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PROPERTIES OF THE FIRST EIGENVALUE OF A MODEL FOR NON NEWTONIAN FLUIDS

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ABSTRACT. We consider a nonlinear Stokes problem on a bounded domain. We prove the existence of the first eigenvalue which is given by a minimization formula. Some properties such as strict monotony and the Fredholm alternative are established.

1. INTRODUCTION

In studies of semi-linear elliptic equations such as

$$\begin{gathered} -\Delta u = f(x,u) + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where Ω is a bounded domain of $\mathbb{R}^n$. It is usual to impose conditions on the asymptotic behavior of the nonlinearity $f(x,u)$ in relation to the spectrum of the linear part of $-\Delta$. In the simplest situations, we consider $f(x,u)$ as a perturbation of λu. According to that λ being or not an eigenvalue of $-\Delta$, the results of such resonance or non-resonance are then obtained. Among the classical references on this subject, we can cite [5] ($\lambda < \lambda_1$), [4] (λ between two consecutive eigenvalues), [6] ($\lambda = \lambda_1$). We also cite the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{k-2}\nabla u) = \lambda m(x)|u|^{k-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The first eigenvalue λ_1 of the Dirichlet problem is simple and isolated. It was proved that it is the unique positive eigenvalue having a non negative eigenfunction, see [2].

Now we consider the eigenvalue problem of a non-linear operator k-Laplacian. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma = \bigcup_{i=1}^4 \overline{\Gamma_i}$, where $\Gamma_1 = \{0\}\times]-1,1[$, $\Gamma_2 = \{1\}\times]-1,1[$ and Γ_3, Γ_4 are symmetrical to the X-axis, see Figure 1. In the interior of this domain, a non-Newtonian liquid is subjected to pressures of known differences between the two sides Γ_1 and Γ_2.

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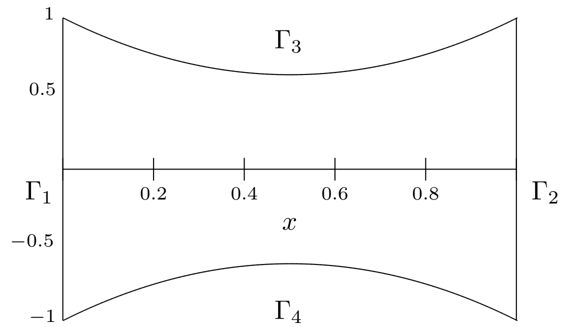

FIGURE 1. Geometry of channel Ω

We denote by V the closure of $\mathcal{V}$ in the space $W^{1,k}(\Omega)$, where

$$\begin{aligned}\mathcal{V} = \big\{&u = (u_1, u_2)^t \in (C^1(\bar{\Omega}))^2 : \operatorname{div} u = 0, u_i(0, y) = u_i(1, y) \text{ on } [-1, 1] \\ &\text{for } i = 1, 2 \text{ and } u = 0 \text{ on } \Gamma_3 \cup \Gamma_4\big\}.\end{aligned}$$

For given $\alpha \in \mathbb{R}$, we consider the eigenvalue problem: Find $(\lambda, u, p) \in \mathbb{R} \times V \setminus \{0\} \times L^2(\Omega)$ such that

$$\begin{gathered}
-\Delta_k u_1 + \frac{\partial p}{\partial x} = \lambda m(x, y)|u_1|^{k-2} u_1 \quad \text{in } \Omega, \\
-\Delta_k u_2 + \frac{\partial p}{\partial y} = \lambda m(x, y)|u_2|^{k-2} u_2 \quad \text{in } \Omega, \\
\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \quad \text{in } \Omega, \\
u_1(0, y) = u_1(1, y) \quad \text{on } [-1, 1], \\
u_2(0, y) = u_2(1, y) \quad \text{on } [-1, 1], \\
\frac{\partial u_1}{\partial x}(0, y) = \frac{\partial u_1}{\partial x}(1, y) \quad \text{on } [-1, 1], \\
|\nabla u_2(0, y)|^{k-2} \frac{\partial u_2}{\partial x}(0, y) = |\nabla u_2(1, y)|^{k-2} \frac{\partial u_2}{\partial x}(1, y) \quad \text{on } [-1, 1], \\
p(1, y) - p(0, y) = -\alpha \quad \text{on } [-1, 1]
\end{gathered} \tag{1.1}$$

