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ENTIRE SOLUTIONS FOR A CLASS OF p-LAPLACE EQUATIONS IN $\mathbb{R}^2$

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Abstract. We study the entire solutions of the p-Laplace equation
$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+a(x,y)W'(u(x,y))=0,\quad (x,y)\in\mathbb{R}^2$$
where $a(x,y)$ is a periodic in x and y, positive function. Here $W:\mathbb{R}\to\mathbb{R}$ is a two well potential. Via variational methods, we show that there is layered solution which is heteroclinic in x and periodic in y direction.

1. Introduction

In this paper we consider the p-Laplacian Allen-Cahn equation
$$\begin{gathered}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+a(x,y)W'(u(x,y))=0,\quad (x,y)\in\mathbb{R}^2\\ \lim_{x\to\pm\infty}u(x,y)=\pm\sigma\quad \text{uniformly w.r.t. } y\in\mathbb{R}.\end{gathered}\tag{1.1}$$
where we assume $2<p<\infty$ and

- (H1) $a(x,y)$ is Hölder continuous on $\mathbb{R}^2$, positive and
  - (i) $a(x+1,y)=a(x,y)=a(x,y+1)$.
  - (ii) $a(x,y)=a(x,-y)$.
- (H2) $W\in C^2(\mathbb{R})$ satisfies
  - (i) $0=W(\pm\sigma)<W(s)$ for any $s\in\mathbb{R}\setminus\{\pm\sigma\}$, and $W(s)=O(|s\mp\sigma|^p)$ as $s\to\pm\sigma$;
  - (ii) there exists $R_0>\sigma$ such that $W(s)>W(R_0)$ for any $|s|>R_0$.

For example, here we may take $W(t)=\frac{p-1}{p}|\sigma^2-t^2|^p$. This is similar with case $p=2$, where the typical examples of W are given by $W(t)=\frac{1}{4}\prod_{i=1}^{k}(t-z_i)^2$, where z_i, $i=1,2,\dots k<\infty$ are zeros of $W(t)$. The case $p=2$ can be viewed as stationary Allen-Cahn equation introduced in 1979 by Allen and Cahn. We recall that the Allen-Cahn equation is a model for phase transitions in binary metallic alloys which corresponds to taking a constant function a and the double well potential $W(t)$. The function u in these models is considered as an order parameter describing pointwise the state of the material. The global minima of W represent energetically favorite pure phases and different values of u depict mixed configurations.

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In 1978, De Giorgi [11] formulated the following question. Assume $N > 1$ and consider a solution $u \in C^2(\mathbb{R}^N)$ of the scalar Ginzburg-Laudau equation:

$$\Delta u = u(u^2 - 1) \tag{1.2}$$

satisfying $|u(x)| \leq 1$, $\frac{\partial u}{\partial x_N} > 0$ for every $x = (x', x_N) \in \mathbb{R}^N$ and $\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1$. Then the level sets of $u(x)$ must be hyperplanes; i.e., there exists $g \in C^2(\mathbb{R})$ such that $u(x) = g(ax' - x_n)$ for some fixed $a \in \mathbb{R}^{N-1}$. This conjecture was first proved for $N = 2$ by Ghoussoub and Gui in [13] and for $N = 3$ by Ambrosio and Cabré in [5]. For $4 \leq N \leq 8$ and assuming an additional limiting condition on u, the conjecture has been proved by Savin in [25] .

Alessio, Jeanjean and Montecchiari [2] studied the equation $-\triangle u + a(x)W'(u) = 0$ and obtained the existence of layered solutions based on the crucial condition that there is some discrete structure of the solutions to the corresponding ODE.

In [3], when $a(x, y) > 0$ is periodic in x and y, the authors got the existence of infinite multibump type solutions, where $a(x, y) = a(x, -y)$ takes an important role [3](see also [3, 20, 21, 22, 23, 24]).

Inherited from the above results, I wonder under what condition p-Laplace type equation (1.1) would have two dimensional layered solutions periodical in y. Adapting the renormalized variational introduced in [2, 3] (see also [21, 22]) to the p-Laplace case, we prove

Theorem 1.1. *Assume* (H1)–(H2). *Then there exists entire solution for* (1.1), *which behaves heteroclinic in x and periodic in y direction.*

2. The periodic problem

To prove Theorem 1.1, we first consider the equation

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a(x,y)W'(u(x,y)) = 0, \quad (x,y) \in \mathbb{R}^2 \\ u(x,y) = u(x, y+1) \\ \lim_{x \to \pm\infty} u(x,y) = \pm\sigma \quad \text{uniformly w.r.t. } y \in \mathbb{R}. \end{gathered} \tag{2.1}$$

The main feature of this problem is that it has mixed boundary conditions, requiring the solution to be periodic in the y variable and of the heteroclinic type in the x variable.

