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ASYMPTOTIC BEHAVIOUR FOR A DIFFUSION EQUATION GOVERNED BY NONLOCAL INTERACTIONS

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Abstract. In this article, we study the asymptotic behaviour of a nonlocal nonlinear parabolic equation governed by a parameter. After giving the existence of unique branch of solutions composed by stable solutions in stationary case, we gives for the parabolic problem L^∞ estimates of solution based on using the Moser iterations and existence of global attractor. We finish our study by the issue of asymptotic behaviour in some cases when $t \to \infty$.

1. Introduction

The non-local problems are important in studying the behavior of certain physical phenomena and population dynamics. A major difficulty in studying these problems often lie in the absence of well-known properties as maximum principle, regularity and properties of Lyapunov (see [5, 6]) and also the difficulty to characterize and determine the stationary solutions associated thus making study the asymptotic behavior of these solutions very difficult.

In this article, we study the solution to the nonlocal equation

$$\begin{gathered} u_t - \operatorname{div}(a(l_r(u(t)))\nabla u) = f \quad \text{in } \mathbb{R}^+ \times \Omega \\ u(x,t) = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \\ u(.,0) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{1.1}$$

In the above problem u_0 and f are such that

$$u_0 \in L^2(\Omega), \quad f \in L^2(0,T,L^2(\Omega)), \tag{1.2}$$

with T an arbitrary positive number, a is a continuous function for which there exist m, M such that

$$0 < m \le a(\epsilon) \le M \quad \forall \epsilon \in \mathbb{R}. \tag{1.3}$$

The nonlocal functional $l_r(.)(x) : L^2(\Omega) \to \mathbb{R}$, is defined by

$$u \to l_r(u(t))(x) = \int_{\Omega \cap B(x,r)} g(y)u(t,y)dy. \tag{1.4}$$

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Here $B(x,r)$ is the closed ball of $\mathbb{R}^n$ with radius r and $g \in L^2(\Omega)$. It is sometimes possible to consider g more generally, especially when one is interested in the study of stationary solutions (see [3]).

From the physical point of view problem (1.1) gives many applications especially where $g = 1$ in population dynamics. Indeed, in this situation u may represent a population density and $l_r(u)$ the total mass of the subdomain $\Omega \cap B(x,r)$ of Ω. Hence (1.1) can describe the evolution of a population whose diffusion velocity depends on the total mass of a subdomain of Ω. For more details of modelling we refer the reader to [7]. This type of equations can be applied more generally to other models including the study of propagation of mutant gene (see [11, 12, 13]). A very recent study of this propagation was made by Bendahmane and Sepúlveda [4] in which they analyze using a finite volume scheme adapted, the transmission of this gene through 3 types of people: susceptible, infected and recovered.

From the mathematical point of view, when $r = d$ where d is the diameter of Ω, problem (1.1) has been studied in various forms [6, 8, 9, 15].

However, when $0 < r < d$, several questions from the theory of bifurcations have arisen concerning the structure of stationary solutions including the existence of a principle of comparison of different solutions depending on the parameter r and the existence of branches (local and global) of solutions. A large majority of these issues has been resolved in [3]. It shows that when a is decreasing the existence of a unique global branch of solutions and existence of branch of solutions that are purely local. Some questions may then arise:

(i) The unique branch described in [3] it is composed of stable solutions?
(i) What about stability properties of the corresponding parabolic problem?

The plan for this work is the following. In section 2 we give some existence and uniqueness results. Section 3 is devoted to stationary problem corresponding to (1.1). In particular, we study in a radial case, a generalization of Chipot-Lovat results about determination of the number of solutions. We also establish that the unique global branch of solutions described in [3] is composed by stable solutions (theorem 3.12). In section 4 firstly we address an L^∞ estimate taking to account L^p estimate based on Moser iterations. Secondly we prove existence of absorbing set in H_0^1, which allows us to prove the existence of a global attractor associated to (1.1) (see remark 4.5). Finally we obtain a result of stability properties of the corresponding parabolic problem.

2. Existence and uniqueness results

In this section we show a result of existence. We set $V = H_0^1(\Omega)$ and V' its dual. The norm in V is

$$\|u\|_V^2 = \int_\Omega |\nabla u|^2 dx,$$

and the duality bracket of V' and V is $\langle \cdot, \cdot \rangle$.

