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REGULARITY FOR 3D NAVIER-STOKES EQUATIONS IN TERMS OF TWO COMPONENTS OF THE VORTICITY

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Abstract. We establish regularity conditions for the 3D Navier-Stokes equation via two components of the vorticity vector. It is known that if a Leray-Hopf weak solution u satisfies

$$\tilde{\omega} \in L^{2/(2-r)}(0,T;L^{3/r}(\mathbb{R}^3)) \quad \text{with } 0<r<1,$$

where $\tilde{\omega}$ form the two components of the vorticity, $\omega = \operatorname{curl} u$, then u becomes the classical solution on $(0,T]$ (see [5]). We prove the regularity of Leray-Hopf weak solution u under each of the following two (weaker) conditions:

$$\tilde{\omega} \in L^{2/(2-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)) \quad \text{for } 0<r<1,$$
$$\nabla\tilde{u} \in L^{2/(2-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)) \quad \text{for } 0\le r<1,$$

where $\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ is the Morrey-Campanato space. Since $L^{3/r}(\mathbb{R}^3)$ is a proper subspace of $\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$, our regularity criterion improves the results in Chae-Choe [5].

1. Introduction

We consider the Navier-Stokes equations, in $\mathbb{R}^3$,

$$\begin{gathered}
\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = 0, \quad (x,t)\in\mathbb{R}^3\times(0,T),\\
\operatorname{div} u = 0, \quad (x,t)\in\mathbb{R}^3\times(0,T),\\
u(x,0)=u_0(x), \quad x\in\mathbb{R}^3,
\end{gathered} \tag{1.1}$$

where $u=u(x,t)$ is the velocity field, $p=p(x,t)$ is the scalar pressure and $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included in the pressure gradient.

In their well known articles, Leray [15] and Hopf [10] independently constructed a weak solution u of (1.1) for arbitrary $u_0\in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. The solution is called the Leray-Hopf weak solution. Regularity of such Leray-Hopf weak solutions is one of the most significant open problems in mathematical fluid mechanics. Here we mean by the weak solution a function $u\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ satisfying (1.1) in sense of distributions.

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A weak solution of the Navier-Stokes equation that belongs to $L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3))$ is called a strong solution. Introducing the class $L^\alpha(0,T;L^q(\mathbb{R}^3))$, it is shown that if a Leray-Hopf weak solution u belongs to $L^\alpha((0,T);L^q(\mathbb{R}^3))$ with the exponents α and q satisfying $\frac{2}{\alpha}+\frac{3}{q}\le 1$, $2\le\alpha<\infty$, $3<q\le\infty$, then $u(x,t)\in C^\infty(\mathbb{R}^3\times(0,T))$ [20, 18, 19, 8, 21, 22, 9], while the limit case $\alpha=\infty$, $q=3$ was covered much later Escauriaza, Seregin and Sverak in [7]. See also [12] for recent improvements of the criteria, using the negative order Triebel-Lizorkin spaces.

On the other hand, Beirão da Veiga [1] obtained a sufficient condition for regularity using the vorticity rather than velocity. His result says that if the vorticity $\omega=$curl u of a weak solution u belongs to the space $L^\alpha(0,T;L^q(\mathbb{R}^3))$ with $\frac{2}{\alpha}+\frac{3}{q}\le 2$ and $1\le\alpha<\infty$, then u becomes the strong solution on $(0,T]$. Later, Chae-Choe [5] obtained an improved regularity criterion of [1] imposing condition on only two components of the vorticity, namely if

$$\tilde\omega=(\omega_1,\omega_2,0)\in L^\alpha(0,T;L^q(\mathbb{R}^3))\quad\text{with } \frac{2}{\alpha}+\frac{3}{q}\le 2,\ 1\le\alpha<\infty, \tag{1.2}$$

then the weak solution becomes smooth.