Letting $S_0 = \mathbb{R} \times [0, 1]$, we look for minima of the Euler-Lagrange functional

$$I(u) = \int_{S_0} \frac{1}{p}|\nabla u(x,y)|^p + a(x,y)W(u(x,y))\,dx\,dy$$

on the class

$$\Gamma = \{u \in W^{1,p}_{\rm loc}(S_0) : \|u(x,\cdot) \mp \sigma\|_{L^p(0,1)} \to 0.\ x \to \pm\infty\}$$

where $\|u(x_1,\cdot) - u(x_2,\cdot)\|^p_{L^p(0,1)} = \int_0^1 |u(x_1,y) - u(x_2,y)|^p dy$. Setting

$$\Gamma_p = \{u \in \Gamma : u(x,0) = u(x,1) \text{for a.e. } x \in \mathbb{R}\}$$

$$c_p = \inf_{\Gamma_p} I \quad \text{and} \quad \mathcal{K}_p = \{u \in \Gamma_p : I(u) = c_p\}$$

Then we use the reversibility assumption (H1)-(ii) to show that the minima c on Γ equals minima c_p on Γ_p, and so solutions of (2.1).

Note the assumptions on a and W are sufficient to prove that I is lower semicontinuous with respect to the weak convergence in $W^{1,p}_{\rm loc}(S_0)$; i.e., if $u_n \to u$ weakly in $W^{1,p}_{\rm loc}(\Omega)$ for any Ω relatively compact in S_0, then $I(u) \le \liminf_{n\to\infty} I(u_n)$. Moreover we have

Lemma 2.1. *If $(u_n) \subset W^{1,p}_{\rm loc}(S_0)$ is such that $u_n \to u$ weakly in $W^{1,p}_{\rm loc}(S_0)$ and $I(u_n) \to I(u)$, then $I(u) \le \liminf_{n\to\infty} u_n$ and*

$$\int_{S_0} a(x,y)W(u_n)\,dx\,dy \to \int_{S_0} a(x,y)W(u)\,dx\,dy$$
$$\int_{S_0} |\nabla u_n|^p\,dx\,dy \to \int_{S_0} |\nabla u|^p\,dx\,dy$$

Proof. Since $u_n \to u$ weakly in $W^{1,p}_{\rm loc}(S_0)$, $\|\nabla u\|_{L^p(S_0)} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^p(S_0)}$ by the lower semicontinuous of the norm. By compact embedding theorem, we have $u_n \to u$ in $L^p_{\rm loc}(S_0)$, using pointwise convergence and Fatou lemma, we have $\int_{S_0} a(x,y)W(u)\,dx\,dy \le \liminf_{n\to\infty} \int_{S_0} a(x,y)W(u_n)\,dx\,dy$, then

$$\begin{aligned}
\int_{S_0} a(x,y)W(u)\,dx\,dy &\le \limsup_{n\to\infty} \int_{S_0} a(x,y)W(u_n)\,dx\,dy\\
&= \limsup_{n\to\infty} \Big[I(u_n) - \int_{S_0} \frac{1}{p}|\nabla u_n|^p\,dx\,dy\Big]\\
&= I(u) - \liminf_{n\to\infty} \int_{S_0} \frac{1}{p}|\nabla u_n|^p\,dx\,dy\\
&\le \int_{S_0} a(x,y)W(u)\,dx\,dy.
\end{aligned}$$

Thus, $\int_{S_0} a(x,y)W(u_n)\,dx\,dy \to \int_{S_0} a(x,y)W(u)\,dx\,dy$, and since $I(u_n) \to I(u)$, we have $\int_{S_0} |\nabla u_n|^p\,dx\,dy \to \int_{S_0} |\nabla u|^p\,dx\,dy$. $\square$

