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REPRESENTATION OF THE NORMING CONSTANTS BY TWO SPECTRA

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Abstract. The representation of the norming constants by 2 spectra was studied by Levitan, Gasymov (and others) for the Sturm-Liouville problem with boundary conditions $y(0)\cos\alpha + y'(0)\sin\alpha = 0$, $y(\pi)\cos\beta + y'(\pi)\sin\beta = 0$, when $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. We investigate the representation by 2 spectra without these restrictions.

1. Introduction and statement of the main result

Let $L(q,\alpha,\beta)$ denote the Sturm-Liouville (S.-L.) problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{1.3}$$

where q is a real-valued, summable on $[0,\pi]$ function (we write $q \in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by the problem (1.1)-(1.3), (see [6, 8]). It is known, that the spectra of the operator $L(q,\alpha,\beta)$ is discrete and consists of simple eigenvalues, which we denote by $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\dots$, emphasizing the dependence of μ_n on q, α and β.

The eigenvalues $\mu_n(q,\alpha,\beta)$ are enumerated in increasing order; i.e.,

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \dots < \mu_n(q,\alpha,\beta) < \mu_{n+1}(q,\alpha,\beta) < \dots .$$

In what follows, for brevity, we often use the notation $\mu_n = \mu_n(\alpha,\beta) = \mu_n(q,\alpha,\beta)$.

By $\varphi(x,\mu,\gamma)$ and $\psi(x,\mu,\delta)$ we denote the solutions of (1.1), satisfying the initial conditions

$$\varphi(0,\mu,\gamma) = \sin\gamma, \quad \varphi'(0,\mu,\gamma) = -\cos\gamma, \quad \gamma \in \mathbb{C}, \tag{1.4}$$

$$\psi(\pi,\mu,\delta) = \sin\delta, \quad \psi'(\pi,\mu,\delta) = -\cos\delta, \quad \delta \in \mathbb{C}, \tag{1.5}$$

correspondingly. The eigenvalues $\mu_n = \mu_n(q,\alpha,\beta)$, $n = 0,1,2,\dots$, of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\Phi(\mu) = \Phi(\mu,\alpha,\beta) := \varphi(\pi,\mu,\alpha)\cos\beta + \varphi'(\pi,\mu,\alpha)\sin\beta = 0, \tag{1.6}$$

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or the equation

$$\Psi(\mu) = \Psi(\mu, \alpha, \beta) := \psi(0, \mu, \beta)\cos\alpha + \psi'(0, \mu, \beta)\sin\alpha = 0. \tag{1.7}$$

According to the Liouville formula, the Wronskian $W(x) = W(x, \varphi, \psi) = \varphi \cdot \psi' - \varphi'\psi$ of the solutions φ and ψ is constant. It follows that $W(0) = W(\pi)$ and, consequently,

$$\Psi(\mu, \alpha, \beta) = -\Phi(\mu, \alpha, \beta).$$

The functions $\varphi_n(x) = \varphi(x, \mu_n, \alpha)$ and $\psi_n(x) = \psi(x, \mu_n, \beta)$, $n = 0, 1, 2, \dots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. The squares of the L^2-norms of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) = \int_0^\pi |\varphi_n(x)|^2 dx, \quad b_n = b_n(q, \alpha, \beta) = \int_0^\pi |\psi_n(x)|^2 dx, \tag{1.8}$$

are called the norming constants.

Our aim is to prove the following theorem.

Theorem 1.1. *(1) For arbitrary $\varepsilon \in (0, \pi)$, $\varepsilon \neq \alpha$, the following representations are true ($n = 0, 1, 2, \dots$):*

$$a_n(q, \alpha, \beta) = \frac{\sin\alpha}{\sin\varepsilon} \frac{\sin(\alpha - \varepsilon)}{\mu_n(\alpha, \beta) - \mu_n(\varepsilon, \beta)} \prod_{k=0,\, k\neq n}^{\infty} \frac{\mu_k(\alpha, \beta) - \mu_n(\alpha, \beta)}{\mu_k(\varepsilon, \beta) - \mu_n(\alpha, \beta)} \tag{1.9}$$

if $\alpha \in (0, \pi)$, $\beta \in [0, \pi)$.

$$\begin{aligned} a_n(q, \pi, \beta) = &\frac{4n^2}{\pi\left(n + \frac{1}{2}\right)^2} \cdot \frac{\mu_0(\pi, \beta) - \mu_n(\pi, \beta)}{\mu_0(\varepsilon, \beta) - \mu_n(\pi, \beta)} \\ &\cdot \frac{1}{\mu_n(\pi, \beta) - \mu_n(\varepsilon, \beta)} \cdot \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{k^2}{\left(k + \frac{1}{2}\right)^2} \cdot \frac{\mu_k(\pi, \beta) - \mu_n(\pi, \beta)}{\mu_k(\varepsilon, \beta) - \mu_n(\pi, \beta)} \end{aligned} \tag{1.10}$$

if $\beta \in (0, \pi)$, $n \neq 0$.