where the weight function $m(x, y) \in L^\infty(\Omega)$ can change the sign and it is positive in a subset of Ω,

$$-\Delta_k u_i = -\operatorname{div}(|\nabla u_i|^{k-2} \nabla u_i)$$

is a k-Laplacian, $i = 1, 2$ and $1 < k < \infty$. In the particular case $k = 2$; i.e., $\Delta_k = \Delta$, and $\lambda = 0$, the above problem has been studied by many authors, we cite for example Amrouche et al. [1]. Here, we give an extension to previous work in the nonlinear case by applying new methods to characterize the first eigenvalue for this kind of problem such as minimization and as application is to solving the problem of Fredholm alternative. This note is organized as follows. In Section 2, we give the existence and the characterization of the first eigenvalue. In Section 3, we prove the Fredholm alternative and we justify all the given properties. In Section 4, we give a conclusion.

2. Existence and characterization of the first eigenvalue

Theorem 2.1. *There exists one principal eigenvalue λ_1 for Problem* (1.1)*. It is characterized by*

$$\lambda_1 = k\beta + (k-1)\alpha \int_{-1}^{1} (\varphi^1)_1(0,y)dy, \tag{2.1}$$

where φ^1 is the principal corresponding eigenfunction and

$$\begin{gathered}\beta = \min\Big\{\frac{1}{k}\int_\Omega |\nabla u_1|^k + |\nabla u_2|^k - \alpha\int_{-1}^{1} u_1(0,y)dy;\\ \int_\Omega m(x,y)(|u_1|^k + |u_2|^k) = 1, u \in V\Big\},\end{gathered}$$

$$\begin{gathered}\beta = \frac{1}{k}\int_\Omega |\nabla(\varphi^1)_1|^k + |\nabla(\varphi^1)_2|^k - \alpha\int_{-1}^{1} (\varphi^1)_1(0,y)dy,\\ \int_\Omega m(x,y)[|(\varphi^1)_1|^k + |(\varphi^1)_2|^k] = 1.\end{gathered}$$

Furthermore, for all α, $\alpha' \in \mathbb{R}$ such that $\alpha\alpha' > 0$, $\lambda_1(\alpha)$ is an eigenvalue of Problem (1.1)*.*

For the sake of simplicity, in what follows, we denote $\lambda_1 = \lambda_1(m) = \lambda_1(\alpha, m) = \lambda_1(\alpha)$.

Theorem 2.2. *(i) λ_1 defined by*

$$\frac{1}{\lambda_1} = \max\{\int_\Omega m(x,y)(|u_1|^k + |u_2|^k);\ \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k = 1, u \in V\} \tag{2.2}$$

is the first eigenvalue of Problem (1.1) *with $\alpha = 0$ in the sense $\Sigma \subset [\lambda_1, +\infty[$, where Σ is the set of the positive eigenvalues. Moreover, u is the eigenfunction associated with λ_1 if and only if $\int_\Omega |\nabla u_1|^k + |\nabla u_2|^k - \lambda_1 \int_\Omega m(x,y)(|u_1|^k + |u_2|^k) = \inf\{\int_\Omega |\nabla v_1|^k + |\nabla v_2|^k - \lambda_1 \int_\Omega m(x,y)(|v_1|^k + |v_2|^k)\ \ v \in V\} = 0$.*

(ii) $\lambda_1(.)$ is strictly monotone in $L^\infty(\Omega)$; i.e., if m_1, m_2 are in the set

$$\big\{m \in L^\infty(\Omega); \mathrm{measure}\{(x,y) \in \Omega;\ m(x,y) > 0\} \neq 0\big\}$$

such that $m_1(x,y) < m_2(x,y)$ a.e., then $\lambda_1(m_1) > \lambda_1(m_2)$.

