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EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION FOR A THIRD-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In this work we study a third-order three-point boundary-value problem (BVP). We derive sufficient conditions that guarantee the positivity of the solution of the corresponding linear BVP Then, based on the classical Guo-Krasnosel'skii's fixed point theorem, we obtain positive solutions to the nonlinear BVP. Additional hypotheses guarantee the uniqueness of the solution.

1. Introduction

In this article, we are concerned with a certain class of third-order differential equations, known as the three-point boundary-value problem (BVP), given by

$$\begin{gathered} u'''(t) = -f(t,u(t)), \quad 0 < t < 1, \\ u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0, \end{gathered} \tag{1.1}$$

where

$$q \geq \frac{1}{2\eta}(1-2\eta), \quad 0 < \eta < 1/2. \tag{1.2}$$

The problem (1.1) consists of a new set of boundary conditions but it is closely related with several boundary conditions.

Recently, Sun, Cho and O'Regan [15] proved the existence of positive solutions to the third-order boundary-value problem

$$\begin{gathered} z''' + q(t)f(t,z) = 0, \quad 0 < t < 1, \\ z(0) = z'(0) = z(1) = 0, \end{gathered}$$

mainly under a local (at $z = 0$) monotone condition and sublinearity (at $z = +\infty$) of the nonlinearity. In that paper, they constructed the corresponding Green function and then applied the Krasnosel'skiĭ's fixed point theorem.

Lately there have been several papers on third-order boundary value problems. Hopkins and Kosmatov [6], Infante and Webb [7], Li [9], Liu et al [10, 11], Guo et al [5] and Kang et al [12] have all considered third-order problems. Graef and

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Yang [4] and Wong [16] considered three-point focal problems. Also Henderson et al [2, 3] studied higher order boundary value problem.

Anderson et al [1] proved the existence of Green's function and found an explicit formula for it, associated with the homogeneous BVP:

$$
\begin{gathered}
x'''(t) = 0, \quad 0 \le t \le 1 \\
ax(0) - bx'(0) = 0, \quad \gamma x(\eta) - \delta x'(\eta) = 0, \quad x''(1) = 0.
\end{gathered}
$$

Finally, Palamides and Smyrlis [13] proved the existence of positive solutions for the general nonlinear boundary-value problem

$$
\begin{gathered}
x'''(t) = a(t)F(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\
x(0) = x'(\eta) = x''(1) = 0.
\end{gathered}
$$

In this article, mainly motivated by the above mentioned papers we consider a new set of boundary conditions and assume similar hypothesis as in [15]. Initially we construct the Green's function to the homogeneous BVP corresponding to (1.1) and then we derive the sufficient condition (1.2) which guarantees that the Green's function is positive. Then, based on the Guo-Krasnosel'skii's fixed point theorem, we obtain a positive solution to the nonlinear BVP (1.1), under superlinear (or sublinear) type growth-rate on the nonlinearity. Straightforwardly, we also conclude existence of a negative solution of the above BVP, under the hypothesis of negativity of the nonlinearity. Finally, we give conditions under which the existing solution is unique.

2. Constructing the Green's function

Consider first the homogeneous third-order boundary-value problem

$$
\begin{gathered}
u'''(t) = 0, \quad 0 < t < 1, \\
u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0.
\end{gathered} \tag{2.1}
$$

Lemma 2.1. *If $\eta \neq 1/(2(1+q))$, the boundary value problem* (1.1) *has the unique solution*

$$
u(t) = 0, \quad 0 \le t \le 1.
$$

Proof. The general solution of the BVP (2.1) has the form $u(t) = at^2 + bt + c$. The conditions at $t = \eta$ and $t = 1$ imply that $2a\eta + b = 0$ and $a + b + c = 0$. Moreover the condition at $t = 0$ yields $c - qb = 0$. Hence we immediately obtain the expected result. □

Consider now the inhomogeneous BVP

$$
\begin{gathered}
u'''(t) = -1, \quad 0 \le t \le 1, \\
u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0.
\end{gathered} \tag{2.2}
$$