Theorem 2.1. *Let $T > 0$, $f \in L^2(0,T,V')$ and $u_0 \in L^2(\Omega)$, we assume that a is a continuous function and the assumption* (1.3) *checked then for every r fixed,*

$r \in [0, \operatorname{diam}(\Omega)]$, *there exists a function* u *such that*

$$\begin{gathered} u \in L^2(0,T,V), \quad u_t \in L^2(0,T,V') \\ u(0,.) = u_0 \quad \text{in } \Omega \\ \frac{d}{dt}(u,\phi) + \int_\Omega a(l_r(u(t)))\nabla u \nabla \phi \, dx = \langle f, \phi \rangle \quad \text{in } D'(0,T)\ \forall \phi \in H_0^1(\Omega). \end{gathered} \tag{2.1}$$

Moreover, if a *is locally Lipschitz; i.e, for each* c *there exists* γ_c *such that*

$$|a(\epsilon) - a(\epsilon')| \le \gamma_c |\epsilon - \epsilon'| \quad \forall \epsilon, \epsilon' \in [-c, c], \tag{2.2}$$

then the solution of (2.1) *is unique.*

Remark 2.2. Before to do the proof, we note that for $r = 0$, problem (2.1) is linear and the proof follows a well-known result [10]. It is also valid when $r = \operatorname{diam}(\Omega)$ (see [7]). Therefore, we will focus in the case $r \in]0, \operatorname{diam}(\Omega)[$.

Proof. For the proof of existence, we will use the Schauder fixed point theorem. Let $w \in L^2(0,T,L^2(\Omega))$. Then the mapping $t \mapsto l_r(w(t))$ is measurable. As a is continuous then $t \mapsto a(l_r(w(t)))$ is also continuous. The problem of finding solution $u = u(x,t)$ of

$$\begin{gathered} u \in L^2(0,T,V) \cap C([0,T], L^2(\Omega)) \quad u_t \in L^2(0,T,V') \\ u(0,.) = u_0 \\ \frac{d}{dt}(u,\phi) + \int_\Omega a(l_r(w(t)))\nabla u \nabla \phi dx = \langle f, \phi \rangle \quad \text{in } D'(0,T)\ \forall \phi \in H_0^1(\Omega), \end{gathered} \tag{2.3}$$

is linear, and admits a unique solution $u = F_r(w)$ [10, 7]. Thus we show that the application

$$w \mapsto F_r(w) = u, \tag{2.4}$$

admits a fixed point. Taking $w = u$ in (2.3), using (1.3) and using the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + m\|u\|_V^2 \le |f|_\star \|u\|_V, \tag{2.5}$$

where $\|\cdot\|_V$ is the usual norm in V and $|f|_\star$ is the dual norm of f. We take

$$|u|_{L^2(0,T,V)} = \Big\{ \int_0^T \|u\|_V^2 dt \Big\}^{\frac{1}{2}}.$$

Using Young's inequality to the right-hand side of (2.5), it follows that

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{m}{2}\|u\|_V^2 \le \frac{1}{2m}|f|_\star^2. \tag{2.6}$$

By integrating (2.6) on $(0,t)$ for $t \le T$, we obtain

$$\frac{1}{2}|u(t)|_2^2 + \frac{m}{2}\int_0^t \|u\|_V^2 dt \le \frac{1}{2}|u_0|_2^2 + \frac{1}{2m}\int_0^t |f|_\star^2. \tag{2.7}$$

We deduce that there exists a constant $C = C(m, u_0, f)$ such that

$$|u|_{L^2(0,T,V)} \le C \tag{2.8}$$

Moreover

$$< u_t, v\rangle + \langle -\operatorname{div}(a(l_r(u(t)))\nabla u), v\rangle = \langle f, v\rangle \quad \forall v \in V,$$

This gives us

$$|u_t|_\star \le M\|u\|_V + |f|_\star. \tag{2.9}$$

By squaring both sides and using the Young inequality, we have

$$|u_t|_\star^2 \le 2M^2\|u\|_V^2 + 2|f|_\star^2. \tag{2.10}$$

By integrating on $(0,t)$ and assuming (2.8) we obtain

$$|u_t|_{L^2(0,T,V')} \le C', \tag{2.11}$$

with $C' = C'(m, M, f, u_0)$, independent to w. It follows from (2.8) and (2.11) that

$$|u_t|^2_{L^2(0,T,V')} + |u|^2_{L^2(0,T,V)} \le R, \tag{2.12}$$

with $R = C^2 + C'^2$. From (2.8) and the Poincaré inequality it follows that

$$|u|_{L^2(0,T,L^2(\Omega))} \le R', \tag{2.13}$$

By setting

$$R_1 = \max(R', R), \tag{2.14}$$

and associating (2.13) and (2.14), it follows that the application F maps the ball $B(0,R_1)$ of $L^2(0,T,L^2(\Omega))$ into itself. Moreover the balls of $H^1(0,T,V,V')$ are relatively compact in $L^2(0,T,L^2(\Omega))$ (see [10] for more details). (2.12) clearly shows us that $F(B(0,R_1)$ is relatively compact in $B(0,R_1)$ with

$$B(0,R_1) = \{u \in L^2(0,T,L^2(\Omega)) : |u|_{L^2(0,T,L^2(\Omega))} \le R_1\}.$$

To apply the Schauder fixed point theorem, as announced, we just need to show that F is continuous from $B(0,R_1)$ to itself. This is actually the case and completes the proof of existence.