(iii) $\lambda_1(.)$ is continuous in $L^\infty(\Omega)$.

Theorem 2.3 (Fredholm alternative)**.** *Suppose that $\lambda < \lambda_1$, then for $f \in (C(\overline{\Omega}))^2$ the problem: Find $(u,p) \in V \times L^2(\Omega)$ such that*

$$\begin{gathered}
-\Delta_k u_1 + \frac{\partial p}{\partial x} = \lambda m(x,y)|u_1|^{k-2}u_1 + f_1 \quad \text{in } \Omega,\\
-\Delta_k u_2 + \frac{\partial p}{\partial y} = \lambda m(x,y)|u_2|^{k-2}u_2 + f_2 \quad \text{in } \Omega,\\
\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \quad \text{in } \Omega,\\
u_1(0,y) = u_1(1,y) \quad \text{on } [-1,1],\\
u_2(0,y) = u_2(1,y) \quad \text{on } [-1,1],\\
\frac{\partial u_1}{\partial x}(0,y) = \frac{\partial u_1}{\partial x}(1,y) \quad \text{on } [-1,1],\\
|\nabla u_2(0,y)|^{k-2}\frac{\partial u_2}{\partial x}(0,y) = |\nabla u_2(1,y)|^{k-2}\frac{\partial u_2}{\partial x}(1,y) \quad \text{on } [-1,1],\\
p(1,y) - p(0,y) = -\alpha \quad \text{on } [-1,1]
\end{gathered} \tag{2.3}$$

has a solution.

3. Proof of the main theorems

For proving Theorem 2.1, we need the following results.

Proposition 3.1. *$u = (u_1,u_2)^t$ is a solution of problem: Find $(u,p) \in V \setminus \{0\} \times L^2(\Omega)$ such that*

$$\begin{gathered}
-\Delta_k u_1 + \frac{\partial p}{\partial x} = f_1 \quad \text{in } \Omega,\\
-\Delta_k u_2 + \frac{\partial p}{\partial y} = f_2 \quad \text{in } \Omega,\\
\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \quad \text{in } \Omega,\\
u_1(0,y) = u_1(1,y) \quad \text{on } [-1,1],\\
u_2(0,y) = u_2(1,y) \quad \text{on } [-1,1],\\
\frac{\partial u_1}{\partial x}(0,y) = \frac{\partial u_1}{\partial x}(1,y) \quad \text{on } [-1,1],\\
|\nabla u_2(0,y)|^{k-2}\frac{\partial u_2}{\partial x}(0,y) = |\nabla u_2(1,y)|^{k-2}\frac{\partial u_2}{\partial x}(1,y) \quad \text{on } [-1,1],\\
p(1,y) - p(0,y) = -\alpha \quad \text{on } [-1,1]
\end{gathered} \tag{3.1}$$

where $f = (f_1,f_2)^t \in (C(\overline{\Omega}))^2$, if and only if u is a solution of problem: Find $u \in V$ such that

$$\sum_{i=1}^{2} \int_\Omega |\nabla u_i|^{k-2}\nabla u_i \nabla v_i - \alpha \int_{-1}^{1} v_1(0,y)dy = \int_\Omega (f_1v_1 + f_2v_2) \tag{3.2}$$

for all $v \in V$.

Remark 3.2. If we take $f_i = \lambda m(x,y)|u_i|^{k-2}u_i$, $i=1,2$. Then (λ,u,p) is a solution of (1.1) if and only if

$$\sum_{i=1}^{2}\int_\Omega |\nabla u_i|^{k-2}\nabla u_i\nabla v_i - \alpha\int_{-1}^{1} v_1(0,y)dy = \lambda\sum_{i=1}^{2}\int_\Omega m(x,y)|u_i|^{k-2}u_iv_i$$

for all $v\in V$. For a proof of this remark see [3].