The purpose of this article is to prove the result of [5] in the other cases, proving that if $\tilde\omega\in L^{2/(2-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$ with $0<r<1$, then the weak solution becomes smooth. Here $\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ is the Morrey-Campanato space, which is strictly bigger than $L^{3/r}(\mathbb{R}^3)$ (see the next section for the related embedding relations). We remark that in the limiting case $r=0$, Kozono-Yatsu [13] previously weakened the condition $\tilde\omega\in L^1(0,T;L^\infty(\mathbb{R}^3))$ of [5] into $\tilde\omega\in L^1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^3))$, where $\dot B^0_{\infty,\infty}$ is the Besov space.

2. Preliminaries and the main theorems

Now, we recall the definition and some properties of the space that we are going to use. These spaces play an important role in studying the regularity of solutions to partial differential equations (see e.g. [11, 23]).

Definition 2.1. For $1<p\le q\le+\infty$, the Morrey-Campanato space $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is defined as

$$\begin{aligned}&\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)\\&=\big\{f\in L^p_{\rm loc}(\mathbb{R}^3):\|f\|_{\dot{\mathcal{M}}_{p,q}}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}R^{3/q-3/p}\|f(y)1_{B(x,R)}(y)\|_{L^p(dy)}<\infty\big\}\end{aligned}$$

It is easy to check that

$$\begin{gathered}\|f(\lambda\cdot)\|_{\dot{\mathcal{M}}_{p,q}}=\frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}},\quad\lambda>0,\\ \dot{\mathcal{M}}_{p,\infty}(\mathbb{R}^3)=L^\infty(\mathbb{R}^3)\quad\text{for } 1\le p\le\infty.\end{gathered}$$

Additionally, for $2\le p\le 3/r$ and $0\le r<3/2$ we have the following embedding relations:

$$L^{3/r}(\mathbb{R}^3)\hookrightarrow L^{3/r,\infty}(\mathbb{R}^3)\hookrightarrow\dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3),$$

where $L^{p,\infty}$ denotes the weak L^p-space. The second relation

$$L^{3/r,\infty}(\mathbb{R}^3)\hookrightarrow\dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^3)$$

is shown as follows.

$$
\begin{aligned}
\|f\|_{\dot{\mathcal{M}}_{p,3/r}} &\le \sup_E |E|^{\frac{r}{3}-\frac{1}{p}}\Big(\int_E |f(y)|^p dy\Big)^{1/p} \quad (f \in L^{3/r,\infty}(\mathbb{R}^3))\\
&= \Big(\sup_E |E|^{\frac{pr}{3}-1}\int_E |f(y)|^p dy\Big)^{1/p}\\
&\cong \Big(\sup_{R>0} R|\{x \in \mathbb{R}^3 : |f(y)|^p > R\}|^{pr/3}\Big)^{1/p}\\
&= \sup_{R>0} R|\{x \in \mathbb{R}^p : |f(y)| > R\}|^{r/3}\\
&\cong \|f\|_{L^{3/r,\infty}}.
\end{aligned}
$$

Remark 2.2. For the case $q = 3/2$ in (1.2), we can show that there exists an absolute constant δ such that if the weak solution u of (1.1) on $(0,T)$ with energy inequality satisfies

$$
\sup_{0<t<T} \|\tilde{\omega}(t)\|_{L^{3/2,\infty}} \le \delta,
$$

then u is actually regular (see [13, p. 60], [3]). As another type of criterion, Neustupa-Novotny-Penel [17] considered suitable weak solution $u = (u_1, u_2, u_3)$ introduced by Caffarelli-Kohn-Nirenberg [4] and showed regularity of u under the hypothesis that $u_3 \in L^\alpha(0,T; L^q(\mathbb{R}^3))$ with $\frac{2}{\alpha} + \frac{3}{q} = \frac{1}{2}$ (see [6]). The corresponding result for u to the above case was obtained by Berselli [2] who proved regularity under the assumption that $\sup_{0<t<T} \|\widetilde{u}(t)\|_{L^{3,\infty}}$ is sufficiently small, where $\widetilde{u} = (u_1, u_2, 0)$.