By Fubini's Theorem, if $u \in W^{1,p}_{\rm loc}(S_0)$, then $u(x,\cdot) \in W^{1,p}(0,1)$, and for all $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\int_0^1 |u(x_1,y) - u(x_2,y)|^p dy &= \int_0^1 |\int_{x_1}^{x_2} \partial_x u(x,y) dx|^p dy\\
&\le |x_1 - x_2|^{p-1} \int_0^1 \int_{x_1}^{x_2} |\partial_x u(x,y) dx|^p\,dx\,dy\\
&\le pI(u)|x_1 - x_2|^{p-1}.
\end{aligned}$$

If $I(u) < +\infty$, the function $x \to u(x,\cdot)$ is Hölder continuous from a dense subset of $\mathbb{R}$ with values in $L^p(0,1)$ and so it can be extended to a continuous function on $\mathbb{R}$. Thus, any function $u \in W^{1,p}_{\rm loc}(S_0) \cap \{I < +\infty\}$ defines a continuous trajectory in $L^p(0,1)$ verifying

$$\begin{aligned}
\mathrm{d}(u(x_1,\cdot), u(x_2,\cdot))^p &= \int_0^1 |u(x_1,y) - u(x_2,y)|^p dy\\
&\le pI(u)|x_1 - x_2|^{p-1}, \forall x_1, x_2 \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

Lemma 2.2. *For all $r > 0$, there exists $\mu_r > 0$, such that if $u \in W^{1,p}_{\rm loc}(S_0)$ satisfies $\min \|u(x,\cdot) \pm \sigma\|_{W^{1,p}(0,1)} \geq r$ for a.e. $x \in (x_1, x_2)$, then*

$$\begin{aligned}
&\int_{x_1}^{x_2} \Big[\int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u(x,y))dy\Big]dx \\
&\geq \frac{1}{p(x_2-x_1)^{p-1}} \mathrm{d}(u(x_1,\cdot),u(x_2,\cdot))^p + \frac{p-1}{p}\mu_r^{\frac{p}{p-1}}(x_2-x_1) \\
&\geq \mu_r \mathrm{d}(u(x_1,\cdot),u(x_2,\cdot))
\end{aligned} \tag{2.3}$$

Proof. We define the functional

$$F(u(x,\cdot)) = \int_0^1 \frac{1}{p}|\partial_y u(x,y)|^p + \underline{a}W(u(x,y))dy$$

on $W^{1,p}(0,1)$, where $\underline{a} = \min_{\mathbb{R}^2} a(x,y) > 0$. To prove the lemma, we first to claim that:

For any $r > 0$, there exists $\mu_r > 0$, such that if $q(y) \in W^{1,p}(0,1)$ is such that $\min \|q(y) \pm \sigma\|_{W^{1,p}(0,1)} \geq r$, then$F(q(y)) \geq \frac{p-1}{p}\mu_r^{\frac{p}{p-1}}$. Namely, if $q_n(\cdot) \in W^{1,p}(0,1)$ and $F(q_n) \to 0$, then $\min \|q_n \pm \sigma\|_{W^{1,p}(0,1)} \to 0$.

Assume by contradiction that if $F(q_n) \to 0$ and $\min \|q_n \pm \sigma\|_{L^\infty(0,1)} \geq \varepsilon_0 > 0$. Then there exists a sequence $(y_n^1) \subset [0,1]$ such that $\min |q_n(y_n^1) \pm \sigma| \geq \varepsilon_0$. Since $\int_0^1 \underline{a}W(q_n)dy \to 0$ there exists a sequence $(y_n^2) \subset [0,1]$ such that $|q_n(y_n^2) \pm \sigma| < \frac{\varepsilon_0}{2}$. Then

$$\begin{aligned}
\frac{\varepsilon_0}{2} &\leq |q_n(y_n^2) - q_n(y_n^1)| \\
&\leq \Big| \int_{y_n^1}^{y_n^2} |\dot{q}_n(t)|dt \Big| \\
&\leq |y_n^2 - y_n^1|^{1-\frac{1}{p}} \Big[\int_0^1 |\dot{q}_n(t)|^p dt\Big]^{1/p} \\
&\leq p^{\frac{1}{p}}(F(q_n))^{1/p} \to 0.
\end{aligned}$$

It is a contradiction.

Since $\min \|q_n \pm \sigma\|_{L^\infty(0,1)} \to 0$ as $F(q_n) \to 0$, then $\int_0^1 |\dot{q}_n(y)|^p dy \to 0$, and it follows that $\|q_n - \sigma\|_{W^{1,p}(0,1)} \to 0$ as $F(q_n) \to 0$.