Proof of Theorem 2.1. Since for all $v\in V$, $v=0$ on $\Gamma_3\cup\Gamma_4$, $u\in V\to(\int_\Omega|\nabla u_1|^k+|\nabla u_2|^k)^{1/k}$ define a norm in V according to the Poincaré inequality in the space V: There exists $c>0$ such that

$$c\int_\Omega |u_1|^k+|u_2|^k \le \int_\Omega |\nabla u_1|^k+|\nabla u_2|^k. \tag{3.3}$$

Suppose by contradiction that for all $n\in\mathbb{N}^*$ there exists $u_n=(u_1^n,u_2^n)^t\in V$ such that $\frac{1}{n}\int_\Omega |u_1^n|^k+|u_2^n|^k > \int_\Omega|\nabla u_1^n|^k+|\nabla u_2^n|^k$, then we put

$$v_n=(v_1^n,v_2^n)^t$$

where $v_i^n=\frac{u_i^n}{(\int_\Omega|u_1^n|^k+|u_2^n|^k)^{1/k}}$, $i=1,2$. Thus $\int_\Omega|v_1^n|^k+|v_2^n|^k=1$, so

$$\frac{1}{n}>\int_\Omega|\nabla v_1^n|^k+|\nabla v_2^n|^k. \tag{3.4}$$

As $(v_n)_n$ is bounded in V, we have for a subsequence also denoted $(v_n)_n$, $v_n\rightharpoonup v$ in V and $v_n\to v$ in $L^k(\Omega)$. Therefore $\|v\|_{L^k(\Omega)}=1$, so $v\neq 0$. By passing to the limit in (3.4), we have

$$0\ge\liminf_n\int_\Omega|\nabla v_1^n|^k+|\nabla v_2^n|^k\ge\int_\Omega|\nabla v_1|^k+|\nabla v_2|^k.$$

So $\int_\Omega|\nabla v_1|^k=\int_\Omega|\nabla v_2|^k=0$, hence $v=cst$, therefore $v=0$ because $v=0$ on $\Gamma_3\cup\Gamma_4$, is a contradiction. By using (3.3) and the Holder's inequality, we easily prove that β is well defined. Let $(u_n)=((u_{n1},u_{n2}))$ be a suitable minimization of β, then we have

$$\beta=\lim_{n\to\infty}\frac{1}{k}\int_\Omega|\nabla u_{n1}|^k+|\nabla u_{n2}|^k-\alpha\int_{-1}^{1}u_{n1}(0,y)dy$$

and

$$\int_\Omega m(x,y)(|u_{n1}|^k+|u_{n2}|^k)=1.$$

The sequence $(X_n):=(\frac{1}{k}\int_\Omega|\nabla u_{n1}|^k+|\nabla u_{n2}|^k)$ is bounded, if we have not for a subsequence, also denoted (X_n), $X_n\to+\infty$. Using the Holder's inequality and the fact that $V\hookrightarrow L^k(\Gamma_1)$ we get

$$\alpha\int_{-1}^{1}u_{n1}(0,y)dy\le|\alpha|c(\frac{1}{k}\int_\Omega|\nabla u_{n1}|^k+|\nabla u_{n2}|^k)^{1/k}=|\alpha|cX_n^{1/k}$$

where $c\in\mathbb{R}$. Thus $\frac{1}{k}\int_\Omega\sum_{i=1}^{2}|\nabla u_{ni}|^k-\alpha\int_{-1}^{1}u_{n1}(0,y)dy\ge X_n-|\alpha|cX_n^{1/k}$, this prove that $\beta=+\infty$, which is impossible. According to the reflexivity of the space V and the compact injections $V\hookrightarrow L^k(\Omega)$ and $V\hookrightarrow L^k(\Gamma_1)$, there exists a subsequence of $(u_n)=((u_{n1},u_{n2}))$, which is also denoted by $(u_n)=((u_{n1},u_{n2}))$, such that

$$\begin{gathered}
u_n=(u_{n1},u_{n2})\rightharpoonup \varphi^1=((\varphi^1)_1,(\varphi^1)_2)\quad \text{in } V,\\
u_n=(u_{n1},u_{n2})\to \varphi^1=((\varphi^1)_1,(\varphi^1)_2)\quad \text{in } L^k(\Omega),\\
u_{n1}|_{\Gamma_1}\to (\varphi^1)_1|_{\Gamma_1}\quad \text{in } L^k(\Gamma_1).
\end{gathered}$$

