p(t) > 0$ and $q(t) > 0$;
- $h(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $h(r) > 0$;
- $g(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $rg(r) > 0$ for $r \neq 0$;
- $f(r) : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and $rf(r) > 0$ for $r \neq 0$.

(H1) There exists a constant $M_1 > 0$ such that

$$|f^{-1}(uv)| \leq M_1 |f^{-1}(u)||f^{-1}(v)|, \quad \forall u, v \in \mathbb{R}.$$

Remark 1.1. Assumption (H1) holds for $f(r) = \Phi_p(r)$ with $M_1 = 1$. In fact, we have

$$f^{-1}(uv) = f^{-1}(u)f^{-1}(v), \quad \forall u, v \in \mathbb{R}. \tag{1.8}$$

Remark 1.2. Let

$$f(r)=\begin{cases} r, & |r|\le 1,\\ \sqrt[3]{r}, & |r|>1.\end{cases}$$

Then

$$f^{-1}(r)=\begin{cases} r, & |r|\le 1,\\ r^3, & |r|>1.\end{cases}$$

It is easy to see that (H1) holds with $M_1=1$, but (1.8) does not hold in this case.

We will prove that the monotonicity and boundedness properties of solutions to (1.1) can be characterized by means of the convergence of the following two integrals

$$\begin{gathered} J_1:=\int_a^\infty f^{-1}\Big(\frac{1}{p(t)}\int_a^t q(s)ds\Big)dt,\\ J_2:=\int_a^\infty f^{-1}\Big(-\frac{1}{p(t)}\int_a^t q(s)ds\Big)dt. \end{gathered}$$

This article is organized as follows: Section 1 is the introduction. The background, motivation, and the main contributions of the paper are briefly addressed in this section. Continuability of solutions is discussed in Section 2. Section 3 deals with the existence of class A and class B solutions. In Section 4, necessary and sufficient conditions for boundedness of all solutions are established. Also, several examples and remarks are provided in this section to compare our results with some known results in the literature.

2. Continuability of Solutions

In this section we discuss the continuability of solutions to (1.1). First of all, we give two lemmas that will be used later on. The first lemma is a minor extension of Proposition 1 in [15].

Lemma 2.1. *If $x(\cdot)$ is a solution of (1.1) with maximal existence interval $[a,\alpha_x)$, $0<\alpha_x\le\infty$, then $x(\cdot)$ is eventually monotonic.*

Proof. Let $F(t)=p(t)h(x(t))f(x'(t))x(t)$. Note that $F(t)$ is continuous on $[a,\alpha_x)$ and $F'(t)=q(t)g(x(t))x(t)+p(t)h(x(t))f(x'(t))x'(t)\ge 0$, then $F(t)$ is nondecreasing on $[a,\alpha_x)$. The rest part of the proof is omitted. □

Lemma 2.2. *If a solution $x(\cdot)$ of (1.1) is bounded on every finite subinterval of $[a,\alpha_x)$, the maximal existence interval, then $\alpha_x=\infty$.*

Proof. Assume that α_x is a finite number. By Lemma 2.1 there exists a $b\ge a$ such that $x(t)$ is monotone on $[b,\alpha_x)$. We assume $x(t)>0$, $t\in[b,\alpha_x)$, without loss of generality. Since $x(t)$ is bounded on any finite subinterval of $[b,\alpha_x)$, then $\lim_{t\to\alpha_x-}x(t)$ exists finitely and $\lim_{t\to\alpha_x-}x'(t)=\infty$. Integrating (1.1) from a to t implies

$$p(t)h(x(t))f(x'(t))=p(a)h(x(a))f(x'(a))+\int_a^t q(s)g(x(s))ds.$$

Hence,

$$\begin{aligned}\lim_{t\to\alpha_x-} p(t)h(x(t))f(x'(t)) &= p(a)h(x(a))f(x'(a)) + \int_a^{\alpha_x} q(s)g(x(s))ds \\ &:= A \in (-\infty,\infty).\end{aligned}$$