Observe that if $(x_1, x_2) \subset \mathbb{R}$ and $u \in W^{1,p}_{\rm loc}(S_0)$ are such that $F(u(x,\cdot)) \geq \frac{p-1}{p}\mu_r^{\frac{p}{p-1}}$ for a.e. $x \in (x_1, x_2)$, by Hölder's and Yung's inequalities we have

$$\begin{aligned}
&\int_{x_1}^{x_2} \Big[\int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u(x,y))dy\Big]dx \\
&\geq \int_{x_1}^{x_2}\int_0^1 \frac{1}{p}|\partial_x u|^p \,dy\,dx + \int_{x_1}^{x_2}\int_0^1 \frac{1}{p}|\partial_y u|^p + \underline{a}W(u)\,dy\,dx \\
&= \frac{1}{p}\int_0^1\int_{x_1}^{x_2} |\partial_x u|^p \,dx\,dy + \int_{x_1}^{x_2} F(u(x,\cdot))dx \\
&\geq \frac{1}{p(x_2-x_1)^{p-1}} \mathrm{d}(u(x_1,\cdot),u(x_2,\cdot))^p + \frac{p-1}{p}\mu_r^{\frac{p}{p-1}}(x_2-x_1) \\
&\geq \mu_r \mathrm{d}(u(x_1,\cdot),u(x_2,\cdot)).
\end{aligned}$$

The proof is complete. ∎

As a direct consequence of Lemma 2.2, we have the following result.

Lemma 2.3. *If* $u \in W^{1,p}_{\rm loc}(S_0) \cap \{I < +\infty\}$, *then* $\mathrm{d}\big(u(x,\cdot), \pm\sigma\big) \to 0$ *as* $x \to \pm\infty$.

Proof. Note that since

$$I(u) = \int_{S_0} \frac{1}{p}|\nabla u|^p + a(x,y)W(u(x,y))\,dx\,dy < +\infty,$$

$W(u(x,y)) \to 0$ as $|x| \to +\infty$. Then by Lemma 2.2, $\liminf_{x\to+\infty} \mathrm{d}\big(u(x,\cdot),\sigma\big) = 0$. Next we show that $\limsup_{x\to+\infty} \mathrm{d}\big(u(x,\cdot),\sigma\big) = 0$ by contradiction. We assume that there exists $r \in (0,\sigma/4)$ such that $\limsup_{x\to+\infty} \mathrm{d}\big(u(x,\cdot),\sigma\big) > 2r$, by (2.2) there exists infinite intervals $(p_i,s_i), i \in \mathbb{N}$ such that $\mathrm{d}\big(u(p_i,\cdot),\sigma\big) = r$, $\mathrm{d}\big(u(s_i,\cdot),\sigma\big) = 2r$ and $r \le \mathrm{d}\big(u(x,\cdot),\sigma\big) \le 2r$ for $x \in \cup_i(p_i,s_i)$, $i \in \mathbb{N}$ by Lemma 2.2, this implies $I(u) = +\infty$, it's a contradiction. Similarly, we can prove that $\lim_{x\to-\infty} \mathrm{d}\big(u(x,\cdot),-\sigma\big) = 0$. ☐

Now we consider the functional on the class

$$\Gamma = \{u \in W^{1,p}_{\rm loc}(S_0) : I(u) < +\infty,\ \mathrm{d}\big(u(x,\cdot),\pm\sigma\big) \to 0 \text{ as x} \to \pm\infty\}$$

Let

$$c = \inf_{\Gamma} I \quad \text{and} \quad \mathcal{K} = \{u \in \Gamma : I(u) = c\} \tag{2.4}$$

We will show that $\mathcal{K}$ is not empty, and we start noting that the trajectory in Γ with action close to the minima has some concentration properties.