We need the following lemma which is essentially due to Lemarié -Rieusset [14].

Lemma 2.3. *For $0 \le r < 3/2$, the space $\dot{Z}_r(\mathbb{R}^3)$ is defined as the space of $f(x) \in L^2_{\rm loc}(\mathbb{R}^3)$ such that*

$$
\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \le 1} \|fg\|_{L^2} < \infty.
$$

Then $f \in \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ if and only if $f \in \dot{Z}_r(\mathbb{R}^3)$ with equivalence of norms.

Since $L^{3/r}(\mathbb{R}^3) \subsetneq \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$, the above regularity criterion is an improvement on Chae-Choe' s result and hence our regularity criterion covers the recent result given by Chae-Choe [5]. Our result on (1.1) reads as follows.

Theorem 2.4. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla.u_0 = 0$ and $\omega_0 =$curl $u_0 \in L^2(\mathbb{R}^3)$. If the first two components of the vorticity $\tilde{\omega} = \omega_1 e_1 + \omega_2 e_2$ of the Leray-Hopf weak solution u, satisfies $\tilde{\omega} \in L^{2/(2-r)}(0,T,\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$ with $0 < r < 1$, then u becomes the classical solution on $(0,T]$.*

Remark 2.5. As an immediate consequence of the above theorem, we find that if the classical solution of the Navier-Stokes equations blow-up at time T, then

$$
\|\tilde{\omega}\|_{L^{2/(2-r)}(0,T,\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))} = \infty,
$$

where $\tilde{\omega}$ is any two component vector of ω.

Our second theorem concerns the regularity criterion in terms of gradients of the components of velocity.

Theorem 2.6. *Let $\tilde{u} = u_1e_1 + u_2e_2$ be the first two components of a Leray-Hopf weak solution of the Navier-Stokes equation corresponding to $u_0 \in H^1(\mathbb{R}^3)$ with* $\operatorname{div} u_0 = 0$. *Suppose that $\nabla\tilde{u} \in L^{2/(2-r)}(0,T,\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$ with $0 \le r < 1$, then u becomes the classical solution on $(0,T]$.*

3. Proof of Theorem 2.4

Now we are in a position to prove our main result.

Proof. Taking the curl on (1.1), we obtain

$$\partial_t\omega - \Delta\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0. \tag{3.1}$$

Multiplying (3.1) by ω in $L^2(\mathbb{R}^3)$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(t,\cdot)\|_{L^2}^2 + \|\nabla\omega(t,\cdot)\|_{L^2}^2 = \langle\omega\cdot\nabla u,\omega\rangle. \tag{3.2}$$

Here we have used the identity

$$\langle u\cdot\nabla\omega,\omega\rangle = 0.$$

Using the Biot-Savart law, u is written in terms of ω:

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times\omega(x,t)}{|x-y|^3}dy.$$

Substituting this into the right hand side of (3.2), we obtain

$$\langle\omega\cdot\nabla u,\omega\rangle = \frac{3}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y}{|y|}\cdot\omega(x,t)\big\{\frac{y}{|y|^4}\times\omega(x+y,t)\cdot\omega(x,t)\big\}\,dy\,dx.$$

We decompose ω for the vorticities in $\{\cdot\}$ as follows

$$\omega = \tilde{\omega} + \omega', \quad \tilde{\omega} = \omega_1e_1 + \omega_2e_2, \quad \omega' = \omega_3e_3.$$