Lemma 2.2. *Assume that $\eta \neq 1/(2(1+q)$. Then, the Green's function of the BVP* (2.2) *is given by: for $s < \eta$,*

$$
G(t,s) = \begin{cases} u_1^*(t,s), & 0 \le t \le s \\ v_1^*(t,s), & s \le t \le 1 \end{cases}
$$

and for $s > \eta$,

$$
G(t,s) = \begin{cases} u_2^*(t,s), & 0 \le t \le s \\ v_2^*(t,s), & s \le t \le 1, \end{cases}
$$

where

$$
\begin{aligned}
u_1^*(t,s) &= C_0\Big((1-2\eta-2q\eta+s^2+2qs)t^2-2(s-2s\eta+s^2\eta)t \\
&\quad -2q(s-2s\eta+s^2\eta)\Big) \\
v_1^*(t,s) &= C_0\Big((s^2+2qs)t^2-2(2qs\eta+s^2\eta)t-(s^2+2qs-4qs\eta-2s^2\eta)\Big), \\
u_2^*(t,s) &= C_0\Big((1+s^2-2s)t^2-2(\eta-2s\eta+s^2\eta)t-2q(\eta-2s\eta+s^2\eta)\Big), \\
v_2^*(t,s) &= C_0\Big((2\eta+2q\eta+s^2-2s)t^2-2(\eta-s+2qs\eta+s^2\eta)t \\
&\quad -2q\eta+s^2-4qs\eta-2s^2\eta\Big), \\
C_0 &= -\frac{1}{2(-1+2\eta+2q\eta)}.
\end{aligned}
$$

Proof. To obtain the solution of (2.2), we proceed by cases on the two branches of the solution, via the above Green's function $G(t,s)$:

If $t<\eta$,

$$
\begin{aligned}
u_1(t) = C_0\Big(&\int_0^t \Big((s^2+2qs)t^2-2(2qs\eta+s^2\eta)t-(s^2+2qs-4qs\eta-2s^2\eta)\Big)ds \\
&+\int_t^\eta \Big((1-2\eta-2q\eta+s^2+2qs)t^2-2(s-2s\eta+s^2\eta)t \\
&-2q(s-2s\eta+s^2\eta)\Big)ds \\
&+\int_\eta^1 \Big((1+s^2-2s)t^2-2(\eta-2s\eta+s^2\eta)t-2q(\eta-2s\eta+s^2\eta)\Big)ds\Big)
\end{aligned}
\tag{2.3}
$$

Hence an easy computation ensures that $u_1(0)-qu_1'(0)=0$ and $u_1'(\eta)=0$.

For $\eta\le t\le 1$,

$$
\begin{aligned}
u_2(t) = C_0\Big(&\int_0^\eta \Big((s^2+2qs)t^2-2(+2qs\eta+s^2\eta)t-(s^2+2qs-4qs\eta-2s^2\eta)\Big)ds \\
&+\int_\eta^t \Big((2\eta+2q\eta+s^2-2s)t^2-2(\eta-s+2qs\eta+s^2\eta)t \\
&-(2q\eta+s^2-4qs\eta-2s^2\eta)\Big)ds \\
&+\int_t^1 \Big((1+s^2-2s)t^2-2(\eta-2s\eta+s^2\eta)t-2q(\eta-2s\eta+s^2\eta)\Big)ds\Big)
\end{aligned}
$$

By another calculation, we may obtain that $u_2'(\eta)=0$ and $u_2(1)=0$. Furthermore,

$$
u_1'''(t)=-1 \quad\text{and}\quad u_1(t)=u_2(t)=u(t),\quad 0\le t\le 1.
$$

Hence the obtained function $u(t)$, $0\le t\le 1$, is a solution of (2.2). □

Lemma 2.3. *Assume hypothesis* (1.2). *Then the Green function is nonnegative.*

Proof. By condition (1.2), we have $C_0\le 0$. Then

$$
4s\eta-2s-2s^2\eta-t^2(2\eta-2s)=-2s(-2\eta+s\eta+1)-t^2(2\eta-2s)\le 0,
$$

since by the definition of $u_1^*(t,s)$, $-2\eta+1\geq 0$ and $s\leq\eta$. Consequently

$$\begin{aligned}
u_1^*(t,s) &= C_0\Big((4s\eta-2s-2s^2\eta-t^2(2\eta-2s))q \\
&\quad + (t^2(s^2-2\eta+1)-2t(s-2s\eta+s^2\eta))\Big) \\
&\geq C_0\Big((4s\eta-2s-2s^2\eta-t^2(2\eta-2s))(-\frac{1}{2\eta}(2\eta-1)) \\
&\quad + (t^2(s^2-2\eta+1)-2t(s-2s\eta+s^2\eta))\Big).
\end{aligned}$$

