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# EXISTENCE OF NON-OSCILLATORY SOLUTIONS FOR A HIGHER-ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATION

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Abstract. This article concerns the solvability of the higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn}\dots\Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n+b_nx_{n-d}))\big)\Big)+\sum_{j=1}^{s}p_{jn}f_j(x_{n-r_{jn}})=q_n,$$

where $n\geq n_0\geq 0$, $d,k,j,s$ are positive integers, $f_j:\mathbb{R}\to\mathbb{R}$ and $xf_j(x)\geq 0$ for $x\neq 0$. Sufficient conditions for the existence of non-oscillatory solutions are established by using Krasnoselskii fixed point theorem. Five theorems are stated according to the range of the sequence $\{b_n\}$.

## 1. Introduction and preliminaries

Interest in the solvability of difference equations has increased lately, as inferred from the number of related publications; see for example the references in this article and their references. Authors have examined various types difference equations, as follows:

$$\Delta(a_n\Delta x_n)+p_nx_{g(n)}=0,\quad n\geq 0,\quad \text{in [14]},\tag{1.1}$$

$$\Delta(a_n\Delta x_n)=q_nx_{n+1},\quad \Delta(a_n\Delta x_n)=q_nf(x_{n+1}),\quad n\geq 0,\quad \text{in [11]},\tag{1.2}$$

$$\Delta^2(x_n+px_{n-m})+p_nx_{n-k}-q_nx_{n-l}=0,\quad n\geq n_0,\quad \text{in [6]},\tag{1.3}$$

$$\Delta^2(x_n+px_{n-k})+f(n,x_n)=0,\quad n\geq 1,\quad \text{in [10]},\tag{1.4}$$

$$\Delta^2(x_n-px_{n-\tau})=\sum_{i=1}^{m}q_if_i(x_{n-\sigma_i}),\quad n\geq n_0,\quad \text{in [9]},\tag{1.5}$$

$$\Delta(a_n\Delta(x_n+bx_{n-\tau}))+f(n,x_{n-d_{1n}},x_{n-d_{2n}},\dots,x_{n-d_{kn}})=c_n,\quad n\geq n_0,\quad \text{in [8]},\tag{1.6}$$

$$\Delta^m(x_n+cx_{n-k})+p_nx_{n-r}=0,n\geq n_0,\quad \text{in [15]},\tag{1.7}$$

$$\Delta^m(x_n+c_nx_{n-k})+p_nf(x_{n-r})=0,\quad n\geq n_0,\quad \text{in [3, 4, 12, 13]},\tag{1.8}$$

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$$\Delta^m(x_n + cx_{n-k}) + \sum_{s=1}^{u} p_n^s f_s(x_{n-r_s}) = q_n, \quad n \geq n_0, \quad \text{in [16]}, \tag{1.9}$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \quad n \geq n_0, \quad \text{in [17]}. \tag{1.10}$$

Motivated by the above publications, we investigate the higher-order nonlinear neutral difference equation

$$\Delta\Big(a_{kn} \dots \Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n + b_n x_{n-d}))\big)\Big) + \sum_{j=1}^{s} p_{jn} f_j(x_{n-r_{jn}}) = q_n, \tag{1.11}$$

where $n \geq n_0 \geq 0$, $d, k, j, s$ are positive integers, $\{a_{in}\}_{n \geq n_0}$ $(i = 1, 2, \dots, k)$, $\{b_n\}_{n \geq n_0}$, $\{p_{jn}\}_{n \geq n_0}$ $(1 \leq j \leq s)$ and $\{q_n\}_{n \geq n_0}$ are sequences of real numbers, $r_{jn} \in \mathbb{Z}$ $(1 \leq j \leq s, n_0 \leq n)$, $f_j : \mathbb{R} \to \mathbb{R}$ and $x f_j(x) \geq 0$ for $x \neq 0$ $(j = 1, 2, \dots, s)$. Clearly, difference equations (1.1)–(1.10) are special cases of (1.11), for which we use Krasnoselskii fixed point theorem to obtain non-oscillatory solutions.