Since $\omega' = (0,0,\omega_3)$, there holds

$$\frac{3}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y}{|y|}\cdot\omega(x,t)\big\{\frac{y}{|y|^4}\times\omega'(x+y,t)\cdot\omega'(x,t)\big\}\,dy\,dx = 0$$

for all $0 < t < T$. Then, it follows

$$\begin{aligned}\langle\omega\cdot\nabla u,\omega\rangle &= \frac{3}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y}{|y|}\cdot\omega(x,t)\{\frac{y}{|y|^4}\times\tilde{\omega}(x+y,t)\cdot\omega'(x,t)\}\,dy\,dx\\ &\quad+\frac{3}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y}{|y|}\cdot\omega(x,t)\{\frac{y}{|y|^4}\times\tilde{\omega}(x+y,t)\cdot\tilde{\omega}(x,t)\}\,dy\,dx\\ &\quad+\frac{3}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{y}{|y|}\cdot\omega(x,t)\{\frac{y}{|y|^4}\times\omega'(x+y,t)\cdot\tilde{\omega}(x,t)\}\,dy\,dx,\end{aligned} \tag{3.3}$$

where all the integrations with respect to y are in the sense of principal value. Using the following interpolation inequality [16]:

$$\|w\|_{\dot{B}^r_{2,1}} \le C\|w\|_{L^2}^{1-r}\|\nabla w\|_{L^2}^r, \quad r\in(0,1)$$

it is easy to see that by Lemma 2.3

$$
\begin{aligned}
&|\langle \omega\cdot\nabla u,\omega\rangle| \\
&\le C\int_{\mathbb{R}^3}|\omega(x,t)||P(\tilde\omega)||\omega'(x,t)|dx \\
&\quad + C\int_{\mathbb{R}^3}|\omega(x,t)||P(\tilde\omega)||\tilde\omega(x,t)|dx \\
&\quad + C\int_{\mathbb{R}^3}|\omega(x,t)||P(\omega')||\tilde\omega(x,t)|dx \\
&\le C\int_{\mathbb{R}^3}|\omega|^2|P(\tilde\omega)|dx + C\int_{\mathbb{R}^3}|\omega||P(\omega')||\tilde\omega|dx \\
&\le C\|\omega\|_{L^2}\|\omega\cdot P(\tilde\omega)\|_{L^2} + C\|\tilde\omega\cdot P(\omega')\|_{L^2}\|\omega\|_{L^2} \\
&\le C\|\omega\|_{L^2}\|\omega\|_{\dot B^r_{2,1}}\|P(\tilde\omega)\|_{\dot{\mathcal{M}}_{2,3/r}} + C\|\omega\|_{L^2}\|P(\omega')\|_{\dot B^r_{2,1}}\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}} \\
&\le C\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}}\|\omega\|_{L^2}\|\omega\|_{\dot B^r_{2,1}} + C\|\omega\|_{L^2}\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}}\|\omega'\|_{\dot B^r_{2,1}} \\
&\le C\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}}\|\omega\|_{L^2}^{2-r}\|\nabla\omega\|_{L^2}^r
\end{aligned}
$$

where $P(.)$ denotes the singular integral operator defined by the integrals with respect to y in (3.3).

By Young' s inequality, we find

$$
|\langle \omega\cdot\nabla u,\omega\rangle| \le C\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\omega\|_{L^2}^2 + \frac{1}{2}\|\nabla\omega\|_{L^2}^2. \tag{3.4}
$$

Substituting (3.4) in (3.2), we have

$$
\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2 + \|\nabla\omega(\cdot,t)\|_{L^2}^2 \le C\|\tilde\omega\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\omega\|_{L^2}^2. \tag{3.5}
$$

By Gronwall' s inequality we have that

$$
\|\omega(\cdot,t)\|_{L^2} \le \|\omega(0,\cdot)\|_{L^2}\exp\Big(C\int_0^T\|\tilde\omega(\cdot,\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{2/(2-r)}d\tau\Big). \tag{3.6}
$$

This implies

$$
\omega \in L^\infty([0,T);L^2(\mathbb{R}^3)) \cap L^2([0,T);H^1(\mathbb{R}^3))
$$

provided that $\tilde\omega$ satisfies the condition $\tilde\omega \in L^{\frac{2}{2-r}}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$. This proves Theorem 2.4. $\square$

4. Proof of Theorem 2.6

Now we are in a position to prove our second result.