That is,

$$u_1^*(t,s)\geq C_0 s(t-1)(t-2\eta+1)\frac{-2\eta+s\eta+1}{\eta}\geq 0. \tag{2.4}$$

Similarly,

$$\begin{aligned}
u_2^*(t,s) &= C_0((4\eta s-2\eta s^2-2\eta)q+(t^2(s^2-2s+1)-2t(\eta s^2-2\eta s+\eta))) \\
&\geq C_0\Big((4\eta s-2\eta s^2-2\eta)(-\frac{1}{2\eta}(2\eta-1))+(t^2(s^2-2s+1) \\
&\quad -2t(\eta s^2-2\eta s+\eta))\Big)
\end{aligned}$$

Hence

$$u_2^*(t,s)\geq C_0(s-1)^2(t-1)(t-2\eta+1)\geq 0. \tag{2.5}$$

In the same way, we may verify that

$$\begin{gathered}
v_1^*(t,s)\geq C_0 s(t-1)(t-2\eta+1)\frac{-2\eta+s\eta+1}{\eta}\geq 0, \\
v_2^*(t,s)\geq C_0(s-1)^2(t-1)(t-2\eta+1)\geq 0.
\end{gathered} \tag{2.6}$$

□

For convenience, we set: For $s\leq\eta$,

$$\frac{\partial}{\partial t}G(t,s)=2C_0\times\begin{cases}(s^2+2qs+1-2\eta-2q\eta)t-(s-2s\eta+s^2\eta), & t\leq s\\ (s^2+2qs)t-(2q\eta s+s^2\eta), & s\leq t\leq 1\end{cases}$$

and for $s>\eta$,

$$\frac{\partial}{\partial t}G(t,s)=2C_0\times\begin{cases}(s^2-2s+1)t-(\eta s^2-2\eta s+\eta), & 0\leq t\leq s\\ (s^2-2s+2\eta+2q\eta)t-(\eta-s+2qs\eta+s^2\eta), & s\leq t\leq 1\end{cases}$$

Moreover, for $s<\eta$,

$$\frac{\partial^2}{\partial t^2}G(t,s)=2C_0\times\begin{cases}s^2+2qs+1-2\eta-2q\eta, & 0\leq t\leq s\\ s^2+2qs, & s\leq t\leq 1\end{cases}$$

and for $s>\eta$,

$$\frac{\partial^2}{\partial t^2}G(t,s)=2C_0\times\begin{cases}s^2-2s+1, & 0\leq t\leq s\\ s^2-2s+2\eta+2q\eta, & s\leq t\leq 1.\end{cases}$$

Consider the Banach space $C=C([0,1],\mathbb{R})$ of continuous maps, equipped with the standard norm

$$\|y\|=\max\{|y(t)|:0\leq t\leq 1\},$$

$0 < \theta \le \eta < 1/2$ and let

$$K_0 = \Big\{ y \in C : y(t) \ge 0,\ t \in [0,1],\ y''(t) \le 0,\ t \in [\theta, 1-\theta], \\ \max_{0 \le t \le 1} y(t) = y(\eta) \text{ and } y(1) = 0 \Big\}.$$

It is obvious that K_0 is a cone in C. We define furthermore the subcone

$$K = \big\{ y \in K_0 : \min_{t \in [\theta, 1-\theta]} y(t) \ge \theta \|y\| \big\}$$

Lemma 2.4. *For any $y \in K_0$,*

$$\min_{t \in [\theta, 1-\theta]} y(t) \ge \theta \|y\| = \theta y(\eta).$$

Proof. Since $y \in K_0$, $y(t) \ge 0$ for $0 \le t \le 1$ and moreover it is concave downward on the interval $[\theta, 1-\theta]$. Thus for any $t_1, t_2 \in [\theta, 1-\theta]$ and $\lambda \in [0,1]$,

$$y(\lambda t_1 + (1-\lambda)t_2) \ge \lambda y(t_1) + (1-\lambda) y(t_2).$$

Therefore,

$$y(t) \ge \|y\| \min_{t \in [\theta, 1-\theta]} \big\{ \frac{t}{\eta}, \frac{1-t}{1-\eta} \big\} \ge \|y\| \min_{t \in [\theta, 1-\theta]} \{t, 1-t\} \ge \theta \|y\|.$$

□

The next result is very useful.