Define $H(t) = p(t)h(x(t))f(x'(t))$. Then

$$x'(t) = f^{-1}\Big(\frac{H(t)}{p(t)h(x(t))}\Big).$$

The continuity of $f^{-1}(r)$ implies

$$\lim_{t\to\alpha_x-} x'(t) = f^{-1}\Big(\frac{A}{p(\alpha_x)h(x(\alpha_x))}\Big) < \infty.$$

This is a contradiction. Therefore, $\alpha_x = \infty$ and the proof is complete. $\square$

It follows from Lemmas 2.1 and 2.2 that all solutions of (1.1) except the trivial solution can be divided into two classes:

$$\begin{gathered}A = \{x \text{ solution of (1.1) defined on } [a,\alpha_x) : x(t)x'(t) > 0 \\ \text{in a left neighborhood of } \alpha_x\}, \\ B = \{x \text{ solution of (1.1) defined on } [a,\infty) : x(t)x'(t) < 0 \text{ for } t \geq a\}.\end{gathered}$$

It is well-known that for some equations of type (1.1), class A solutions are not continuable at infinity; see [13] for the discussion of the binomial equations of type $x'' = q(t)|x|^\gamma \operatorname{sgn} x$.

In the next we consider the continuability of solutions to (1.1). Let

(H2) $g(r)$ is nondecreasing for $|r| \geq m$ where $m > 0$ is a real number.

(H3)
$$\int_1^\infty \frac{dr}{f^{-1}(z(r))} = \infty, \quad \int_{-\infty}^{-1} \frac{dr}{f^{-1}(z(r))} = -\infty,$$
where $z(r) = g(r)/h(r)$.

Theorem 2.3. *Under assumptions* (H2),(H3)*, all solutions of* (1.1) *can be extended to $[a,\infty)$.*

Proof. We consider class A solutions only since all class B solutions can be extended to $[a,\infty)$. Let $x(\cdot)$ be a class A solution of (1.1) and without loss of generality we assume that $x(t) > 0$ and $x'(t) > 0$ for all $t \in [b,\alpha_x)$. If $\alpha_x < \infty$, by Lemma 2.2, $x(t) \to \infty$ as $t \to \alpha_x-$. Hence, there exists a real number $d > b$ such that $x(t) \geq m$ for $d \leq t < \alpha_x$.

Integrating (1.1) from d to t we have

$$p(t)h(x(t))f(x'(t)) = p(d)h(x(d))f(x'(d)) + \int_d^t q(s)g(x(s))ds.$$

It follows from (H2) that

$$\begin{aligned}
f(x'(t)) &= \frac{p(d)h(x(d))f(x'(d))}{p(t)h(x(t))} + \frac{1}{p(t)h(x(t))}\int_d^t q(s)g(x(s))ds \\
&\le \frac{p(d)h(x(d))f(x'(d))}{p(t)h(x(t))} + \frac{g(x(t))}{p(t)h(x(t))}\int_d^t q(s)ds \\
&= \frac{g(x(t))}{p(t)h(x(t))}\Big(\frac{p(d)h(x(d))f(x'(d))}{g(x(t))} + \int_d^t q(s)ds\Big) \\
&\le \frac{g(x(t))}{p(t)h(x(t))}\Big(\frac{p(d)h(x(d))f(x'(d))}{g(x(d))} + \int_d^t q(s)ds\Big).
\end{aligned}$$

Since $q(t) > 0$, we can choose $k > 1$ and $t_1 \ge d$ such that for $t \ge t_1$,

$$\frac{p(d)h(x(d))f(x'(d))}{g(x(d))} + \int_d^t q(s)ds \le k\int_d^t q(s)ds.$$

Then

$$\begin{gathered}
f(x'(t)) \le \frac{kg(x(t))}{p(t)h(x(t))}\int_d^t q(s)ds, \\
x'(t) \le f^{-1}\Big(\frac{kg(x(t))}{p(t)h(x(t))}\int_d^t q(s)ds\Big).
\end{gathered}$$

Taking into account (H1) we have

$$\begin{aligned}
x'(t) &\le f^{-1}\Big(kz(x(t))\frac{1}{p(t)}\int_d^t q(s)ds\Big) \\
&\le M_1^2 f^{-1}(k) f^{-1}\big(z(x(t))\big) f^{-1}\Big(\frac{1}{p(t)}\int_d^t q(s)ds\Big).
\end{aligned}$$