$$a_0(q, \pi, \beta) = \frac{4}{\pi} \frac{1}{\mu_0(\pi, \beta) - \mu_0(\varepsilon, \beta)} \cdot \prod_{k=1}^{\infty} \frac{k^2}{\left(k + \frac{1}{2}\right)^2} \cdot \frac{\mu_k(\pi, \beta) - \mu_0(\pi, \beta)}{\mu_k(\varepsilon, \beta) - \mu_0(\pi, \beta)} \tag{1.11}$$

if $\beta \in (0, \pi)$.

$$\begin{aligned} &a_n(q, \pi, 0) \\ &= \frac{\left(n + \frac{1}{2}\right)^2}{(n+1)^2} \cdot \frac{\pi}{\mu_n(\pi, 0) - \mu_n(\varepsilon, 0)} \cdot \prod_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\left(k + \frac{1}{2}\right)^2}{(k+1)^2} \cdot \frac{\mu_k(\pi, 0) - \mu_n(\pi, 0)}{\mu_k(\varepsilon, 0) - \mu_n(\pi, 0)}. \end{aligned} \tag{1.12}$$

(2) For arbitrary $\eta \in (0, \pi)$, $\eta \neq \beta$, ($n = 0, 1, 2, \dots$):

$$b_n(q, \alpha, \beta) = \frac{\sin\beta}{\sin\eta} \cdot \frac{\sin(\beta - \eta)}{\mu_n(\alpha, \beta) - \mu_n(\alpha, \eta)} \prod_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\mu_k(\alpha, \beta) - \mu_n(\alpha, \beta)}{\mu_k(\alpha, \eta) - \mu_n(\alpha, \beta)}, \tag{1.13}$$

if $\alpha \in (0,\pi]$, $\beta \in (0,\pi)$.

$$\begin{aligned} b_n(q,\alpha,0) &= \frac{4n^2}{\pi\left(n+\frac12\right)^2}\cdot\frac{\mu_0(\alpha,0)-\mu_n(\alpha,0)}{\mu_n(\alpha,\eta)-\mu_n(\alpha,0)} \\ &\quad\cdot\frac{1}{\mu_n(\alpha,0)-\mu_n(\alpha,\eta)}\cdot\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{k^2}{\left(k+\frac12\right)^2}\cdot\frac{\mu_k(\alpha,0)-\mu_n(\alpha,0)}{\mu_k(\alpha,\eta)-\mu_n(\alpha,0)}, \end{aligned} \tag{1.14}$$

if $\alpha \in (0,\pi)$, $n \neq 0$.

$$b_0(q,\alpha,0) = \frac{4}{\pi\left(\mu_0(\alpha,0)-\mu_0(\alpha,\eta)\right)}\cdot\prod_{k=1}^{\infty}\frac{k^2}{\left(k+\frac12\right)^2}\cdot\frac{\mu_k(\alpha,0)-\mu_n(\alpha,0)}{\mu_k(\alpha,\eta)-\mu_n(\alpha,0)}, \tag{1.15}$$

if $\alpha \in (0,\pi)$.

$$\begin{aligned} &b_n(q,\pi,0) \\ &= \frac{\left(n+\frac12\right)^2}{(n+1)^2}\cdot\frac{\pi}{\mu_n(\pi,0)-\mu_n(\pi,\eta)}\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\left(k+\frac12\right)^2}{(k+1)^2}\cdot\frac{\mu_k(\pi,0)-\mu_n(\pi,0)}{\mu_k(\pi,\eta)-\mu_n(\pi,0)}. \end{aligned} \tag{1.16}$$

The representation of the norming constants by two spectra was investigated in [1, 2, 7, 10] with the purpose of using it for (constructive) solution of the inverse Sturm-Liouville problem by two spectra. They do it by reducing the inverse problem by 2 spectra to the solution of the inverse problem by spectral function and solve it by Gelfand-Levitan method. In these papers the Sturm-Liouville problem $L(q,\alpha,\beta)$ considered under conditions $\sin\alpha \neq 0$ and $\sin\beta \neq 0$ and as the norming constants the authors consider (in our notation) $\tilde a_n(q,\alpha,\beta) = \frac{a_n(q,\alpha,\beta)}{\sin^2\alpha}$.