Proposition 2.5. *Assume condition* (1.2) *holds and let $y : [0,1] \to [0, +\infty))$ be a continuous map. Then the BVP*

$$\begin{gathered} u'''(t) = -y(t), \quad 0 \le t \le 1, \\ u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0. \end{gathered} \tag{2.7}$$

admits the unique positive solution $u \in K$, where

$$u(t) = \int_0^1 G(t,s) y(s) ds.$$

Proof. We notice firstly that $u(t) \ge 0$, $0 \le t \le 1$. Indeed, this fact follows directly by the nonnegativity of the Green's function (see Lemma 2.3). On the other hand, for $0 \le t \le \eta$, we have

$$\begin{aligned} u'(t) &= \int_0^1 \frac{\partial}{\partial t} G(t,s) y(s) ds \\ &= \int_0^t 2C_0((s^2 + 2qs)t - (\eta s^2 + 2q\eta s)) y(s) ds \\ &\quad + \int_t^\eta 2C_0\Big((s^2 + 2qs + 1 - 2\eta - 2q\eta)t - (s - 2s\eta + s^2\eta)\Big) y(s) ds \\ &\quad + \int_\eta^1 2C_0\big((s^2 - 2s + 1)t - (\eta s^2 - 2\eta s + \eta)\big) y(s) ds. \end{aligned}$$

Consequently,

$$\begin{aligned}
u'(\eta) &= \int_0^\eta (2C_0((s^2+2qs)\eta - (\eta s^2 + 2q\eta s)))y(s)ds \\
&\quad + \int_\eta^1 (2C_0((s^2-2s+)\eta - (\eta s^2 - 2\eta s + \eta)))y(s)ds \\
&= \int_0^1 2C_0[(s^2+2qs)\eta - (\eta s^2 + 2q\eta s)]y(s)ds \\
&= \int_0^1 0y(s)ds = 0.
\end{aligned}$$

Similarly, we may prove that $u(0) - qu'(0) = 0$ and $u(1) = 0$. Furthermore,

$$\begin{aligned}
u''(t) &= \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s)y(s)ds \\
&= \int_0^t 2C_0(s^2+2qs)y(s)ds \\
&\quad + \int_t^\eta 2C_0(s^2+2qs+1-2\eta-2q\eta)y(s)ds \\
&\quad + \int_\eta^1 2C_0(s^2-2s+1)y(s)ds.
\end{aligned}$$

Hence, recalling that $C_0 = 1/2(1-2\eta-2q\eta)$,

$$u'''(t) = 2C_0(t^2+2qt)y(t) - 2C_0(t^2+2qt+1-2\eta-2q\eta)y(t) = -y(t).$$

Finally, by the nonnegativity of the solution $u(t)$ and the boundary conditions $u(0) - qu'(0) = 0$ and $u(1) = 0$, we may assume that $u''(0) \leq 0$. Otherwise, if $u''(0) > 0$, we get $u'(t) > 0$, in a right neighborhood of 0, due to the differential equation $u'''(t) = -y(t)$, $0 \leq t \leq 1$, and since $u'(\eta) = 0$). Hence there is a $\theta \in [0, \eta)$, such that $u''(\theta) = 0$. Thus in both the cases, we conclude that

$$u''(t) \leq 0, \quad \theta \leq t \leq 1-\theta.$$

Consequently, in view of Lemma 2.4, we obtain that $u \in K$. $\square$

Corollary 2.6. *Assume that hypotheses of Proposition 2.5 are satisfied. Consider the BVP*

$$\begin{gathered}
u'''(t) = y(t), \quad 0 \leq t \leq 1, \\
u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0.
\end{gathered} \tag{2.8}$$

Then, the map

$$u(t) = -\int_0^1 G(t,s)y(s)ds$$

is clearly a non-positive solution of (2.8).

3. Main Results

In this section we prove the existence of at least one positive solution of (1.1). We assume that

$$f \in C([0,1] \times [0,+\infty), [0,+\infty)) \tag{3.1}$$

In view of Proposition 2.5, we consider the positive solution $u_1(t)$ of (2.2) and set

$$A_0 = \max\{u_1(t) : 0 \le t \le 1\} = \max_{0\le t\le 1}\Big(\int_0^1 G(t,s)ds\Big),$$
$$B_0 = \max\{u_1(t) : \theta \le t \le 1-\theta\} = \max_{\theta\le t\le 1-\theta}\Big(\int_\theta^{1-\theta} G(t,s)ds\Big).$$

In view of Lemma 2.3, we get $A_0 \ge B_0 > 0$. We define the operator

$$\mathcal{T}u(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Obviously, BVP (1.1) has a solution $u = u(t)$, if and only if u is a fixed point of T. Moreover, recalling that the operator $\mathcal{T} : K \to C([0,1])$ is called *completely continuous*, if it is continuous and maps bounded sets into precompact sets we state the next well-known result [14].