Dividing both sides by $f^{-1}(z(x(t)))$ and integrating from t_1 to t we have

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k)\int_{t_1}^t f^{-1}\Big(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\Big)ds. \tag{2.1}$$

Letting $t \to \alpha_x-$, we have

$$\int_{x(t_1)}^{\infty} \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k)\int_{t_1}^{\alpha_x} f^{-1}\Big(\frac{1}{p(s)}\int_d^s q(\sigma)d\sigma\Big)ds < \infty,$$

which is a contradiction to (H3). Therefore, all solutions of (1.1) can be extended to $[a, \infty)$. □

Without the monotonic condition (H2), we have the following theorem. Let

(H4) There exists a constant $M_2 > 0$ such that $|g(r)| \le M_2$ for all $r \in \mathbb{R}$.

(H5)
$$\int_1^{\infty} \frac{dr}{f^{-1}(\frac{1}{h(r)})} = \infty, \quad \int_{-\infty}^{-1} \frac{dr}{f^{-1}(\frac{1}{h(r)})} = \infty.$$

Theorem 2.4. *Under assumptions* (H4), (H5), *all solutions of* (1.1) *can be extended to* $[a, \infty)$.

Proof. Similar to the proof of Theorem 2.3, we consider positive class A solutions only. Let x be a positive class A solution of (1.1) with maximum existence interval $[a, \alpha_x)$ such that $x(t) > 0$ and $x'(t) > 0$ for all $t \in [b, \alpha_x)$. If $\alpha_x < \infty$, then $x(t) \to \infty$ as $t \to \alpha_x -$. From

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds$$

and (H4) we have

$$p(t)h(x(t))f(x'(t)) \le p(b)h(x(b))f(x'(b)) + M_2 \int_b^t q(s)ds.$$

Choosing $k > 1$ and $t_1 \ge b$ such that for $t \ge t_1$

$$p(b)h(x(b))f(x'(b)) + M_2 \int_b^t q(s)ds \le k \int_b^t q(s)ds.$$

By (H1), we have

$$x'(t) \le M_1^2 f^{-1}(k) f^{-1}\Big(\frac{1}{h(x(t))}\Big) f^{-1}\Big(\frac{1}{p(t)} \int_b^t q(s)ds\Big).$$

Dividing both sides by $f^{-1}\big(\frac{1}{h(x(t))}\big)$, integrating from t_1 to t, and letting $t \to \alpha_x -$, we have

$$\int_{x(t_1)}^{\infty} \frac{dr}{f^{-1}\Big(\frac{1}{h(r)}\Big)} \le M_1^2 f^{-1}(k) \int_{t_1}^{\alpha_x} f^{-1}\Big(\frac{1}{p(s)} \int_d^s q(\sigma)d\sigma\Big)ds < \infty,$$

which is a contradiction to (H5). Therefore, $x(\cdot)$ can be extended to infinity. □

3. Existence of Class A & B Solutions

In this section we discuss the existence of class A and class B solutions of (1.1). We assume that (1.1) has a unique solution for any initial conditions $(x(a), x'(a))$ with $x(a) \neq 0$.

Theorem 3.1. *Equation* (1.1) *has both positive and negative class A solutions.*

Proof. Let x be the solution of (1.1) with initial conditions $x(a) > 0$ and $x'(a) > 0$. From the proof of Lemma 2.1 we have $F(t) > 0$ for $t \in [a, \alpha_x)$ because of $F(a) > 0$ in this case. Therefore, $x(t)x'(t) > 0$ for $t \in [a, \alpha_x)$ and x is a positive class A solution. Similarly, let $\tilde{x}$ be the solution of (1.1) with initial conditions $\tilde{x}(a) < 0$ and $\tilde{x}'(a) < 0$. We can show that $\tilde{x}$ is a negative class A solution. □

Now, we discuss sufficient conditions for the existence of class B solutions. The following simple lemma is needed for the proof.