**Lemma 1.1** (Krasnoselskii Fixed Point Theorem)**.** *Let $\Omega$ be a bounded closed convex subset of a Banach space $X$ and $T_1, T_2 : S \to X$ satisfy $T_1 x + T_2 y \in \Omega$ for each $x, y \in \Omega$. If $T_1$ is a contraction mapping and $T_2$ is a completely continuous mapping, then the equation $T_1 x + T_2 x = x$ has at least one solution in $\Omega$.*

As usual, the forward difference $\Delta$ is defined as $\Delta x_n = x_{n+1} - x_n$, and for a positive integer $m$ the higher-order difference is defined as

$$\Delta^m x_n = \Delta(\Delta^{m-1} x_n), \quad \Delta^0 x_n = x_n.$$

In this article, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the sets of all integers, $\alpha = \inf\{n - r_{jn} : 1 \leq j \leq s, n_0 \leq n\}$, $\beta = \min\{n_0 - d, \alpha\}$, $\lim_{n\to\infty}(n - r_{jn}) = +\infty$, $1 \leq j \leq s$, $l_\beta^\infty$ denotes the set of real-valued bounded sequences $x = \{x_n\}_{n \geq \beta}$. It is well known that $l_\beta^\infty$ is a Banach space under the supremum norm $\|x\| = \sup_{n \geq \beta} |x_n|$.

For $N > M > 0$, let

$$A(M, N) = \big\{x = \{x_n\}_{n \geq \beta} \in l_\beta^\infty : M \leq x_n \leq N, n \geq \beta\big\}.$$

Obviously, $A(M, N)$ is a bounded closed and convex subset of $l_\beta^\infty$. Put

$$\bar{b} = \limsup_{n\to\infty} b_n \quad \text{and} \quad \underline{b} = \liminf_{n\to\infty} b_n.$$

**Definition 1.2** ([5])**.** A set $\Omega$ of sequences in $l_\beta^\infty$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer $N_0$ such that

$$|x_i - x_j| < \varepsilon,$$

whenever $i, j > N_0$ for any $x = x_k$ in $\Omega$.

**Lemma 1.3** (Discrete Arzela-Ascoli's theorem [5])**.** *A bounded, uniformly Cauchy subset $\Omega$ of $l_\beta^\infty$ is relatively compact.*

By a solution of (1.11), we mean a sequence $\{x_n\}_{n \geq \beta}$ with a positive integer $N_0 \geq n_0 + d + |\alpha|$ such that (1.11) is satisfied for all $n \geq N_0$. As is customary, a solution of (1.11) is said to be oscillatory about zero, or simply oscillatory, if the terms $x_n$ of the sequence $\{x_n\}_{n \geq \beta}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called non-oscillatory.

## 2. Existence of non-oscillatory solutions

In this section, we will give five sufficient conditions of the existence of non-oscillatory solutions of (1.11).

**Theorem 2.1.** *If there exist constants $M$ and $N$ with $N > M > 0$ and such that*

$$|b_n| \le b < \frac{N-M}{2N}, \quad \text{eventually}, \tag{2.1}$$

$$\sum_{t=n_0}^{\infty} \max\Big\{\frac{1}{|a_{it}|}, |p_{jt}|, |q_t| : 1 \le i \le k, 1 \le j \le s\Big\} < +\infty, \tag{2.2}$$