For any $\delta > 0$, we set

$$\lambda_\delta = \frac{1}{p}\delta^p + \max_{\mathbb{R}^2} a(x,y) \cdot \max_{|s\pm\sigma| \le p^{1/p}\delta} W(s). \tag{2.5}$$

Lemma 2.4. *There exists* $\bar\delta_0 \in (0,\sigma/2)$ *such that for any* $\delta \in (0,\bar\delta_0)$ *there exists* $\rho_\delta > 0$ *and* $l_\delta > 0$, *for which, if* $u \in \Gamma$ *and* $I(u) \le c + \lambda_\delta$, *then*

- (i) $\min \|u(x,\cdot) \pm \sigma\|_{W^{1,p}(0,1)} \ge \delta$ *for a.e.* $x \in (s,p)$ *then* $p - s \le l_\delta$.
- (ii) *if* $\|u(x_-,\cdot) + \sigma\|_{W^{1,p}(0,1)} \le \delta$, *then* $\mathrm{d}(u(x_-,\cdot),-\sigma) \le \rho_\delta$ *for any* $x \le x_-$, *and if* $\|u(x_+,\cdot) - \sigma\|_{W^{1,p}(0,1)} \le \delta$, *then* $\mathrm{d}(u(,\cdot),\sigma) \le \rho_\delta$ *for any* $x \ge x_+$.

Proof. By Lemma 2.2, as in this case, there exists $\mu_\delta > 0$ such that

$$\int_s^p \int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u)\,dx\,dy \ge \mu_\delta(p-s).$$

Since $I(u) \le c + \lambda_\delta$ there exists $l_\delta < +\infty$ such that $p - s < l_\delta$.

To prove (ii), we first do some preparation, $\mu_{r_\delta} \ge \frac{p-1}{p}\lambda_\delta$, $\rho_\delta = \max\{\delta, r_\delta\} + 3(\frac{p-1}{p\mu_{r_\delta}})^{\frac{p-1}{p}}\lambda_\delta$. Let $\bar\delta_0 \in (0,\sigma/2)$ be such that $\rho_\delta < \sigma/2$ for all $\delta \in (0,\bar\delta_0)$. Let $\delta \in (0,\bar\delta_0)$, $u \in \Gamma, I(u) \le +\infty$ and $x_- \in \mathbb{R}$ be such that $\|u(x_-,\cdot) + \sigma\|_{W^{1,p}(0,1)} \le \delta$. Define

$$u_-(x,y) = \begin{cases} -1 & \text{if } x < x_- - 1, \\ x - x_- + (x - x_- + 1)u(x_-,y) & \text{if } x_- - 1 \le x_-, \\ u(x,y) & \text{if } x \ge x_-. \end{cases}$$

and note that $u_- \in \Gamma$ and $I(u_-) \ge c$, then $\|u_- + \sigma\|_{W^{1,p}(0,1)} = |x - x_- + 1| \cdot \|u(x_-,\cdot) + \sigma\|_{W^{1,p}(0,1)} \le \delta$ when $x_- - 1 \le x \le x_-$. Recall that $\|q\|_{L^\infty(0,1)} \le p^{1/p}\|q\|_{W^{1,p}(0,1)}$

for any $q \in W^{1,p}(0,1)$, then $\|u_- + \sigma\|_{L^\infty(0,1)} \le p^{1/p}\|u_- + \sigma\|_{W^{1,p}(0,1)} \le p^{1/p}\delta$, by definition (2.5) of λ_δ, we have

$$\int_{x_- -1}^{x_-}[\int_0^1 \frac{1}{p}|\nabla u_-|^p + a(x,y)W(u_-)dy]dx \le \lambda_\delta.$$

Since

$$\begin{aligned} I(u_-) = I(u) &- \int_{-\infty}^{x_-}\int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u)\,dy\,dx \\ &+ \int_{x_- -1}^{x_-}\int_0^1 \frac{1}{p}|\nabla u_-|^p + a(x,y)W(u_-)\,dy\,dx \end{aligned}$$

we obtain

$$\int_{-\infty}^{x_-}\int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u)\,dy\,dx \le 2\lambda_\delta. \tag{2.6}$$

Now, assume by contradiction that there exists $x_1 < x_-$ such that $\mathrm{d}(u(x_1,\cdot),-\sigma) \ge \rho_\delta$, by (2.2) there exists $x_2 \in (x_1, x_-)$ such that $\mathrm{d}(u(x,\cdot),-\sigma) \ge \max\{\delta, r_\delta\}$ for $x \in (x_1,x_2)$ and $\mathrm{d}(u(x_1,\cdot),u(x_1,\cdot)) \ge \rho_\delta - \max\{\delta, r_\delta\}$. By Lemma 2.2, we have

$$\int_{-\infty}^{x_-}\int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u)\,dy\,dx \ge (\frac{p\mu_{r_\delta}}{p-1})^{\frac{p-1}{p}}\big(\rho_\delta - \max\{\delta, r_\delta\}\big) \ge 3\lambda_\delta$$

which contradicts (2.6). Thus $\mathrm{d}(u(x,\cdot),-\sigma) \le \rho_\delta$ for any $x \le x_-$. Analogously, we can prove if $\|u(x_+,\cdot) - \sigma\|_{W^{1,p}(0,1)} \le \delta$, then $\mathrm{d}(u(x,\cdot),\sigma) \le \rho_\delta$ as $x \ge x_+$. □