Levitan [7] studied the representation of $\tilde a_n(q,\alpha,\beta)$ by spectra $\{\mu_k(q,\alpha,\beta)\}_{k=0}^{\infty}$ and $\{\mu_k(q,\alpha,\beta_1)\}_{k=0}^{\infty}$, $\beta\neq\beta_1$, $\alpha,\beta,\beta_1 \in (0,\pi)$, when $q \in C_{\mathbb R}[0,\pi]$.

Levitan and Gasymov [2] study the representation of $\tilde a_n(q,\alpha,\beta)$ by spectra $\{\mu_k(q,\alpha,\beta)\}_{k=0}^{\infty}$ and $\{\mu_k(q,\varepsilon,\beta)\}_{k=0}^{\infty}$, $\alpha\neq\varepsilon$, $\alpha,\varepsilon,\beta \in (0,\pi)$, when $q \in L^1_{\mathbb R}[0,\pi]$. The formula, obtained in [2], coincides with (1.9). The cases, when $\sin\alpha = 0$ ($\alpha=\pi$) or $\sin\beta = 0$ ($\beta = 0$), in [7] and [2] are not considered. The representation of norming constants $b_n(q,\alpha,\beta)$ (or $\tilde b_n(q,\alpha,\beta) = \frac{b_n(q,\alpha,\beta)}{\sin^2\beta}$) are usually not considered (since it is very similar to investigation of a_n, and also it is not required in their solution of inverse problem by Gelfand-Levitan method). One of our aims is the solution of inverse problem by "the eigenvalues function (EVF) of the family of Sturm-Liouville operators" (see [3]). The statement of the problem and its solution will be the subject of our future articles. However in this solution we will need to simultaneously use the representations for both a_n and b_n, and this is the main reason for such a detailed formulation of the Theorem of this article.

Technically the consideration of the cases $\sin\alpha\neq 0$, $\sin\beta\neq 0$ is distinguished by the fact that in these cases the (rough) asymptotic behavior of the eigenvalues is

$$\mu_n(q,\alpha,\beta) = n^2 + O(1), \tag{1.17}$$

and thus

$$\mu_n(q,\alpha,\beta) - \mu_n(q,\varepsilon,\beta) = O(1), \tag{1.18}$$

$$\mu_n(q,\alpha,\beta) - \mu_n(q,\alpha,\beta_1) = O(1). \tag{1.19}$$

These relations are important for the convergence in the infinite product (1.9) and (1.13). Meanwhile, when $\sin\alpha = 0$ ($\alpha = \pi$) or/and $\sin\beta = 0$ ($\beta = 0$) the (rough) asymptotic behavior of the eigenvalues have the form ($n \to \infty$) (in [3] we have obtained the common formula for the asymptotic behavior of the eigenvalues $\mu_n(q, \alpha, \beta)$ in all the cases $\alpha \in (0, \pi]$ and $\beta \in [0, \pi)$, from which (1.17), (1.20)-(1.22) follow as particular cases):

$$\mu_n(q, \pi, \beta) = \Big(n + \frac{1}{2}\Big)^2 + O(1), \quad \text{when } \sin\beta \neq 0 \tag{1.20}$$

$$\mu_n(q, \alpha, 0) = \Big(n + \frac{1}{2}\Big)^2 + O(1), \quad \text{when } \sin\alpha \neq 0 \tag{1.21}$$

$$\mu_n(q, \pi, 0) = (n + 1)^2 + O(1). \tag{1.22}$$

The relations (1.18) and (1.19) do not hold in the general case (for example, $\mu_n(q, \pi, \beta) - \mu_n(q, \alpha, \beta) = O(n)$ when $\alpha \in (0, \pi)$).

Thus to obtain the representation of a_n and b_n for all the cases by two spectra we need some improvement of the method in [7] and [2].

These improvement are the causes that the formulae (1.9) and (1.10)-(1.12) (similarly (1.13) and (1.14)-(1.16)) are different and depend on the cases:

- $\alpha, \beta \in (0, \pi)$,
- $\alpha = \pi$, $\beta \in (0, \pi)$,
- $\alpha \in (0, \pi)$, $\beta = 0$,
- $\alpha = \pi$, $\beta = 0$.

Whether it is possible to join all these formulae in one (may be more complicated, but one), today we do not know.

It was obtained in [3], that the lowest eigenvalue $\mu_0(q, \alpha, \beta)$ has the property: for arbitrary $\beta \in [0, \pi)$, $\lim_{\alpha \to 0} \mu_0(q, \alpha, \beta) = -\infty$ and for arbitrary $\alpha \in (0, \pi]$, $\lim_{\beta \to \pi} \mu_0(q, \alpha, \beta) = -\infty$.