*then (1.11) has a non-oscillatory solution in $A(M,N)$.*

*Proof.* Choose $L \in (M + bN, N - bN)$. By (2.1) and (2.2), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$|b_n| \le b < \frac{N-M}{2N},\ \forall n \ge N_0, \tag{2.3}$$

and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} \le \min\{L - bN - M, N - bN - L\}, \tag{2.4}$$

where $F = \max_{M\le x\le N}\{f_j(x) : 1 \le j \le s\}$. Define two mappings $T_1, T_2 : A(M,N) \to X$ by

$$(T_1x)_n = \begin{cases} L - b_n x_{n-d}, & n \ge N_0,\\ (T_1x)_{N_0}, & \beta \le n < N_0, \end{cases} \tag{2.5}$$

$$(T_2x)_n = \begin{cases} (-1)^k \sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots \\ \sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{\sum_{j=1}^{s} p_{jt} f_j(x_{t-r_{jt}}) - q_t}{\prod_{i=1}^{k} a_{it_i}}, & n \ge N_0,\\ (T_2x)_{N_0}, & \beta \le n < N_0, \end{cases} \tag{2.6}$$

for all $x \in A(M,N)$.

(i) Note that $T_1x + T_2y \in A(M,N)$ for all $x, y \in A(M,N)$. In fact, for every $x, y \in A(M,N)$ and $n \ge N_0$, by (2.4), we have

$$\begin{aligned} (T_1x + T_2y)_n &\ge L - bN - \sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{\big|\sum_{j=1}^{s} p_{jt} f_j(y_{t-r_{jt}}) - q_t\big|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} \\ &\ge L - bN - \sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} \ge M \end{aligned}$$

and

$$\begin{aligned} (T_1x + T_2y)_n &\le L + bN + \sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} \\ &\le N. \end{aligned}$$

That is, $(T_1x + T_2y)(A(M,N)) \subseteq A(M,N)$.

(ii) W show that $T_1$ is a contraction mapping on $A(M,N)$. For any $x, y \in A(M,N)$ and $n \geq N_0$, it is easy to derive that

$$\big|(T_1x)_n - (T_1y)_n\big| \leq |b_n||x_{n-d} - y_{n-d}| \leq b\|x-y\|,$$

which implies

$$\|T_1x - T_1y\| \leq b\|x-y\|.$$

Then $b < \frac{N-M}{2N} < 1$ ensures that $T_1$ is a contraction mapping on $A(M,N)$.

(iii) We show that $T_2$ is completely continuous. First, we show $T_2$ that is continuous. Let $x^{(u)} = \{x_n^{(u)}\} \in A(M,N)$ be a sequence such that $x_n^{(u)} \to x_n$ as $u \to \infty$. Since $A(M,N)$ is closed, $x = \{x_n\} \in A(M,N)$. Then, for $n \geq N_0$,

$$\big|T_2x_n^{(u)} - T_2x_n\big| \leq \sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\big|\sum_{j=1}^{s}p_{jt}\big|\big|f_j(x_{t-r_{jt}}^{(u)}) - f_j(x_{t-r_{jt}})\big|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}.$$

Since

$$\begin{aligned}\frac{\big|\sum_{j=1}^{s}p_{jt}\big|\big|f_j(x_{t-r_{jt}}^{(u)}) - f_j(x_{t-r_{jt}})\big|}{\big|\prod_{i=1}^{k}a_{it_i}\big|} &\leq \frac{\big|\sum_{j=1}^{s}p_{jt}\big|\big(|f_j(x_{t-r_{jt}}^{(u)})| + |f_j(x_{t-r_{jt}})|\big)}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\ &\leq \frac{2F\big|\sum_{j=1}^{s}p_{jt}\big|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\end{aligned}$$

and $|f_j(x_{t-r_{jt}}^{(u)}) - f_j(x_{t-r_{jt}})| \to 0$ as $u \to \infty$ for $j = 1, 2, \dots, s$, it follows from (2.2) and the Lebesgue dominated convergence theorem that $\lim_{u\to\infty}\|T_2x^{(u)} - T_2x\| = 0$, which means that $T_2$ is continuous.