Proof. We set $\tilde u = (u_1,u_2,0)$. Then, taking the first two components of the vorticity equation (3.1), we obtain

$$
\partial_t\tilde\omega - \Delta\tilde\omega + (u\cdot\nabla)\tilde\omega - (\omega\cdot\nabla)\tilde u = 0.
$$

Multiplying (3.1) by $\tilde\omega$ in $L^2(\mathbb{R}^3)$ and integrating by parts, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\tilde\omega(t,\cdot)\|_{L^2}^2 + \|\nabla\tilde\omega(t,\cdot)\|_{L^2}^2 = \langle \omega\cdot\nabla\tilde u,\tilde\omega\rangle. \tag{4.1}
$$

We first consider the case $0 < r < 1$. Using the Hölder inequality and Lemma 2.3, we estimate

$$\begin{aligned} |\langle \omega \cdot \nabla \tilde{u}, \tilde{\omega} \rangle| &= |\langle (\tilde{\omega} + \omega') \cdot \nabla \tilde{u}, \tilde{\omega} \rangle| \le \|\tilde{\omega}\|_{L^2} \|\tilde{\omega} \cdot \nabla \tilde{u}\|_{L^2} \\ &\le C \|\nabla \tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}} \|\tilde{\omega}\|_{L^2} \|\tilde{\omega}\|_{\dot{B}^r_{2,1}} \\ &\le C \|\nabla \tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}} \|\tilde{\omega}\|_{L^2} \|\tilde{\omega}\|_{L^2}^{1-r} \|\nabla \tilde{\omega}\|_{L^2}^r \\ &= C (\|\nabla \tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)} \|\tilde{\omega}\|_{L^2}^2)^{\frac{2-r}{2}} \|\nabla \tilde{\omega}\|_{L^2}^r \\ &\le C \|\nabla \tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)} \|\tilde{\omega}\|_{L^2}^2 + \frac{C}{2} \|\nabla \tilde{\omega}\|_{L^2}^2, \end{aligned} \tag{4.2}$$

where we used

$$\langle \omega' \cdot \nabla \tilde{u}, \tilde{\omega} \rangle = 0.$$

Estimates (4.2) combined with (4.1), yield

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\omega}(t, \cdot)\|_{L^2}^2 + \|\nabla \tilde{\omega}(t, \cdot)\|_{L^2}^2 \le C \|\nabla \tilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)} \|\tilde{\omega}\|_{L^2}^2.$$

By Gronwall' s inequality we have

$$\|\tilde{\omega}(t, \cdot)\|_{L^2} \le \|\tilde{\omega}(0, \cdot)\|_{L^2} \exp\Big(C \int_0^T \|\nabla \tilde{u}(\cdot, \tau)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)} d\tau \Big).$$

Next we consider the case $r = 0$. In this case we estimate

$$|\langle \omega \cdot \nabla \tilde{u}, \tilde{\omega} \rangle| \le \|\tilde{\omega}\|_{L^2} \|\tilde{\omega} \cdot \nabla \tilde{u}\|_{L^2} \le \|\tilde{\omega}\|_{L^2}^2 \|\nabla \tilde{u}\|_{L^\infty}. \tag{4.3}$$

This estimate combined with (4.1), yield

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\omega}(t, \cdot)\|_{L^2}^2 + \|\nabla \tilde{\omega}(t, \cdot)\|_{L^2}^2 \le C \|\nabla \tilde{u}\|_{L^\infty} \|\tilde{\omega}\|_{L^2}^2.$$

By Gronwall' s inequality we have

$$\|\tilde{\omega}(t, \cdot)\|_{L^2} \le \|\tilde{\omega}(0, \cdot)\|_{L^2} \exp\Big(C \int_0^T \|\nabla \tilde{u}\|_{L^\infty} d\tau \Big).$$

This completes the proof of Theorem 2.6. □

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