In combination with the property

$$\frac{1}{a_n(q, \alpha, \beta)} = \frac{\partial \mu_n(q, \alpha, \beta)}{\partial \alpha}$$

(and $\frac{1}{b_n(q,\alpha,\beta)} = -\frac{\partial \mu_n(q,\alpha,\beta)}{\partial \beta}$ [3]) this leads to the fact that the formulae (1.11) and (1.15) for $a_0(q, \pi, \beta)$ and $b_0(q, \alpha, 0)$ have a special form.

2. Proof of the main theorem

The proof of the main theorem is based on the following lemmas. Here we give the statements of these lemmas and after that we prove the Theorem. The proofs of the lemmas are presented in §3.

Lemma 2.1. *As $t \to \infty$, we have the following:*

$$\Phi(-t^2, \alpha, \beta) = \frac{te^{\pi t}}{2}[\sin\alpha \cdot \sin\beta + O\big(\frac{1}{t}\big)], \quad \textit{when } \alpha, \beta \in (0, \pi), \tag{2.1}$$

$$\Phi(-t^2, \pi, \beta) = \frac{e^{\pi t}}{2}[\sin\beta + O\big(\frac{1}{t}\big)], \quad \textit{when } \beta \in (0, \pi), \tag{2.2}$$

$$\Phi(-t^2, \alpha, 0) = \frac{e^{\pi t}}{2}[\sin\alpha + O\big(\frac{1}{t}\big)], \quad \textit{when } \alpha \in (0, \pi), \tag{2.3}$$

$$\Phi(-t^2, \pi, 0) = \frac{e^{\pi t}}{2t}[1 + O\big(\frac{1}{t}\big)]. \tag{2.4}$$

Lemma 2.2. *The specification of the spectra* $\{\mu_n(q,\alpha,\beta)\}_{n=0}^{\infty}$ *uniquely determines the characteristic functions* $\Phi(\mu,\alpha,\beta)$ *by the formulae*

$$\Phi(\mu,\alpha,\beta)=\pi\left(\mu_0(\alpha,\beta)-\mu\right)\cdot\sin\alpha\cdot\sin\beta\cdot\prod_{k=1}^{\infty}\frac{\mu_k(\alpha,\beta)-\mu}{k^2}, \tag{2.5}$$

when $\alpha,\beta\in(0,\pi)$.

$$\Phi(\mu,\pi,\beta)=\sin\beta\cdot\prod_{k=0}^{\infty}\frac{\mu_k(\pi,\beta)-\mu}{(k+\frac{1}{2})^2},\quad \text{when } \beta\in(0,\pi), \tag{2.6}$$

$$\Phi(\mu,\alpha,0)=\sin\alpha\cdot\prod_{k=0}^{\infty}\frac{\mu_k(\alpha,0)-\mu}{(k+\frac{1}{2})^2},\quad \text{when } \alpha\in(0,\pi), \tag{2.7}$$

$$\Phi(\mu,\pi,0)=\pi\prod_{k=0}^{\infty}\frac{\mu_k(\pi,0)-\mu}{(k+1)^2}. \tag{2.8}$$

In what follows, by $\dot{f}(x,\mu)$ we denote the derivative by μ; i.e., $\dot{f}(x,\mu)=\frac{\partial f(x,\mu)}{\delta\mu}$.

Lemma 2.3. *The following formulae hold*

$$\dot{\Phi}(\mu,\alpha,\beta)\big|_{\mu=\mu_0(\alpha,\beta)}=-\pi\sin\alpha\cdot\sin\beta\prod_{k=1}^{\infty}\frac{\mu_k(\alpha,\beta)-\mu}{k^2}, \tag{2.9}$$

when $\alpha,\beta\in(0,\pi)$.

$$\begin{aligned}\dot{\Phi}(\mu,\alpha,\beta)\big|_{\mu=\mu_n(\alpha,\beta)}&=-\frac{\pi}{n^2}\left[\mu_0(\alpha,\beta)-\mu_n(\alpha,\beta)\right]\\&\quad\cdot\sin\alpha\cdot\sin\beta\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{\mu_k(\alpha,\beta)-\mu_n(\alpha,\beta)}{k^2},\end{aligned} \tag{2.10}$$

when $\alpha,\beta\in(0,\pi)$ *and* $n\neq 0$.

$$\dot{\Phi}(\mu,\pi,\beta)\big|_{\mu=\mu_n(\pi,\beta)}=-\frac{\sin\beta}{\left(n+\frac{1}{2}\right)^2}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\mu_k(\pi,\beta)-\mu_n(\pi,\beta)}{(k+\frac{1}{2})^2}, \tag{2.11}$$

when $\beta\in(0,\pi)$.