Lemma 3.2. *If x is a solution of* (1.1) *on $[t_1, t_2]$ such that $x(t_1) = x(t_2) = 0$, then $x(t) = 0$ for all $t \in [t_1, t_2]$.*

Notice that if otherwise $x(t_1) = x(t_2) = 0$, then $F(t_1) = F(t_2) = 0$ which contradicts the monotonicity of F. Let

(H2A) $g(r)$ is nondecreasing on $(-\infty, \infty)$.

(H6) There exists $r_0 > 0$ such that

$$\int_0^{\pm r_0} \frac{dr}{f^{-1}(z(r))} = \infty.$$

Theorem 3.3. *Assume* (H2A), (H6)*. Then*

(a) *Equation* (1.1) *has both positive and negative class B solutions.*
(b) *Equation* (1.1) *has no solution which is nonzero but eventually identically equal to zero.*

Proof. (a) We prove that (1.1) has a positive solution, the case of negative solution is similar. Assume $x_0 > 0$. The solution of (1.1) with initial conditions $x(a) = x_0$ and $x'(a) = c$ is denoted by $x(t) := x(t, c)$ that has the form

$$x(t) = x_0 + \int_a^t f^{-1}\Big(\frac{p(a)h(x_0)f(c)}{p(s)h(x(s))} + \frac{1}{p(s)h(x(s))}\int_a^s q(\sigma)g(x(\sigma))d\sigma\Big)ds.$$

Define two sets U and L as

$$U = \{c \in \mathbb{R} : \text{ there exists some } \bar{t} \geq a \text{ such that } x'(\bar{t}, c) > 0\},$$
$$L = \{c \in \mathbb{R} : \text{ there exists some } \bar{t} \geq a \text{ such that } x(\bar{t}, c) < 0\}.$$

Then $U \cap L = \emptyset$. Clearly, $U \neq \emptyset$. With the same argument as Theorem 4 in [20] we are able to prove that U is open. Next we show $L \neq \emptyset$. Define $M_2 := \max_{0\leq r\leq x_0} h(r) > 0$ and $M_3 := \max_{a\leq t\leq a+1} p(t) > 0$. Let

$$c < f^{-1}\Big(\frac{M_2M_3f^{-1}(-x_0) - g(x_0)\int_a^{a+1} q(s)ds}{p(a)h(x_0)}\Big). \tag{3.1}$$

We claim $x'(t, c) < 0$ for $a \leq t \leq a + 1$. Otherwise, there exists $t_1 \in (a, a + 1]$ such that $x'(t_1, c) = 0$ and $x'(t, c) < 0$ for $t \in [a, t_1)$. It follows from (3.1) that

$$\begin{aligned} 0 = p(t)h(x(t_1, c))f(x'(t_1, c)) &= p(a)h(x_0)f(c) + \int_a^{t_1} q(s)g(x(s, c))ds \\ &\leq p(a)h(x_0)f(c) + g(x_0)\int_a^{a+1} q(s)ds < 0. \end{aligned}$$

This is a contradiction and hence $x(t, c)$ is decreasing on $[a, a + 1]$.

If there exists a $b \in (a, a + 1]$ such that $x(b, c) < 0$, then $c \in L$ and $L \neq \emptyset$. Otherwise, $x(t) \geq 0$ on $[a, a + 1]$. Hence, we have from (3.1) that

$$\begin{aligned} x(a + 1, c) &= x_0 + \int_a^{a+1} f^{-1}\Big(\frac{p(a)h(x_0)f(c)}{p(t)h(x(t))} + \frac{1}{p(t)h(x(t))}\int_a^t q(s)g(x(s))ds\Big)dt \\ &\leq x_0 + \int_a^{a+1} f^{-1}\Big(\frac{p(a)h(x_0)f(c) + g(x_0)\int_a^{a+1} q(s)ds}{M_2M_3}\Big)dt < 0. \end{aligned}$$