Proposition 3.1. *Assume that* (1.2)-(3.1) *hold. Then $\mathcal{T} : K \to K$ is a completely continuous operator.*

Proof. It is sufficient to show that $\mathcal{T}(K) \subset K$. This is easily derived from Lemma 2.4 and Proposition 2.5, due to assumption (1.2) and the definition of the cone K. $\square$

We will employ the following fixed point theorem due to Krasnosel'skii [8].

Theorem 3.2. *Let E be a Banach space, $K \subseteq E$ be a cone and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Furthermore, suppose that $\mathcal{T} : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either :$\|\mathcal{T}u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|\mathcal{T}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; or $\|\mathcal{T}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|\mathcal{T}u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$ holds. Then $\mathcal{T}$ admits a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Now we are ready to formulate and prove our main result.

Theorem 3.3. *Assume that* (1.2)-(3.1) *hold and there exist positive constants $r \ne R$ such that*

$$|f(t,x)| \le \frac{r}{A_0}, \quad (t,x) \in [0,1] \times [0,r]; \tag{3.2}$$
$$|f(t,x)| \ge \frac{R}{B_0}, \quad (t,x) \in [0,1] \times [\theta R, R]. \tag{3.3}$$

Then the boundary value problem (1.1) *admits a positive solution $u = u(t)$, $0 \le t \le 1$, such that*

$$\min\{r,R\} \le \|u\| \le \max\{r,R\}.$$

Moreover, the obtained solution $u = u(t)$, $0 \le t \le 1$ is concave downward.

Proof. Assuming first that $r < R$, we consider the open balls

$$\Omega_1 = \{u \in C([0,1]) : \|u\| < r\}, \quad \Omega_2 = \{u \in C([0,1]) : \|u\| < R\}.$$

Let $u \in K \cap \partial\Omega_1$ be any function. By noticing the sign of nonlinearity, the assumption (3.2) yields

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\ &\le \max_{0\le t\le 1} \Big(\int_0^1 G(t,s)\frac{r}{A_0}ds\Big) = r = \|u\|.\end{aligned}$$

Therefore, the first part of the assumption of Theorem 3.3, is fulfilled. Similarly, for every $u \in K \cap \partial\Omega_2$, in view of Lemma 2.4, it obvious that $\theta R \le u(s) \le R$, $\theta \le s \le 1-\theta$. Thus the assumption (3.3) implies

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\ &\ge \max_{0\le t\le 1} \Big|\int_\theta^{1-\theta} G(t,s)f(s,u(s))ds\Big| \\ &\ge \max_{\theta\le t\le 1-\theta} \Big(\int_\theta^{1-\theta} G(t,s)\frac{R}{B_0}ds\Big) \\ &= R = \|u\|.\end{aligned}$$

Therefore, $\|\mathcal{T}u\| \ge \|u\|$, for $u \in K \cap \partial\Omega_2$.

Finally, we may apply Theorem 3.2, to obtain a solution $u = u(t)$, $0 \le t \le 1$ of BVP (1.1). Additionally by the definition of $K \subset K_0$ and the fact that $u \in K$, we conclude that $u(t)$ is a positive solution. Noticing that $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, it is obvious that

$$r \le \|u\| \le R.$$

We assume now that $r > R$. We consider the open balls

$$\Omega_1 = \{u \in C([0,1]) : \|u\| < R\}, \quad \Omega_2 = \{u \in C([0,1]) :\ \|u\| < r\}.$$

and let $u \in K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$\min_{t\in[\theta,1-\theta]} u(t) \ge \theta\|u\| = \theta R.$$

Then from assumption (3.3), we conclude that

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\ &\ge \max_{0\le t\le 1} \Big(\int_0^1 G(t,s)\frac{R}{B_0}ds\Big) \\ &\ge \max_{\theta\le t\le 1-\theta} \Big(\int_\theta^{1-\theta} G(t,s)\frac{R}{B_0}ds\Big) \\ &= R = \|u\|.\end{aligned}$$