Next, we show that $T_2A(M,N)$ is relatively compact. By (2.2), for any $\varepsilon > 0$, take $N_1 \geq N_0$ large enough,

$$\sum_{t_1=N_1}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\big|\sum_{j=1}^{s}p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k}a_{it_i}\big|} < \frac{\varepsilon}{2}. \tag{2.7}$$

Then, for any $x = \{x_n\} \in A(M,N)$ and $n_1, n_2 \geq N_1$, (2.7) ensures that

$$\begin{aligned}\big|T_2x_{n_1} - T_2x_{n_2}\big| &\leq \sum_{t_1=n_1}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\big|\sum_{j=1}^{s}p_{jt}f_j(y_{t-r_{jt}}) - q_t\big|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\ &\quad + \sum_{t_1=n_2}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\big|\sum_{j=1}^{s}p_{jt}f_j(y_{t-r_{jt}}) - q_t\big|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\ &\leq \sum_{t_1=N_1}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\big|\sum_{j=1}^{s}p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\ &\quad + \sum_{t_1=N_1}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\big|\sum_{j=1}^{s}p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which implies $T_2A(M,N)$ begin uniformly Cauchy. Therefore, by Lemma 1.3, the set $T_2A(M,N)$ is relatively compact. By Lemma 1.1, there exists $x = \{x_n\} \in A(M,N)$ such that $T_1x + T_2x = x$, which is a bounded non-oscillatory solution to (1.11). This completes the proof. □

**Theorem 2.2.** *If* (2.2) *holds,*

$$b_n \geq 0 \text{ eventually}, \quad 0 \leq \underline{b} \leq \bar{b} < 1, \tag{2.8}$$

*and there exist constants* $M$ *and* $N$ *with* $N > \frac{2-\underline{b}}{1-\bar{b}} M > 0$ *then* (1.11) *has a non-oscillatory solution in* $A(M,N)$.

*Proof.* Choose $L \in (M + \frac{1+\bar{b}}{2} N, N + \frac{\underline{b}}{2} M)$. By (2.2) and (2.8), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b}}{2} \leq b_n \leq \frac{1+\bar{b}}{2}, \ \forall n \geq N_0 \tag{2.9}$$

and

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\leq \min\Big\{L - M - \frac{1+\bar{b}}{2} N, N - L + \frac{\underline{b}}{2} M\Big\}, \end{aligned} \tag{2.10}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$. Then define $T_1, T_2 : A(M,N) \to X$ as (2.5) and (2.6). The rest proof is similar to that of Theorem 2.1, and it is omitted. □

**Theorem 2.3.** *If* (2.2) *holds,*

$$b_n \leq 0 \text{ eventually}, \quad -1 < \underline{b} \leq \bar{b} \leq 0, \tag{2.11}$$

*and there exist constants* $M$ *and* $N$ *with* $N > \frac{2+\bar{b}}{1+\underline{b}} M > 0$*, then* (1.11) *has a non-oscillatory solution in* $A(M,N)$.

*Proof.* Choose $L \in (\frac{2+\bar{b}}{2} M, \frac{1+\underline{b}}{2} N)$. By (2.2) and (2.11), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b}-1}{2} \leq b_n \leq \frac{\bar{b}}{2}, \ \forall n \geq N_0, \tag{2.12}$$

and

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\leq \min\Big\{L - \frac{2+\bar{b}}{2} M, \frac{1+\underline{b}}{2} N - L\Big\}, \end{aligned} \tag{2.13}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$. Then define $T_1, T_2 : A(M,N) \to X$ by (2.5) and (2.6). The rest proof is similar to that of Theorem 2.1, and is omitted. □

**Theorem 2.4.** *If* (2.2) *holds,*

$$b_n > 1 \text{ eventually}, \quad 1 < \underline{b}, \text{ and } \bar{b} < \underline{b}^2 < +\infty, \tag{2.14}$$

*and there exist constants* $M$ *and* $N$ *with* $N > \frac{\underline{b}(\bar{b}^2 - \underline{b})}{\bar{b}(\underline{b}^2 - \bar{b})} M > 0$*, then* (1.11) *has a non-oscillatory solution in* $A(M,N)$.