This shows that $c \in L$. Clearly, L is open. Therefore, $\mathbb{R} - (U \cup L) \neq \emptyset$. For any $c \in \mathbb{R} - (U \cup L)$, $x(t, c)$ is a non-increasing nonnegative solution on $[a, \infty)$. We will show that $x(t, c) > 0$ on $[a, \infty)$. If not, there exists $t_0 > a$ such that $x(t_0) = 0$ and $x(t) = 0$ for $t \geq t_0$ and $x'(t_0) = 0$. Note that for $t \in [a, t_0]$ we have

$$\begin{aligned} x'(t) &= f^{-1}\Big(-\frac{1}{p(t)h(x(t))}\int_t^{t_0} q(s)g(x(s))ds\Big) \\ &\geq M_1f^{-1}(z(x(t)))f^{-1}\Big(-\frac{1}{p(t)}\int_t^{t_0} q(s)ds\Big). \end{aligned}$$

Dividing both sides by $f^{-1}(z(x(t)))$ and integrating from a to t_0, we have

$$\int_a^{t_0} \frac{x'(t)}{f^{-1}(z(x(t)))}dt \geq M_1\int_a^{t_0} f^{-1}\Big(-\frac{1}{p(t)}\int_t^{t_0} q(s)ds\Big)dt.$$

That is,

$$\int_0^{x_0} \frac{1}{f^{-1}(z(r))} dr \le -M_1 \int_a^{t_0} f^{-1}\Big(-\frac{1}{p(t)} \int_t^{t_0} q(s) ds\Big) dt < \infty,$$

a contradiction to (H6). Therefore, $x(t) > 0$ for $t \ge a$. Note that $x'(t) \le 0$ for $t \ge a$. It follows from equation (1.1) and $x(t) > 0$ that $x'(t) \ne 0$ for $t \ge a$. Hence, $x'(t) < 0$ for $t \ge a$ and $x \in B$.

The proof of part (b) follows from the end part of the proof of part (a). □

4. Boundedness of Solutions

In this section we discuss the boundedness of all solutions of (1.1), some necessary and sufficient conditions are obtained.

Theorem 4.1. *Assume* (H2), (H3). *Then all positive (negative) solutions of* (1.1) *are bounded if and only if $J_1 < \infty$ ($J_2 > -\infty$).*

Proof. We consider positive solutions only since the case of negative solutions can be handled similarly.

Necessity. Let $x(\cdot)$ be a positive bounded class A solution. Then $x(t) > 0$ and $x'(t) > 0$ for $t \ge b > a$ and $\lim_{t\to\infty} x(t) = l \in (0, \infty)$. By the Extreme Value Theorem we have $L_1 := \min_{x(b)\le r\le l} g(r) > 0$. Hence

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds \ge L_1 \int_b^t q(s)ds.$$

Since $x(\cdot)$ is continuous and bounded and $h(r)$ is continuous, $h(x(\cdot))$ is bounded. Let $h(x(t)) \le K$ for $t \in [a, \infty)$. Then

$$Kp(t)f(x'(t)) \ge p(t)h(x(t))f(x'(t)) \ge L_1 \int_b^t q(s)ds,$$
$$\frac{K}{L_1} f(x'(t)) \ge \frac{1}{p(t)} \int_b^t q(s)ds.$$

By (H1) we have

$$f^{-1}\Big(\frac{1}{p(t)} \int_b^t q(s)ds\Big) \le f^{-1}\Big(\frac{K}{L_1} f(x'(t))\Big) \le M_1 f^{-1}\Big(\frac{K}{L_1}\Big) x'(t).$$

Integrating from b to t and letting $t \to \infty$ we have

$$J_1 = \int_b^\infty f^{-1}\Big(\frac{1}{p(t)} \int_b^t q(s)ds\Big)dt \le M_1 f^{-1}\Big(\frac{K}{L_1}\Big)(l - x(b)) < \infty.$$

Sufficiency. We will prove by contradiction. Let $x(\cdot)$ be a unbounded class A solution of (1.1). Then $x(t) > 0$ and $x'(t) > 0$ on $[b, \infty)$, and there exists a real number $d \ge b$ such that $x(t) \ge m$ for $d \le t < \infty$. Similar to the proof of Theorem 2.3, we have the inequality