Similarly, if $u \in K \cap \partial\Omega_2$, then $0 \le u(s) \le r$, $0 \le s \le 1$. Thus (3.2) implies

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\ &\le \max_{0\le t\le 1} \Big(\int_0^1 G(t,s)\frac{r}{A_0}ds\Big) \\ &= r = \|u\|.\end{aligned}$$

Therefore, the existence result follows. □

Corollary 3.4. *Assume* (1.2)-(3.1) *and in addition we suppose either: The nonlinearity is superlinear at both points $x = 0$ and $x = +\infty$; i.e.,*

$$\lim_{x\to 0+} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0+ \quad \textit{and} \quad \lim_{x\to+\infty} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty; \tag{3.4}$$

or the nonlinearity is sublinear at both points $x = 0$ and $x = +\infty$, i.e.,

$$\lim_{x\to 0+} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0+ . \tag{3.5}$$

Then boundary value problem (1.1) *admits a positive, concave downward solution $u = u(t)$, $0 \le t \le 1$.*

Proof. By the superlinearity of f, there exists an $r > 0$ such that $\frac{f(t,x)}{x} \le \frac{1}{A_0}$, for all $(t, x) \in [0, 1] \times [0, r]$ and this yields assumption (3.2) of previous Theorem 3.3. Similarly by the superlinearity at $+\infty$, we get an $R > r$ such that $\frac{f(t,x)}{x} \ge \frac{1}{\theta B_0}$, for all $(t, x) \in [0, 1] \times [\theta R, R]$. Hence Theorem 3.3 is applicable. On the other hand, when the nonlinearity is sublinear, we examine the following cases: (a) If f is bounded, say by $M > 0$, we may choose any $R \ge A_0 M$ and then we obtain

$$\begin{aligned} \|\mathcal{T}u\| &\le \max_{0\le t\le 1} \Big|\int_0^1 G(t,s) f(s,u(s))ds\Big| \\ &\le \max_{0\le t\le 1} \Big(\int_0^1 G(t,s) M ds\Big) \\ &= MA_0 \le R = \|u\|, \end{aligned} \tag{3.6}$$

for $u \in K$ with $\|u\| = R$.

(b) If f is unbounded, let also an R be large enough such that

$$\frac{|f(t,R)|}{R} \le \frac{1}{A_0} \quad \text{and} \quad |f(t,u)| \le |f(t,R)|, \quad (t,u) \in [0,1] \times [0,R].$$

Therefore,

$$|f(t,u)| \le |f(t,R)| \le \frac{R}{A_0}, \quad (t,u) \in [0,1] \times [0,R].$$

Consequently,

$$\begin{aligned} \|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s) f(s,u(s))ds\Big| \\ &\le \max_{0\le t\le 1} \Big(\int_0^1 G(t,s) \frac{R}{A_0} ds\Big) \\ &\le \frac{R}{A_0} A_0 = \|u\| \end{aligned}$$

for $u \in K$ with $\|u\| = R$. Moreover, by the sublinearity of f at $u = 0$, there exists an $r < R$ such that for any $u \in K$, $\|u\| = r$ (then we know that $r \ge u(s) \ge \theta\|u\| = \theta r$, $\theta \le s \le 1-\theta$)

$$|f(s,u(s))| \ge \frac{u(s)}{\theta B_0} \ge \frac{\theta r}{\theta B_0} = \frac{r}{B_0}, \quad (s,u(s)) \in [\theta, 1-\theta] \times [\theta r, r].$$

Hence, for any $u \in K$ such that $\|u\| = r$, we have

$$\begin{aligned}
\|\mathcal{T}u\| &= \max_{0\le t\le 1} |\int_0^1 G(t,s)f(s,u(s))ds| \\
&\ge \max_{\theta\le t\le 1-\theta} |\int_\theta^{1-\theta} G(t,s)f(s,u(s))ds| \\
&\ge \max_{\theta\le t\le 1-\theta} \Big(\int_\theta^{1-\theta} G(t,s)\frac{r}{B_0}ds\Big) \\
&= r \ge \|u\|.
\end{aligned}$$

This clearly completes the proof. □

Remark 3.5. We notice that the positive solution, $u = u(t)$, obtained above satisfies the properties

$$u'(0) > 0, \quad u''(0) \le 0, \quad u'(1) < 0, \quad u''(1) \le 0.$$

Furthermore the map $u''(t)$, $0 \le t \le 1$ is non-increasing.

