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EXISTENCE OF ENTIRE POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

ZHIJUN ZHANG

Abstract. Under simple conditions on f_i and g_i, we show the existence of entire positive radial solutions for the semilinear elliptic system

$$\Delta u = p(|x|)f_1(v)f_2(u)$$
$$\Delta v = q(|x|)g_1(v)g_2(u),$$

where $x \in \mathbb{R}^N$, $N \geq 3$, and p, q are continuous functions.

1. Introduction

The purpose of this paper is to investigate the existence of entire positive radial solutions to the semilinear elliptic system

$$\begin{aligned} \Delta u &= p(|x|)f_1(v)f_2(u), \quad x \in R^N, \\ \Delta v &= q(|x|)g_1(v)g_2(u), \quad x \in R^N, \end{aligned} \tag{1.1}$$

where $N \geq 3$. We assume that p, q, f_i, g_i $(i = 1, 2)$ satisfy the following hypotheses.

- (H1) The functions $p, q, f_i, g_i : [0, \infty) \to [0, \infty)$ are continuous;
- (H2) the functions f_i and g_i are increasing on $[0, \infty)$.

Denote

$$P(\infty) := \lim_{r\to\infty} P(r), \quad P(r) = \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}p(s)ds\Big)dt, \quad r \geq 0,$$
$$Q(\infty) := \lim_{r\to\infty} Q(r), \quad Q(r) = \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}q(s)ds\Big)dt, \quad r \geq 0,$$
$$F(\infty) := \lim_{r\to\infty} F(r), \quad F(r) = \int_a^r \frac{ds}{f_1(s)f_2(s) + g_1(s)g_2(s)}, \quad r \geq a > 0.$$

We see that $F'(r) = \frac{1}{f_1(r)f_2(r)+g_1(r)g_2(r)} > 0$, for $r > a$ and F has the inverse function F^{-1} on $[a, \infty)$.

This problem arises in many branches of mathematics and physics and has been discussed by many authors; see, for instance, [1]-[8], [10, 11, 12] and the references therein.

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When $f_2 = g_1 \equiv 1$, $f_1(v) = v^\alpha$, $g_2(u) = u^\beta$, $0 < \alpha \le \beta$, Lair and Wood [8] considered the existence and nonexistence of entire positive radial solutions to (1.1). Their results were extended by Cîrstea and Rădulescu [1], Wang and Wood [12], Ghergu and Rădulescu [6], Peng and Song [11], Ghanmi, Mâagli, Rădulescu and Zeddini [5], and the authors of this article in [10].

When $f_1(v) = v^{\alpha_1}$, $f_2(u) = u^{\alpha_2}$, $g_1(v) = v^{\beta_1}$, $g_2(u) = u^{\beta_2}$, where $\alpha_1 > 0$, $\beta_2 > 0$, $\alpha_2 > 1$ and $\beta_1 > 1$, García-Melián and Rossi [3], García-Melián [4] have studied the existence, uniqueness and exact blow-up rate near the boundary of positive solutions to system (1.1) on a bounded domain.

In this paper, we give simple conditions on f_i and g_i to show the existence of entire positive radial solutions to (1.1). Our main results are as the following.

Theorem 1.1. *Under hypotheses* (H1)–(H2) *and*

(H3) $F(\infty) = \infty$,

system (1.1) *has one positive radial solution* $(u, v) \in C^2([0,\infty))$. *Moreover, when* $P(\infty) < \infty$ *and* $Q(\infty) < \infty$, u *and* v *are bounded; when* $P(\infty) = \infty = Q(\infty)$, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$.