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k) \int_{t_1}^t f^{-1}\Big(\frac{1}{p(s)} \int_d^s q(\sigma)d\sigma\Big)ds.$$

Letting $t \to \infty$ and noting that $x(\infty) = \infty$, we have

$$\int_{x(t_1)}^\infty \frac{dr}{f^{-1}(z(r))} \le M_1^2 f^{-1}(k) \int_{t_1}^\infty f^{-1}\Big(\frac{1}{p(s)} \int_b^s q(\sigma)d\sigma\Big)ds \le M_1^2 f^{-1}(k) J_1 < \infty.$$

This is a contradiction to (H3). Therefore, x is bounded. □

Corollary 4.2. *Assume* (H2), (H3)*. If* (1.1) *has a positive (negative) bounded class A solution, then all positive (negative) solutions in class A are bounded. On the other hand, if* (1.1) *has an unbounded positive (negative) class A solution, then all positive (negative) solutions in class A are unbounded.*

Theorem 4.3. *Assume* (H4), (H5)*. Then all positive (negative) solutions of* (1.1) *are bounded if and only if* $J_1 < \infty$ $(J_2 > -\infty)$*.*

Proof. We prove only the case of positive solutions, since the argument is similar for negative solutions.

Necessity. Let $x(\cdot)$ be a positive bounded class A solution, i.e., $x(t) > 0$ and $x'(t) > 0$ for $t \in [b, \infty)$. Then $\lim_{t\to\infty} x(t) = l_1 \in (0, \infty)$. By the Extreme Value Theorem we have $L_2 := \min_{x(b)\le r\le l_1} g(r) > 0$. Hence

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds \ge L_2 \int_b^t q(s)ds.$$

Similar to the proof of Theorem 4.1, let $h(x(t)) \le K$ for $t \in [b, \infty)$. We have

$$\frac{K}{L_2} f(x'(t)) \ge \frac{1}{p(t)} \int_b^t q(s)ds.$$

Hence

$$f^{-1}\Big(\frac{1}{p(t)} \int_b^t q(s)ds\Big) \le M_1 f^{-1}\Big(\frac{K}{L_2}\Big) x'(t).$$

Integrating from b to t and letting $t \to \infty$, we have

$$J_1 = \int_b^\infty f^{-1}\Big(\frac{1}{p(t)} \int_b^t q(s)ds\Big)dt \le M_1 f^{-1}\Big(\frac{K}{L_2}\Big)(l_2 - x(b)) < \infty.$$

Sufficiency. Assume that $x(\cdot)$ is any positive class A solution. It follows from

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds$$

and (H4) that

$$p(t)h(x(t))f(x'(t)) \le p(b)h(x(b))f(x'(b)) + M_2 \int_b^t q(s)ds.$$

Similar to the proof of Theorem 2.4, we have

$$\int_{x(t_1)}^\infty \frac{dr}{f^{-1}\Big(\frac{1}{h(r)}\Big)} \le M_1^2 f^{-1}(k) \int_{t_1}^\infty f^{-1}\Big(\frac{1}{p(t)} \int_d^t q(s)ds\Big)dt \le M_1^2 f^{-1}(k) J_1 < \infty.$$

Therefore, $x(\cdot)$ is bounded and the proof is complete. □

Corollary 4.4. *Let* (H4) *and* (H5) *hold. If* (1.1) *has a positive (negative) bounded class A solution, then all positive (negative) solutions in class A are bounded. On the other hand, if* (1.1) *has an unbounded positive (negative) class A solution, then all positive (negative) solutions in class A are unbounded.*

Remark 4.5. The condition (H3) of Theorem 4.1 is sharp. For example, consider the following equation

$$(t^6(x^2(t) + 1)(x'(t))^3)' = \frac{162}{t^4}(3x^7(t) + 2x^5(t)), \quad t \ge 1, \tag{4.1}$$

where $p(t)=t^6$, $h(r)=r^2+1$, $f(r)=r^3$, $g(r)=3r^7+2r^5$, $q(t)=\frac{162}{t^4}$. Clearly, conditions (H), (H1), and (H2) are satisfied. By simple computation, we have

$$\int_1^\infty \frac{dr}{f^{-1}(z(r))}<\infty, \quad \int_{-\infty}^{-1} \frac{dr}{f^{-1}(z(r))}>-\infty.$$