$$\dot{\Phi}(\mu,\alpha,0)\big|_{\mu=\mu_n(\alpha,0)}=-\frac{\sin\alpha}{\left(n+\frac{1}{2}\right)^2}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\mu_k(\alpha,0)-\mu_n(\alpha,0)}{(k+\frac{1}{2})^2}, \tag{2.12}$$

when $\alpha\in(0,\pi)$.

$$\dot{\Phi}(\mu,\pi,0)\big|_{\mu=\mu_n(\pi,0)}=-\frac{\pi}{(n+1)^2}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\mu_k(\pi,0)-\mu_n(\pi,0)}{(k+1)^2}. \tag{2.13}$$

We consider also the meromorphic functions (see (1.6) and (1.7)):

$$m_{\alpha,\beta,\varepsilon}(\mu):=\frac{\Psi(\mu,\alpha,\beta)}{\Psi(\mu,\varepsilon,\beta)}=\frac{\psi(0,\mu,\beta)\cos\alpha+\psi'(0,\mu,\beta)\sin\alpha}{\psi(0,\mu,\beta)\cos\varepsilon+\psi'(0,\mu,\beta)\sin\varepsilon}, \tag{2.14}$$

$$n_{\alpha,\beta,\delta}(\mu):=\frac{\Phi(\mu,\alpha,\beta)}{\Phi(\mu,\alpha,\delta)}=\frac{\varphi(\pi,\mu,\alpha)\cos\beta+\varphi'(\pi,\mu,\alpha)\sin\beta}{\varphi(\pi,\mu,\alpha)\cos\delta+\varphi'(\pi,\mu,\alpha)\sin\delta}. \tag{2.15}$$

Lemma 2.4. *For arbitrary $\varepsilon \in (0,\pi)$, $\varepsilon \neq \alpha$,*

$$\frac{\partial m_{\alpha,\beta,\varepsilon}(\mu)}{\partial \mu}\Big|_{\mu=\mu_n(\alpha,\beta)} = \frac{a_n(q,\alpha,\beta)}{\sin(\alpha-\varepsilon)}\,, \tag{2.16}$$

and for arbitrary $\delta \in (0,\pi)$, $\delta \neq \beta$,

$$\frac{\partial n_{\alpha,\beta,\delta}(\mu)}{\partial \mu}\Big|_{\mu=\mu_n(\alpha,\beta)} = \frac{b_n(q,\alpha,\beta)}{\sin(\beta-\delta)}\,. \tag{2.17}$$

As it is noted in §1, $\Psi(\mu,\alpha,\beta) = -\Phi(\mu,\alpha,\beta)$. So, (2.14) we can be rewritten as

$$m_{\alpha,\beta,\varepsilon}(\mu) = \frac{\Phi(\mu,\alpha,\beta)}{\Phi(\mu,\varepsilon,\beta)}\,. \tag{2.18}$$

It is easy follows from the (2.18) and (2.15) that

$$\begin{gathered}
\frac{\partial m_{\alpha,\beta,\varepsilon}(\mu)}{\partial \mu}\Big|_{\mu=\mu_n(\alpha,\beta)} = \frac{\dot{\Phi}(\mu_n(\alpha,\beta),\alpha,\beta)}{\Phi(\mu_n(\alpha,\beta),\varepsilon,\beta)},\\
\frac{\partial n_{\alpha,\beta,\delta}(\mu)}{\partial \mu}\Big|_{\mu=\mu_n(\alpha,\beta)} = \frac{\dot{\Phi}(\mu_n(\alpha,\beta),\alpha,\beta)}{\Phi(\mu_n(\alpha,\beta),\alpha,\delta)}\,.
\end{gathered}$$

Then, from (2.16) and (2.17), we obtain

$$a_n(q,\alpha,\beta) = \sin(\alpha-\varepsilon)\,\frac{\dot{\Phi}(\mu_n(\alpha,\beta),\alpha,\beta)}{\Phi(\mu_n(\alpha,\beta),\varepsilon,\beta)}\,, \tag{2.19}$$

$$b_n(q,\alpha,\beta) = \sin(\beta-\delta)\,\frac{\dot{\Phi}(\mu_n(\alpha,\beta),\alpha,\beta)}{\Phi(\mu_n(\alpha,\beta),\alpha,\delta)}\,. \tag{2.20}$$

Substituting corresponding formulae from (2.9)-(2.13) and (2.5)-(2.8) in (2.19) and (2.20), we obtain the formulae (1.9)-(1.12) and (1.13)-(1.16) of Theorem. Thus, our Theorem is proved if the lemmas 2.1-2.4 hold.