To exploit the compactness of I on Γ, we set the function $X : W^{1,p}_{\rm loc}(S_0) \to \mathbb{R} \cup \{+\infty\}$ given by

$$X(u) = \sup\{x : \mathrm{d}(u(x,\cdot),\sigma)\} \ge \sigma/2.$$

Setting $\chi(s) = \min|s \pm \sigma|$, by$(H_3)$, there exist $0 < w_1 < w_2$ such that

$$w_1\chi^p(s) \le W(s) \le w_2\chi^p(s) \text{ when } \chi(s) \le \sigma/2. \tag{2.7}$$

Now, we can get the compactness of the minimizing sequence of I in Γ.

Lemma 2.5. *If $(u_n) \subset \Gamma$ is such that $I(u_n) \to c$ and $X(u_n) \to X_0 \in \mathbb{R}$, then there exists $u_0 \in \mathcal{K}$ such that, along a sequence, $u_n \to u_0$ weakly in $W^{1,p}(S_0)$.*

Proof. We now show that (u_n) is bounded in $W^{1,p}_{\rm loc}(S_0)$, i.e., (u_n) is bounded in $L^p_{\rm loc}(S_0)$, (∇u_n) is bounded in $L^p_{\rm loc}(S_0)$. Since $I(u_n) \to c$and $\int_{S_0}|\nabla u_n|^p\,dx\,dy \le pI(u_n)$, we have that (∇u_n) is bounded in $L^p_{\rm loc}(S_0)$. If we can prove that $u_n(x,\cdot)$ is bounded in $L^p(0,1)$ for a.e. $x \in \mathbb{R}$, then (u_n) is bounded in $L^p_{\rm loc}(S_0)$.

Let $B_r = \{q \in L^p(0,1)/\|q\|_{L^p(0,1)} \le r\}$, we assume by contradiction that for any $R > 2\sigma$, there exists $\bar{x} \in \mathbb{R}$ such that $u(\bar{x},\cdot) \notin B_R$ for $u \in \Gamma \cap \{I(u) \le c+\lambda\}, \lambda > 0$, such that $\|u(\bar{x},\cdot)\|_{L^p(0,1)} \ge R$, then $\mathrm{d}(u(\bar{x},\cdot),\sigma) \ge \|u(\bar{x},\cdot)\|_{L^p(0,1)} - \|\sigma\|_{L^p(0,1)} \ge R - \sigma$. Since $\mathrm{d}(u(x,\cdot),\pm\sigma) \to 0$ as $x \to \pm\infty$, by continuity there exists $x_1 > \bar{x}$ such that $\mathrm{d}(u(x_1,\cdot),\sigma) \le \sigma/2$ and $\mathrm{d}(u(x,\cdot),\sigma) \ge \sigma/2$ for $x \in (\bar{x}, x_1)$. Using Lemma 2.2, we get

$$c + \lambda \ge I(u) \ge \mu_{\sigma/2}\mathrm{d}(u(x_1,\cdot),u(\bar{x},\cdot)) \ge \mu_{\sigma/2}(R - 3\sigma/2).$$

which is a contradiction for R large enough. We conclude that (u_n) is bounded in $W^{1,p}_{\rm loc}(S_0)$, thus there exists $u_0 \in W^{1,p}_{\rm loc}(S_0)$ such that up to a sequence, $u_n \to u_0$ weakly in $W^{1,p}_{\rm loc}(S_0)$. We shall prove that $u_0 \in \Gamma$; i.e., $\mathrm{d}(u_0(x,\cdot),\pm\sigma) \to 0$ as $x \to \pm\infty$. First we claim that:

For any small $\varepsilon > 0$, there exists $\lambda(\varepsilon) \in (0, \lambda_{\bar{\delta}})$ and $l(\varepsilon) > l_{\bar{\delta}}$ such that if $u \in \Gamma \cap \{I(u) \leq c + \lambda(\varepsilon)\}$ then

$$\int_{|x-X(u)|\geq l(\varepsilon)} \int_0^1 W(u(x,y))\,dy\,dx \leq \varepsilon. \tag{2.8}$$