Corollary 3.6. *Under the assumptions of Theorem 3.3 or Corollary 3.4, the BVP*

$$\begin{gathered}
u'''(t) = f(t,u(t)), \quad 0 < t < 1, \\
u(0) - qu'(0) = 0, \quad u'(\eta) = 0, \quad u(1) = 0
\end{gathered} \tag{3.7}$$

admits a negative and concave upward solution $u = u(t)$. Here again the map $u''(t)$, $0 \le t \le 1$ is non-increasing.

Proof. Now the above negative solution satisfies

$$u'(0) < 0, \quad u''(0) > 0, \quad u'(1) > 0, \quad u''(1) \ge 0.$$

□

Corollary 3.7. *Under the assumptions of Theorem 3.3 or Corollary 3.4, BVP* (3.7) *admits a negative, concave upward solution $u = u(t)$, $0 \le t \le 1$.*

Proof. Let $u = u_1(t)$, $0 \le t \le 1$ be a solution of BVP (1.1). Then the function

$$u = -u_1(t), \quad 0 \le t \le 1$$

is obviously the desired solution of (3.7). We notice that

$$u'(0) < 0, \quad u''(0) > 0, \quad u'(1) > 0, \quad u''(1) \ge 0.$$

Moreover, the map $u''(t)$, $0 \le t \le 1$ is nondecreasing. □

Corollary 3.8. *Under the assumptions of Theorem 3.3 or Corollary 3.4, BVP* (3.7) *admits a positive and concave downward solution $u = u(t)$, $0 \le t \le 1$.*

Proof. Obviously the desired solution is given by

$$u(t) = \int_0^1 [-G(t,s)]f(s,u(s))ds.$$

Here also the map $u''(t)$, $0 \le t \le 1$ is nondecreasing. □

Example 3.9. Consider the boundary value problem

$$u'''(t) = -\sqrt[3]{u(t) + (u(t))^2}, \quad 0 < t \le 1$$
$$u(0) = u'(0), \quad u'(3/10) = u(1) = 0.$$

The nonlinearity $f(t,u) = \sqrt[3]{u+t}$ is sublinear. Thus, Corollary 3.4 guarantees the existence of a positive and concave downwards solution to the above BVP.

4. Uniqueness of solution

Theorem 4.1. *Under the assumptions of Theorem 3.3 or Corollary 3.4, BVP* (1.1) *admits a unique solution, provided that the map* $f(t,.) : [0,+\infty) \to [0,+\infty)$ *is nondecreasing for every* $t \in [0,1]$ *and moreover*

$$|f(t,u_2) - f(t,u_1)| \le L|u_2 - u_1|, \quad (t,u_i) \in [0,1] \times [0,+\infty),$$

where

$$\frac{1}{L} > \frac{1}{6}\eta(\eta-1)^2 \frac{2q+\eta+q\eta}{2\eta+2q\eta-1}.$$

Proof. Let $w_i(t)$, $i = 1,2$ be two solutions of BVP ((1.1), Since the Green's function $G(t,s)$ is positive, we have

$$\begin{aligned}
w_2(t) - w_1(t) &= \int_0^1 G(t,s)[f(s,w_2(s)) - f(s,w_1(s))]ds \\
&\le \int_0^1 G(t,s)|f(s,w_2(s)) - f(s,w_1(s))|ds \\
&\le \int_0^1 G(t,s)L|w_2(s) - w_1(s)|ds \\
&\le \int_0^1 G(t,s)L\|w_2 - w_1\|ds \\
&= L\|w_2 - w_1\| \int_0^1 G(t,s)ds \\
&= L\|w_2 - w_1\|u_1(t), \quad 0 \le t \le 1,
\end{aligned}$$

where $u_1(\ t)$, is the unique positive solution of BVP (2.2). Consequently, we obtain the contradiction

$$\begin{aligned}
\|w_2 - w_1\| &\le L\|w_2 - w_1\|\|u_1\| \le L\|w_2 - w_1\|u_1(\eta) \\
&= L\|w_2 - w_1\|\frac{1}{6}\eta(\eta-1)^2\frac{2q+\eta+q\eta}{2\eta+2q\eta-1} \\
&< \|w_2 - w_1\|.
\end{aligned}$$

□

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