Theorem 1.2. *Under hypotheses* (H1)–(H2) *and*

(H4) $F(\infty) < \infty$;
(H5) $P(\infty) < \infty$, $Q(\infty) < \infty$;
(H6) *there exist* $b > a$ *and* $c > a$ *such that* $P(\infty) + Q(\infty) < F(\infty) - F(b + c)$,

system (1.1) *has one positive radial bounded solution* $(u, v) \in C^2([0,\infty))$ *satisfying*

$$b + f_1(c)f_2(b)P(r) \le u(r) \le F^{-1}\Big(F(b + c) + P(r) + Q(r)\Big), \quad \forall r \ge 0;$$
$$c + g_1(c)g_2(b)Q(r) \le v(r) \le F^{-1}\Big(F(b + c) + P(r) + Q(r)\Big), \quad \forall r \ge 0.$$

Remark 1.3. From (H1)–(H2), we see that (H3) implies

$$\int_a^\infty \frac{ds}{f_1(s)f_2(s)} = \int_a^\infty \frac{ds}{g_1(s)g_2(s)} = \infty. \tag{1.2}$$

Remark 1.4. When $f_1(v) = v^{\alpha_1}$, $f_2(u) = u^{\alpha_2}$, $g_1(v) = v^{\beta_1}$, $g_2(u) = u^{\beta_2}$, where α_i and β_i are positive constants, we see that (H3) holds provided $\max\{\alpha_1 + \alpha_2, \beta_1 + \beta_2\} \le 1$ and (H4) holds provided $\alpha_1 + \alpha_2 > 1$ or $\beta_1 + \beta_2 > 1$.

Remark 1.5. By [9], we see that $P(\infty) = \infty$ if and only if $\int_0^\infty sp(s)ds = \infty$.

2. Proof of Theorems 1.1 and 1.2

Note that radial solutions of (1.1) are solutions of the ordinary differential equation system

$$u'' + \frac{N-1}{r}u' = p(r)f_1(v)f_2(u),$$
$$v'' + \frac{N-1}{r}v' = q(r)g_1(v)g_2(u).$$

Thus solutions of (1.1) are simply solutions of

$$
\begin{aligned}
u(r) &= b + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1} p(s) f_1(v(s)) f_2(u(s)) ds\Big) dt, \quad r \ge 0,\\
v(r) &= c + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1} q(s) g_1(v(s)) g_2(u(s)) ds\Big) dt, \quad r \ge 0.
\end{aligned}
$$

Let $\{u_m\}_{m\ge 0}$ and $\{v_m\}_{m\ge 0}$ be the sequences of positive continuous functions defined on $[0,\infty)$ by

$$
\begin{gathered}
u_0(r) \equiv b, \quad v_0(r) \equiv c,\\
u_{m+1}(r) = b + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1} p(s) f_1(v_m(s)) f_2(u_m(s)) ds\Big) dt, \quad r \ge 0,\\
v_{m+1}(r) = c + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1} q(s) g_1(v_m(s)) g_2(u_m(s)) ds\Big) dt, \quad r \ge 0.
\end{gathered}
$$

Obviously, for all $r \ge 0$ and $m \in \mathbb{N}$, $u_m(r) \ge b$, $v_m(r) \ge c$ and

$$
v_0 \le v_1, \quad u_0 \le u_1, \quad \forall r \ge 0.
$$

Hypothesis (H2) yields

$$
u_1(r) \le u_2(r), \quad v_1(r) \le v_2(r), \quad \forall r \ge 0.
$$

Continuing this line of reasoning, we obtain that the sequences $\{u_m\}$ and $\{v_m\}$ are increasing on $[0,\infty)$. Moreover, we obtain by (H1) and (H2) that, for each $r > 0$,

$$
\begin{aligned}
u'_{m+1}(r) &= r^{1-N}\int_0^r s^{N-1} p(s) f_1(v_m(s)) f_2(u_m(s)) ds\\
&\le f_1(v_m(r)) f_2(u_m(r)) P'(r)\\
&\le f_1\big(v_{m+1}(r)+u_{m+1}(r)\big) f_2\big(v_{m+1}(r)+u_{m+1}(r)\big) P'(r)\\
&\le \Big[f_1\big(v_{m+1}(r)+u_{m+1}(r)\big) f_2\big(v_{m+1}(r)+u_{m+1}(r)\big)\\
&\quad + g_1\big(v_{m+1}(r)+u_{m+1}(r)\big) g_2\big(v_{m+1}(r)+u_{m+1}(r)\big)\Big] P'(r)\,,
\end{aligned}
$$