3. Proofs of lemmas

Proof of the lemma 2.1. Let us denoted by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ the solutions of the equation $-y'' + q(x)y = \lambda^2 y$, satisfying the initial conditions

$$\begin{gathered}
y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0,\\
y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1.
\end{gathered}$$

For $y_1(x,\lambda)$ and $y_2(x,\lambda)$ it is well known [1, 6, 8, 9] (the case $q \in L^1_{\mathbb{C}}[0,\pi]$ is considered in detail in [4]) the asymptotic formulae (when $|\lambda| \to \infty$)

$$\begin{gathered}
y_1(\pi,\lambda) = \cos\lambda\pi + O\Big(\frac{e^{|\mathrm{Im}\lambda|\pi}}{|\lambda|}\Big), \quad y_1'(\pi,\lambda) = -\lambda\sin\lambda\pi + O\left(e^{|\mathrm{Im}\lambda|\pi}\right),\\
y_2(\pi,\lambda) = \frac{\sin\lambda\pi}{\lambda} + O\Big(\frac{e^{|\mathrm{Im}\lambda|\pi}}{|\lambda|^2}\Big), \quad y_2'(\pi,\lambda) = \cos\lambda\pi + O\Big(\frac{e^{|\mathrm{Im}\lambda|\pi}}{|\lambda|}\Big).
\end{gathered} \tag{3.1}$$

Since (see (1.4)),

$$\varphi(x,\lambda^2,\alpha) = y_1(x,\lambda)\sin\alpha - y_2(x,\lambda)\cos\alpha \tag{3.2}$$

and $\Phi(\mu,\alpha,\beta)$ is defined by (1.6), if we substitute in (3.1), (3.2) and (1.6) $\mu=\lambda^2=(it)^2=-t^2$, we obtain for $\Phi(-t^2,\alpha,\beta)$ the formula

$$\begin{aligned}\Phi(-t^2,\alpha,\beta) &= \Big(\cos(i\pi t)+O\big(\frac{e^{\pi t}}{t}\big)\Big)\sin\alpha\cos\beta-\Big(\frac{\sin(i\pi t)}{it}+O\big(\frac{e^{\pi t}}{t}\big)\Big)\cos\alpha\cos\beta\\ &\quad+(-it\sin(i\pi t)+O(e^{\pi t}))\sin\alpha\cos\beta-\Big(\cos(i\pi t)+O\big(\frac{e^{\pi t}}{t}\big)\Big)\cos\alpha\sin\beta.\end{aligned}$$

Taking into account the formulae $\cos(i\pi t)=\frac{e^{\pi t}+e^{-\pi t}}{2}$, $\sin(i\pi t)=\frac{e^{-\pi t}-e^{\pi t}}{2i}$ we obtain the assertions of lemma 2.1. □

Proof of the lemma 2.2. The case $\alpha,\beta\in(0,\pi)$ was considered in [1, 2, 5]; see [1, formula (1.1.26)] for details. In this case ($\sin\alpha\neq0$, $\sin\beta\neq0$) they consider as the characteristic function $\frac{\Phi(\mu,\alpha,\beta)}{\sin\alpha\cdot\sin\beta}$ (in our notation).

The remaining three results are presented in [1], but without a proof, and in the case $\alpha=\pi$, $\beta=0$ they write

$$\Phi(\mu,\pi,0)=\pi\prod_{k=1}^{\infty}\frac{\mu_k(\pi,0)-\mu}{k^2},$$

since they begin the enumeration for $\mu_n(\pi,0)$ from $n=1$, but not from $n=0$. We use Hadamard's factorization theorem (it is similar to method of [1]) to represent the entire function $\Phi(\mu,\alpha,\beta)$ of order 1/2 in the form

$$\Phi(\mu,\alpha,\beta)=c\prod_{k=0}^{\infty}\Big(1-\frac{\mu}{\mu_k(\alpha,\beta)}\Big). \tag{3.3}$$

If $\Phi(0,\alpha,\beta)=0$, i.e. for some k_0 $\mu_{k_0}(\alpha,\beta)=0$, then we must change the k_0-th factor by $-\mu$ and it is easy to show that we can take $\mu_{k_0}(\alpha,\beta)=0=\mu_0(\alpha,\beta)$ (change the enumeration for finite number of eigenvalues and correspondingly for finite factors) in

$$\Phi(\mu,\alpha,\beta)=c_1\mu\prod_{\substack{k=0\\k\neq k_0}}^{\infty}\Big(1-\frac{\mu}{\mu_k(\alpha,\beta)}\Big)=c_2\mu\prod_{k=1}^{\infty}\Big(1-\frac{\mu}{\mu_k(\alpha,\beta)}\Big). \tag{3.4}$$

Detailed computations show that results for the cases (3.3) and (3.4) are the same.