Indeed, let $\delta < \bar{\delta}$ be such that $3\lambda_\delta \leq \underline{a} w_1 \varepsilon$ where $\underline{a} = \min_{\mathbb{R}^2} a(x,y)$. Given any $u \in \Gamma \cap \{I(u) \leq c + \lambda_\delta\}$, by Lemma 2.4, there exists $x_- \in (X(u) - l_\delta, X(u))$ and $x_+ \in (X(u), X(u) + l_\delta)$ such that $\|u(x_-, \cdot) + \sigma)\|_{W^{1,p}(0,1)} \leq \delta$ and $\|u(x_+, \cdot) - \sigma\|_{W^{1,p}(0,1)} \leq \delta$. We define the function

$$\tilde{u}(x,y) = \begin{cases} -\sigma & \text{if } x < x_- - 1, \\ \sigma(x - x_-) + (x - x_- + 1)u(x_-, y) & \text{if } x_- - 1 \leq x_-, \\ u(x,y) & \text{if } x_- \leq x \leq x_+, \\ (x_+ - x + 1)u(x_+, y) + \sigma(x - x_+) & \text{if } x_+ \leq x < x_+ + 1, \\ \sigma & \text{if } x > x_+ + 1 \end{cases}$$

which belongs to Γ, and $I(\tilde{u}) \geq c$,

$$\begin{aligned} &\int_{|x-X(u)|\geq l_\delta} \int_0^1 \frac{1}{p}|\nabla u|^p + a(x,y)W(u)\,dy\,dx \\ &\leq I_{-\infty}^{x_-}(u) + I_{x_+}^{+\infty}(u) \\ &= I(u) - I(\tilde{u}) + I_{x_- -1}^{x_-}(\tilde{u}) + I_{x_+}^{x_+ +1}(\tilde{u}) \\ &\leq 3\lambda_\delta \end{aligned}$$

then (2.8) follows setting $l(\varepsilon) = l_{\bar{\delta}}$ and $\lambda(\varepsilon) = \lambda_\delta$.

From (2.8) it is easy to see that $u(x,y) \to \sigma$ as $x \to +\infty$. Combining (2.8) and (2.7) we obtain

$$\int_{|x-X(u)|\geq l(\varepsilon)} \int_0^1 w_1 |u(x,y) - \sigma|^p\,dx\,dy \leq \int_{|x-X(u)|\geq l(\varepsilon)} \int_0^1 W(u(x,y))\,dy\,dx \leq \varepsilon;$$

i.e., $\mathrm{d}\big(u(x,\cdot), \sigma\big) \to 0$ as $x \to +\infty$. Analogously, we can get that $\mathrm{d}\big(u(x,\cdot), -\sigma\big) \to 0$ as $x \to -\infty$, it follows that $u_0 \in \Gamma$. □

As a consequence, we get the following existence result.

Proposition 2.6. *$\mathcal{K} \neq \emptyset$ and any $u \in \mathcal{K}$ satisfies $u \in C^{1,\alpha}(\mathbb{R}^2)$ is a solution of $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a(x,y)W'(u(x,y)) = 0$ on S_0 with $\partial_y u(x,0) = \partial_y u(x,1) = 0$ for all $x \in \mathbb{R}$, and $\|u\|_{L^\infty(S_0)} \leq R_0$. Finally, $u(x,y) \to \pm\sigma$ as $x \to \pm\infty$ uniformly in $y \in [0,1]$.*

Proof. By Lemma 2.5, the set $\mathcal{K}$ is not empty. By (H_2), $\|u\|_{L^\infty(S_0)} \leq R_0$. Indeed, $\tilde{u} = \max\{-R_0, \min\{R_0, u\}\}$ is a fortiori minimizer. Let $\eta \in C_0^\infty(S_0)$ and $\tau \in \mathbb{R}$, then $u + \tau\eta \in \Gamma$ and since $u \in \mathcal{K}$, $I(u + \tau\eta)$ is a C^1 function of τ with a local minima at $\tau = 0$. Therefore,

$$I'(u)\eta = \int_{S_0} |\nabla u|^{p-2}\nabla u \nabla\eta + aW'(u)\eta\,dx\,dy = 0$$

for all such η, namely u is a weak solution of the equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a(x,y)W'(u(x,y)) = 0$ on S_0. Standard regularity arguments show that $u \in C^{1,\alpha}(S_0)$ for some $\alpha \in (0,1)$ and satisfies the Neumann boundary condition (see