$$
\begin{aligned}
v'_{m+1}(r) &= r^{1-N}\int_0^r s^{N-1} q(s) g_1(v_m(s)) g_2(u_m(s)) ds\\
&\le g_1\big(v_m(r)) g_2(u_m(r)\big) Q'(r)\\
&\le g_1\big(v_{m+1}(r)+u_{m+1}(r)\big) g_2\big(v_{m+1}(r)+u_{m+1}(r)\big) Q'(r)\\
&\le \Big[f_1\big(v_{m+1}(r)+u_{m+1}(r)\big) f_2\big(v_{m+1}(r)+u_{m+1}(r)\big)\\
&\quad + g_1\big(v_{m+1}(r)+u_{m+1}(r)\big) g_2\big(v_{m+1}(r)+u_{m+1}(r)\big)\Big] Q'(r)
\end{aligned}
$$

and

$$
\int_{b+c}^{v_{m+1}(r)+u_{m+1}(r)} \frac{d\tau}{f_1(\tau) f_2(\tau) + g_1(\tau) g_2(\tau)} \le Q(r) + P(r).
$$

Consequently,

$$
F\big(u_m(r)+v_m(r)\big) - F(b+c) \le P(r) + Q(r), \quad \forall r \ge 0. \tag{2.1}
$$

Since F^{-1} is increasing on $[0,\infty)$, we have

$$u_m(r)+v_m(r)\le F^{-1}\big(F(b+c)+P(r)+Q(r)\big),\quad \forall r\ge 0. \tag{2.2}$$

(i) When (H3) holds, we see that

$$F^{-1}(\infty)=\infty. \tag{2.3}$$

It follows that the sequences $\{u_m\}$ and $\{v_m\}$ are bounded and equicontinuous on $[0,c_0]$ for arbitrary $c_0>0$. It follows by Arzela-Ascoli theorem that $\{u_m\}$ and $\{v_m\}$ have subsequences converging uniformly to u and v on $[0,c_0]$. By the arbitrariness of $c_0>0$, we see that (u,v) are positive entire solutions of (1.1). Moreover, when $P(\infty)<\infty$ and $Q(\infty)<\infty$, we see by (2.2) that

$$u(r)+v(r)\le F^{-1}\big(F(b+c)+P(\infty)+Q(\infty)\big),\quad \forall r\ge 0;$$

and,when $P(\infty)=\infty=Q(\infty)$, by (H2) and the monotones of $\{u_m\}$ and $\{v_m\}$,

$$u(r)\ge b+f_1(c)f_2(b)P(r),\quad v(r)\ge c+g_1(c)g_2(b)Q(r),\quad \forall r\ge 0.$$

Thus $\lim_{r\to\infty}u(r)=\lim_{r\to\infty}v(r)=\infty$.
(ii) When (H4)–(H6) hold, we see by (2.1) that

$$F(u_m(r)+v_m(r))\le F(b+c)+P(\infty)+Q(\infty)<F(\infty)<\infty. \tag{2.4}$$

Since F^{-1} is strictly increasing on $[0,\infty)$, we have

$$u_m(r)+v_m(r)\le F^{-1}\big(F(b+c)+P(\infty)+Q(\infty)\big)<\infty,\quad \forall r\ge 0. \tag{2.5}$$

The last part of the proof follows from (i). Thus the proof is complete.

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Zhijun Zhang
School of Mathematics and Information Science, Yantai University, Yantai, Shandong, 264005, China
E-mail address: zhangzj@ytu.edu.cn