*Proof.* Take $\varepsilon \in (0, \underline{b} - 1)$ sufficiently small satisfying

$$1 < \underline{b} - \varepsilon < \bar{b} + \varepsilon < (\underline{b} - \varepsilon)^2 \tag{2.15}$$

and

$$\big((\bar{b}+\varepsilon)(\underline{b}-\varepsilon)^2 - (\bar{b}+\varepsilon)^2\big) N > \big((\bar{b}+\varepsilon)^2(\underline{b}-\varepsilon) - (\underline{b}-\varepsilon)^2\big) M. \tag{2.16}$$

Choose $L \in \big((\overline{b}+\varepsilon)M + \frac{\overline{b}+\varepsilon}{\underline{b}-\varepsilon}N, (\underline{b}-\varepsilon)N + \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M\big)$. By (2.2) and (2.15), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\underline{b} - \varepsilon < b_n < \overline{b} + \varepsilon, \quad \forall b \geq N_0 \tag{2.17}$$

and

$$\begin{aligned}&\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\big|\sum_{j=1}^{s}p_{jt}\big|+|q_t|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\&\leq \min\Big\{\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}L - (\underline{b}-\varepsilon)M - N, \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M + (\underline{b}-\varepsilon)N - L\Big\},\end{aligned} \tag{2.18}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$. Define two mappings $T_1, T_2 : A(M,N) \to X$ by

$$(T_1x)_n = \begin{cases}\frac{L}{b_{n+d}} - \frac{x_{n+d}}{b_{n+d}}, & n \geq N_0,\\ (T_1x)_{N_0}, & \beta \leq n < N_0,\end{cases} \tag{2.19}$$

$$T_2x)_n = \begin{cases}\frac{(-1)^k}{b_{n+d}}\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots & \\ \sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\sum_{j=1}^{s}p_{jt}f_j(x_{t-r_{jt}})-q_t}{\prod_{i=1}^{k}a_{it_i}}, & n \geq N_0,\\ (T_2x)_{N_0}, & \beta \leq n < N_0,\end{cases} \tag{2.20}$$

for all $x \in A(M,N)$. The rest proof is similar to that in Theorem 2.1, and is omitted. □

**Theorem 2.5.** *If* (2.2) *holds,*

$$b_n < -1 \text{ eventually}, \quad -\infty < \underline{b},\ \overline{b} < -1 \tag{2.21}$$

*and there exist constants $M$ and $N$ with $N > \frac{1+\underline{b}}{1+\overline{b}}M > 0$, then* (1.11) *has a non-oscillatory solution in $A(M,N)$.*

*Proof.* Take $\epsilon \in \big(0, -(1+\overline{b})\big)$ sufficiently small satisfying

$$\underline{b} - \epsilon < \overline{b} + \epsilon < -1 \tag{2.22}$$

and

$$(1+\overline{b}+\epsilon)N < (1+\underline{b}-\epsilon)M. \tag{2.23}$$

Choose $L \in \big((1+\overline{b}+\epsilon)N, (1+\underline{b}-\epsilon)M\big)$. By (2.2) and (2.22), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\underline{b} - \epsilon < b_n < \overline{b} + \epsilon, \quad \forall n \geq N_0, \tag{2.24}$$

and

$$\begin{aligned}&\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\big|\sum_{j=1}^{s}p_{jt}\big|+|q_t|}{\big|\prod_{i=1}^{k}a_{it_i}\big|}\\&\leq \min\Big\{\Big(\overline{b}+\epsilon+\frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}\Big)M - \frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}L, L - (1+\overline{b}+\epsilon)N\Big\},\end{aligned} \tag{2.25}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$. Then define $T_1, T_2 : A(M,N) \to X$ as (2.19) and (2.20). The rest proof is similar to that in Theorem 2.1, and is omitted. □

**Remark 2.6.** Theorems 2.1–2.5 extend the results in Cheng [6, Theorem 1], Liu, Xu and Kang [8, Theorems 2.3-2.7], Zhou and Huang [16, Theorems 1-5] and corresponding theorems in [3, 4, 9, 10, 11, 12, 13, 14, 15].

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