This shows that (H3) does not hold. We claim $J_1<\infty$ and $J_2>-\infty$. Indeed,

$$J_1=\int_1^\infty \sqrt[3]{\frac{54}{t^6}\Big(-\frac{1}{t^3}+1\Big)}\,dt \le 3\sqrt[3]{2}\int_1^\infty \frac{1}{t^2}dt = 3\sqrt[3]{2},$$

and

$$J_2=-\int_1^\infty \sqrt[3]{\frac{54}{t^6}\Big(-\frac{1}{t^3}+1\Big)}\,dt \ge -3\sqrt[3]{2}\int_1^\infty \frac{1}{t^2}dt = -3\sqrt[3]{2}.$$

However, $x(t)=t^3$ is a positive unbounded class A solution of (4.1) on $[1,\infty)$ and $x(t)=-t^3$ is a negative unbounded class A solution on $[1,\infty)$. Therefore, Theorem 4.1 fails without (H3).

Remark 4.6. The condition (H5) of Theorem 4.3 is sharp. For example, consider the following equation

$$\Big(\frac{t^4}{x^4(t)+1}(x'(t))^3\Big)'=\frac{4t^3}{(t^2+1)^2}\,g(x(t)), \quad t\ge 1, \tag{4.2}$$

where $p(t)=t^4$, $h(r)=\frac{1}{r^4+1}$, $f(r)=r^3$, $q(t)=\frac{4t^3}{(t^2+1)^2}$, and

$$g(r)=\begin{cases} 1, & r\ge 1,\\ |r|, & |r|\le 1,\\ -1, & r\le -1.\end{cases}$$

Clearly, conditions (H), (H1), and (H4) are satisfied. We claim $J_1<\infty$ and $J_2>-\infty$. Indeed,

$$J_1=\int_1^\infty \sqrt[3]{\frac{1}{t^4}\Big(\frac{t^4}{t^4+1}-\frac{1}{2}\Big)}\,dt<\int_1^\infty \sqrt[3]{\frac{1}{t^4+1}}\,dt<\infty,$$

and

$$J_2=-\int_1^\infty \sqrt[3]{\frac{1}{t^4}\Big(\frac{t^4}{t^4+1}-\frac{1}{2}\Big)}\,dt>-\int_1^\infty \sqrt[3]{\frac{1}{t^4+1}}\,dt>-\infty.$$

However, (H5) does not hold since

$$\int_1^\infty \frac{dr}{f^{-1}(\frac{1}{h(r)})}=\int_1^\infty \sqrt[3]{\frac{1}{r^4+1}}\,dr<\infty,$$

and

$$\int_{-\infty}^{-1} \frac{dr}{f^{-1}(\frac{1}{h(r)})}=\int_{-\infty}^{-1} \sqrt[3]{\frac{1}{r^4+1}}\,dr<\infty.$$

It is easy to check that $x(t)=t$ is a positive unbounded class A solution of (4.1) on $[1,\infty)$ and $x(t)=-t$ is a negative unbounded class A solution on $[1,\infty)$. Therefore, Theorem 4.1 fails without (H5).

Remark 4.7. Theorem 2.3 and Theorem 4.1 generalize [15, Theorem 1] since (H3) reduces to (iii) of [15] if $f(r) = r$. Moreover, the differentiability of $p(\cdot)$ and $h(\cdot)$ is not required as of [15]. Theorems 2.3, 3.1, and 4.1 generalize [6, Theorem 8]. Moreover, under (H2), Theorems 2.3, 3.1, and 4.1 improve [6, Theorem 8] since (H3) improves [6, (22)]; see the discussion in [20]. Theorem 2.3 generalizes [17, Theorem 3.9]. Theorems 2.3, 3.1, and 4.1 generalize [20, Theorem 1]. Theorem 3.3 generalizes [17, Theorem 2.1]. Under (H2A), Theorem 3.3 improves [7, Theorem 6] since [7, (hp)] is replaced by a weaker condition (H6).

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