Hence $\int_\Omega m(x,y)(|(\varphi^1)_1|^k+|(\varphi^1)_2|^k)=1$, consequently $\varphi^1\neq 0$ and

$$\begin{aligned}
\beta &\le \frac{1}{k}\int_\Omega |\nabla(\varphi^1)_1|^k+|\nabla(\varphi^1)_2|^k-\alpha\int_0^1(\varphi^1)_1(0,y)dy\\
&\le \frac{1}{k}\int_\Omega |\nabla u_{n1}|^k+|\nabla u_{n2}|^k-\alpha\int_0^1 u_{n1}(0,y)dy,
\end{aligned}$$

so

$$\beta=\frac{1}{k}\int_\Omega |\nabla(\varphi^1)_1|^k+|\nabla(\varphi^1)_2|^k-\alpha\int_0^1(\varphi^1)_1(0,y)dy.$$

On the other hand, for all $t>0$, $v=(v_1,v_2)\in V$, we put $w_t=(w_{thm1},w_{t2})$ where

$$\begin{gathered}
w_{thm1}=\frac{(\varphi^1)_1+tv_1}{(\int_\Omega m(x,y)(|(\varphi^1)_1+tv_1|^k+|(\varphi^1)_2+tv_2|^k))^{1/k}},\\
w_{t2}=\frac{(\varphi^1)_2+tv_2}{(\int_\Omega m(x,y)(|(\varphi^1)_1+tv_1|^k+|(\varphi^1)_2+tv_2|^k))^{1/k}},
\end{gathered}$$

so that $\int_\Omega m(x,y)(|w_{thm1}|^k+|w_{t2}|^k)=1$ and

$$\begin{aligned}
\beta &= \frac{1}{k}\int_\Omega |\nabla(\varphi^1)_1|^k+|\nabla(\varphi^1)_2|^k-\alpha\int_0^1(\varphi^1)_1(0,y)dy\\
&\le \frac{1}{k}\int_\Omega |\nabla w_{thm1}|^k+|\nabla w_{t2}|^k-\alpha\int_0^1 w_{thm1}(0,y)dy.
\end{aligned}$$

By developing to order 1 for $t\to 0$ and by applying the same reasoning to $(-v)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^2\int_\Omega |\nabla(\varphi^1)_i|^{k-2}\nabla(\varphi^1)_i\nabla v_i-\alpha\int_0^1 v_1(0,y)dy\\
&=(k\beta+(k-1)\alpha\int_0^1(\varphi^1)_1(0,y)dy)\times(\sum_{i=1}^2\int_\Omega m(x,y)|(\varphi^1)_i|^{k-2}(\varphi^1)_iv_i).
\end{aligned}$$

Now we suppose that $\alpha\alpha'>0$. We put $\overline{\varphi^1}=(\overline{\varphi^1}_1,\overline{\varphi^1}_2)$, where $\overline{\varphi^1}_i=\eta\varphi^1_i$ with $\eta^{k-1}=\frac{\alpha'}{\alpha}$. Then, by replacing in the equation $(P_1(\alpha))$, we obtain

$$\sum_{i=1}^2\int_\Omega |\nabla\overline{\varphi^1}_i|^{k-2}\nabla\overline{\varphi^1}_i\nabla v_i-\alpha'\int_0^1 v_1(0,y)dy=\lambda_1(\alpha)\sum_{i=1}^2\int_\Omega m(x,y)|\overline{\varphi^1}_i|^{k-2}\overline{\varphi^1}_iv_i,$$

which completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. (i) It is easy to prove that for $\alpha=0$, λ_1 is an eigenvalue of Problem (1.1) with $\alpha=0$ and $u\neq 0$ is a eigenfunction if and only if $\sum_{i=1}^2\int_\Omega|\nabla u_i|^k-\lambda_1(m)\int_\Omega m(x,y)(|u_1|^k+|u_2|^k)=0=\inf\{\sum_{i=1}^2\int_\Omega|\nabla v_i|^k-\lambda_1(m)\int_\Omega m(x,y)(|v_1|^k+|v_2|^k);\quad v\in V\}$. The proofs of (ii) and (iii) follow from (i). □