In the case $\alpha=\pi$, $\beta\in(0,\pi)$ ($\sin\beta\neq0$) we use the formula

$$\cos\lambda\pi=\prod_{k=0}^{\infty}\Big(1-\frac{\lambda^2}{(k+\frac12)^2}\Big) \tag{3.5}$$

and consider the relation

$$\frac{\Phi(\lambda^2,\pi,\beta)}{\cos\lambda\pi}=\frac{c\prod_{k=0}^{\infty}\big(1-\frac{\lambda^2}{\mu_k(\pi,\beta)}\big)}{\prod_{k=0}^{\infty}\frac{(k+\frac12)^2-\lambda^2}{(k+\frac12)^2}}=c\prod_{k=0}^{\infty}\frac{(k+\frac12)^2}{\mu_k(\pi,\beta)}\cdot\frac{\mu_k(\pi,\beta)-\lambda^2}{(k+\frac12)^2-\lambda^2}. \tag{3.6}$$

According to (1.20), in this case the infinite product

$$\prod_{k=0}^{\infty}\frac{\mu_k(\pi,\beta)-\lambda^2}{(k+\frac12)^2-\lambda^2}=\prod_{k=0}^{\infty}\Big(1+\frac{\mu_k(\pi,\beta)-(k+\frac12)^2}{(k+\frac12)^2-\lambda^2}\Big)$$

converges uniformly because $\lambda^2 = -t^2 \in (-\infty, -1]$). The infinite product

$$\prod_{k=0}^{\infty} \frac{(k+\frac{1}{2})^2}{\mu_k(\pi,\beta)}$$

also converses. So we can write the right hand side of (3.6) in the form

$$c\prod_{k=0}^{\infty} \frac{(k+\frac{1}{2})^2}{\mu_k(\pi,\beta)} \cdot \prod_{k=0}^{\infty} \frac{\mu_k(\pi,\beta)-\lambda^2}{(k+\frac{1}{2})^2-\lambda^2}.$$

Thus, if we pass to the limit in (3.6) when $\lambda = it$, $t \to \infty$, then in the left hand side of (3.6), according to lemma 2.1, we obtain

$$\lim_{t\to\infty} \frac{\Phi(-t^2,\alpha,\beta)}{\cos(it\pi)} = \lim_{t\to\infty} \frac{\frac{e^{t\pi}}{2}[\sin\beta + O\left(\frac{1}{t}\right)]}{\frac{e^{\pi t}+e^{-\pi t}}{2}} = \sin\beta,$$

and in the right hand side we obtain $c\prod_{k=0}^{\infty} \frac{(k+\frac{1}{2})^2}{\mu_k(\pi,\beta)}$; i.e., $c = \sin\beta \prod_{k=0}^{\infty} \frac{\mu_k(\pi,\beta)}{(k+\frac{1}{2})^2}$. Substituting this value of c in (3.3), we obtain formula (2.6).

In the case $\alpha \in (0,\pi)$, $\beta = 0$ we follow the same procedure and obtain formula (2.7).

In case $\alpha = \pi$, $\beta = 0$ we consider the relation of

$$\Phi(\lambda^2,\pi,0) = c\prod_{k=0}^{\infty} \Big(1 - \frac{\lambda^2}{\mu_k(\pi,0)}\Big)$$

and $\sin(\lambda\pi)/\lambda$, which we can write in the form

$$\frac{\sin\lambda\pi}{\lambda} = \pi\prod_{k=0}^{\infty} \Big(1 - \frac{\lambda^2}{(k+1)^2}\Big).$$

According to (1.22), the infinite product $\prod_{k=0}^{\infty} \frac{(k+1)^2}{\mu_k(\pi,0)}$ converges and the product

$$\prod_{k=0}^{\infty} \frac{\mu_k(\pi,0)-\lambda^2}{(k+1)^2-\lambda^2} = \prod_{k=0}^{\infty} \Big[1 + \frac{\mu_k(\pi,0)-(k+1)^2}{(k+1)^2-\lambda^2}\Big],$$

converges uniformly by $\lambda^2 = (it)^2 = -t^2 \in (-\infty,-1]$. Thus, we can pass to the limit ($t \to \infty$) in the relation

$$\frac{\Phi(-t^2,\pi,0)}{\frac{\sin i\pi t}{it}} = \frac{c}{\pi}\prod_{k=0}^{\infty} \frac{(k+1)^2}{\mu_k(\pi,0)} \cdot \prod_{k=0}^{\infty} \frac{\mu_k(\pi,0)+t^2}{(k+1)^2+t^2}$$

and according to (2.4) and uniform convergence of the right hand side, we obtain $1 = \frac{c}{\pi}\prod_{k=0}^{\infty} \frac{(k+1)^2}{\mu_k(\pi,0)}$. Thus, we obtain the formula (2.8). Lemma 2.2 is proved. □