[14][17][27]). Since $\|u\|_{L^\infty(S_0)} \le R_0$, there exists $C>0$ such that $\|u\|_{C^{1,\alpha}(S_0)} \le C$, which guarantees that u satisfies the boundary conditions. Indeed, assume by contradiction that u does not verify $u(x,y) \to -\sigma$ as $x \to -\infty$ uniformly with respect to $y \in [0,1]$. Then there exists $\delta > 0$ and a sequence $(x_n, y_n) \in S_0$ with $x_n \to -\infty$ and $|u(x_n,y_n)+\sigma| \ge 2\delta$ for all $n \in \mathbb{N}$. The $C^{1,\alpha}$ estimate of u implies that there exists $\rho > 0$ such that $|u(x,y)+\sigma| \ge \delta$ for $\forall\,(x,y) \in B_\rho(x_n,y_n), n \in \mathbb{N}$. Along a subsequence $x_n \to -\infty$, $y_n \to y_0 \in [0,1]$, $|u(x,y)+\sigma| \ge \delta$ for $(x,y) \in B_{\rho/2}(x_n,y_0)$, which contradicts with the fact that $\mathrm{d}(u(x,\cdot),-\sigma) \to 0$ as $x \to -\infty$ since $u \in \Gamma$. The other case is similar. □

We shall explore the reversibility condition of (H1)-(ii), and we will prove that the minimizer on Γ is in fact a solution of (2.1).

Lemma 2.7. *$c_p = c$.*

Proof. Since $\Gamma_p \subset \Gamma$, $c_p \ge c$. Assume by contradiction that $c_p > c$, then there exists $u \in \Gamma$ such that $I(u) < c_p$. Writing

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\int_0^{1/2} \frac{1}{p}|\nabla u|^p + aW(u)dy\Big]dx + \int_{\mathbb{R}} \Big[\int_{1/2}^1 \frac{1}{p}|\nabla u|^p + aW(u)dy\Big]dx \\ &= I_1 + I_2 \end{aligned}$$

it follows that $\min\{I_1, I_2\} < \frac{c_p}{2}$. Suppose for example $I_1 < c_p/2$, define

$$v(x,y) = \begin{cases} u(x,y) & \text{if } x \in \mathbb{R} \text{ and } 0 \le y \le \frac{1}{2}, \\ u(x,1-y) & \text{if } x \in \mathbb{R} \text{ and } \frac{1}{2} \le y \le 1. \end{cases}$$

Then $v \in \Gamma_p$, by condition (H1)-(ii), $I(v) = 2I_1 < c_p$, this is a contradiction. □

We shall prove that any $u \in \mathcal{K}$ is periodic in y.

Lemma 2.8. *If $u \in \mathcal{K}$ then $u(x,0) = u(x,1)$ for all $x \in \mathbb{R}$.*

Proof. Suppose $u \in \mathcal{K}$ and v as above, then $v(x,y) = u(x,y)$ for $y \in [0,1/2]$. By (H_1)-(ii), $I(u) = c = c_p = I(v)$, so $v \in \mathcal{K}$. Then u and v are solutions of

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + aW'(u(x,y)) = 0, \quad \text{on } S_0, \\ \partial_y u(x,0) = \partial_y u(x,1) = 0 \quad \text{for all } x \in \mathbb{R}. \end{gathered} \tag{2.9}$$

Since $u = v$ for $y \in [0,1/2]$, by the principle of unique continuation (see [8]), we have $u = v$ in $\mathbb{R} \times [0,1]$. i.e. $u(x,0) = u(x,1)$. □

Remark 2.9. It is an open problem for the principle for p-harmonic functions in case $n \ge 3$ and $p \ne 2$. When $p = \infty$, the principle of unique continuation does not hold.

Proof of Theorem 1.1. We now extend u periodically in y direction to the entire space $\mathbb{R}^2$, and write it as $U(x,y)$. As a consequence of the above lemmas and proposition 2.6, $U(x,y)$ is an entire solution of (1.1), which is heteroclinic in x and 1-periodic in y direction. □

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