Proof of Theorem 2.3. It is clear that Problem (2.3) is equivalent to the weak formulation: Find $u \in V$ such that

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega |\nabla u_i|^{k-2} \nabla u_i \nabla v_i - \alpha \int_{-1}^{1} v_1(0,y)dy \\ &= \lambda \sum_{i=1}^{2} \int_\Omega m(x,y)|u_i|^{k-2} u_i v_i + \sum_{i=1}^{2} \int_\Omega f_i v_i \quad \forall v \in V. \end{aligned} \tag{3.5}$$

We consider the energy functional defined on V,

$$\Phi(u) = \frac{1}{k} \sum_{i=1}^{2} \int_\Omega |\nabla u_i|^k - \alpha \int_{-1}^{1} u_1(0,y)dy - \frac{\lambda}{k} \sum_{i=1}^{2} \int_\Omega m(x,y)|u_i|^k - \sum_{i=1}^{2} \int_\Omega f_i u_i. \tag{3.6}$$

We verify that u is a solution of Problem (3.5) if and only if u is a critical point of the function Φ. For the existence, it suffices to prove that there exists $u \in V$ such that

$$\Phi(u) = \inf_{v \in V} \Phi(v).$$

The functional Φ is continuous and convex, it suffices to show that Φ is coercive, indeed for all $u \in V$, using Theorem 2.2, we obtain

$$\lambda_1 \int_\Omega m(x,y)(|u_1|^k + |u_2|^k) \le \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k. \tag{3.7}$$

Since the function $u \mapsto (\int_\Omega |\nabla u_1|^k)^{1/k} + (\int_\Omega |\nabla u_2|^k)^{1/k} := \|u\|_V$ defines a norm in V, then we have successively

$$\begin{aligned} \sum_{i=1}^{2} \int_\Omega f_i u_i &\le \sum_{i=1}^{2} \|f_i\|_{L^{k'}} \|u_i\|_{L^k} \\ &\le c \sum_{i=1}^{2} \|\nabla u_i\|_{L^k} = c\|u\|_V, \end{aligned}$$

where $c > 0$.

$$\begin{aligned} \alpha \int_{-1}^{1} u_1(0,y)dy &\le |\alpha| \int_{\partial\Omega} |u_1| d\sigma \\ &\le |\alpha| c' \Big(\int_{\partial\Omega} |u_1|^k d\sigma \Big)^{1/k} \quad \text{(Holder's inequality)} \\ &\le |\alpha| c' \Big(\int_\Omega |\nabla u_1|^k \Big)^{1/k} \quad (V \hookrightarrow L^k(\partial\Omega) \text{ a continuous injection}) \\ &= c'' \|u\|_V, \end{aligned}$$

where c' and c'' are positive.

$$\begin{aligned} \frac{\lambda}{k} \sum_{i=1}^{2} \int_\Omega m(x,y)|u_i|^k &\le \frac{\tilde{\lambda}}{k} \sum_{i=1}^{2} \int_\Omega m(x,y)|u_i|^k \\ &\le \frac{\widetilde{\lambda}}{\lambda_1 k} \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k, \end{aligned}$$

where $\widetilde{\lambda} := \begin{cases} 0 & \text{if } \lambda < 0 \\ \lambda & \text{if } \lambda \geq 0. \end{cases}$ According to (3.6), we obtain

$$\Phi(u) \geq \frac{1}{k}(1 - \frac{\tilde{\lambda}}{\lambda_1}) \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k - c'''\|u\|_V,$$

where $c''' > 0$. Thus

$$\Phi(u) \geq \frac{1}{k}(1 - \frac{\tilde{\lambda}}{\lambda_1})\|u\|_V^k - c'''\|u\|_V,$$

where $c''' > 0$. Since $\lambda < \lambda_1$, we deduce that $\Phi(u) \to +\infty$ when $\|u\|_V \to +\infty$, so we have proved the existence. □

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