The proof of the lemma 2.3 coincides with simple differentiation with respect to μ at the point $\mu = \mu_n(\alpha,\beta)$ in (2.5)-(2.8). With this aim we represent the infinite product $\prod_{k=0}^{\infty} a_k$ in the form

$$\prod_{k=0}^{\infty} a_k = \prod_{k=0}^{n} a_k \cdot \prod_{k=n+1}^{\infty} a_k = a_n \prod_{k=0}^{n-1} a_k \prod_{k=n+1}^{\infty} a_k = a_n \prod_{\substack{k=0\\k\neq n}}^{\infty} a_k.$$

For example,

$$\Phi(\mu,\alpha,0)=\frac{\mu_n(\pi,0)-\mu}{\left(n+\frac{1}{2}\right)^2}\cdot\sin\alpha\cdot\prod_{k\neq n}\frac{\mu_k(\pi,0)-\mu}{(k+\frac{1}{2})^2}.$$

Now differentiating with respect to μ and taking the result at $\mu=\mu_n(\alpha,0)$ we obtain

$$\begin{aligned}
&\dot\Phi(\mu_n(\alpha,0),\alpha,0)\\
&=\frac{-1}{\left(n+\frac{1}{2}\right)^2}\sin\alpha\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\mu_k(\alpha,0)-\mu}{(k+\frac{1}{2})^2}\Big|_{\mu=\mu_n(\alpha,0)}+\frac{\mu_n(\alpha,0)-\mu}{\left(n+\frac{1}{2}\right)^2}\cdot\mathcal{P}(\mu)\Big|_{\mu=\mu(\alpha,0)}\\
&=-\frac{\sin\alpha}{\left(n+\frac{1}{2}\right)^2}\prod_{\substack{k\neq n\\k=0}}^{\infty}\frac{\mu_k(\alpha,0)-\mu_n(\alpha,0)}{(k+\frac{1}{2})^2},
\end{aligned}$$

where

$$\mathcal{P}(\mu)=\frac{\partial}{\partial\mu}\Big(\sin\alpha\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\mu_k(\alpha,0)-\mu}{(k+\frac{1}{2})^2}\Big).$$

The process is analogous in other cases. Lemma 2.3 is proved.

Proof of lemma 2.4. Since all the eigenvalues of $L(q,\alpha,\beta)$ are simple, there exist the constants $c_n=c_n(q,\alpha,\beta)$, $n=0,1,2,\dots$, such that

$$\varphi_n(x)=c_n\cdot\psi_n(x). \tag{3.7}$$

By direct computation, from (2.14) we obtain

$$\frac{\partial m_{\alpha,\beta,\varepsilon}(\mu)}{\partial\mu}=\frac{[\dot\psi'(0,\mu,\beta)\cdot\psi(0,\mu,\beta)-\dot\psi(0,\mu,\beta)\cdot\psi'(0,\mu,\beta)]\sin(\alpha-\varepsilon)}{[\psi(0,\mu,\beta)\cos\varepsilon+\psi'(0,\mu,\beta)\sin\varepsilon]^2}. \tag{3.8}$$

On the other hand, by standard methods, we obtain

$$\int_0^{\pi}\psi^2(x,\mu,\beta)\,dx=\dot\psi'(0,\mu,\beta)\cdot\psi(0,\mu,\beta)-\dot\psi(0,\mu,\beta)\cdot\psi'(0,\mu,\beta). \tag{3.9}$$

Since for $\mu=\mu_n(q,\alpha,\beta)$ we have (3.7), using (1.4), from (3.8) and (3.9) we obtain

$$\begin{aligned}
\frac{\partial m_{\alpha,\beta,\varepsilon}(\mu)}{\partial\mu}\Big|_{\mu=\mu_n(\alpha,\beta)}&=\frac{\int_0^\infty\psi_n^2(x)dx\cdot\sin(\alpha-\varepsilon)}{[\psi_n(0)\cos\varepsilon+\psi_n'(0)\sin\varepsilon]^2}\\
&=\frac{c_n^2\int_0^\infty\varphi_n^2(x)dx\cdot\sin(\alpha-\varepsilon)}{c_n^2[\varphi_n(0)\cos\varepsilon+\varphi_n'(0)\sin\varepsilon]^2}\\
&=\frac{a_n(q,\alpha,\beta)}{\sin(\alpha-\varepsilon)}.
\end{aligned}$$

We prove (2.17) in a similar way. Lemma 2.4 is proved. □

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