We will now discuss the uniqueness assuming of course that assumption (2.2) be verified. Consider u_1 and u_2 two solutions (2.1), by subtracting one obtains in $D'(0,T)$

$$\frac{d}{dt}(u_1 - u_2, v) + \int_\Omega (a(l_r(u_1(t))\nabla u_1(t) - a(l_r(u_2(t)))\nabla u_2(t))\nabla\phi dx = 0 \tag{2.15}$$

for all $\phi \in H_0^1(\Omega)$. Since

$$\begin{aligned}&a(l_r(u_1(t)))\nabla u_1 - a(l_r(u_2(t)))\nabla u_2(t)\\ &= (a(l_r(u_1(t))) - a(l_r(u_2(t)))\nabla u_1(t) + a(l_r(u_2(t)))\nabla(u_1(t) - u_2(t)),\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}(u_1 - u_2, v) + \int_\Omega a(l_r(u_2(t)))\nabla(u_1(t) - u_2(t))\nabla\phi dx\\ &= -\int_\Omega (a(l_r(u_1(t))) - a(l_r(u_2(t)))\nabla u_1 \nabla\phi dx \quad \forall\phi \in H_0^1(\Omega).\end{aligned} \tag{2.16}$$

Moreover, $u_1, u_2 \in C([0,T], L^2(\Omega))$ and there exist $z > 0$ such that $l_r(u_1(t))$ and $l_r(u_2(t))$ are in $[-z,z]$. Taking $v = u_1 - u_2$ in (2.16), by Cauchy-Schwarz inequality and (2.2), ti follows that

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m\|u_1 - u_2\|_V^2 \le \gamma|l_r(u_1(t)) - l_r(u_2(t))|\|u_1\|_V\|u_1 - u_2\|_V. \tag{2.17}$$

Also we have [3],

$$|l_r(u(t)) \le C|B(x,r) \cap \Omega|^{1/(n\vee 3)}|g|_2|u(t)|_2 \le |\Omega|^{1/(n\vee 3)}|g|_2|u(t)|_2, \tag{2.18}$$

where C a constant, $|\Omega|$ represents the measure of Ω and $n \vee 3$ the maximum between the dimension n of Ω and 3. By using (2.18), (2.17) and the Young inequality

$$ab \leq \frac{1}{2m} b^2 + \frac{m}{2} a^2.$$

We deduce

$$\frac{d}{dt} |u_1 - u_2|_2^2 + m \|u_1 - u_2\|_V^2 \leq p(t) |u_1 - u_2|_2^2, \tag{2.19}$$

with

$$p(t) = \frac{1}{m} (\gamma C |\Omega|^{1/(n \vee 3)} |g|_2 \, \|u_1\|_V \,)^2 \in L^1(0,T),$$

which leads to

$$\frac{d}{dt} |u_1 - u_2|_2^2 \leq \; p(t) |u_1 - u_2|_2^2. \tag{2.20}$$

Multiplying (2.20) by $e^{-\int_0^t p(s) ds}$ it follows that

$$e^{-\int_0^t p(s) ds} \frac{d}{dt} |u_1 - u_2|_2^2 - p(t) e^{-\int_0^t p(s) ds} |u_1 - u_2|_2^2 \leq 0. \tag{2.21}$$

Hence

$$\frac{d}{dt} \{ e^{-\int_0^t p(s) ds} |u_1 - u_2|_2^2 \} \leq 0. \tag{2.22}$$

This shows that $t \mapsto e^{-\int_0^t p(s) ds} |u_1 - u_2|_2^2$ is non-increasing. Since for $t = 0$,

$$u_1(0,.) = u_2(0,.) = u_0.$$

This function vanishes at 0 and nonnegative, we conclude that it is identically zero. This concludes the proof. □

3. Stationary solutions

Consider the weak formulation to the stationary problem associated with (1.1),

$$\begin{gathered} -\operatorname{div}(a(l_r(u)) \nabla u) = f \quad \text{in } \Omega \\ u \in H_0^1(\Omega). \end{gathered} \tag{3.1}$$

3.1. The case $r = d$.

By taking ϕ the weak solution of the problem

$$\begin{gathered} -\Delta \phi = f \quad \text{in } \Omega \\ \phi \in H_0^1(\Omega). \end{gathered}$$

Due to a Chipot-Lovat [8] results we obtain the following result.

Theorem 3.1. *Let a be a mapping from $\mathbb{R}$ into $(0, \infty)$. The problem (3.1) with $r = d$ has as many solutions as the problem, in $\mathbb{R}$,*

$$\mu a(\mu) = l_d(\phi), \tag{3.2}$$

with $\mu = l_d(u_d)$.

Remark 3.2. Theorem 3.1 allows us to see where a is increasing that the problem (3.1) with $r = d$ admits a unique solution and determine for a given a the exact number of solutions (3.1). However it is difficult or impossible to adapt the proof of the theorem 3.1 in the case $0 < r < d$.

3.2. **The case $0 < r < d$.** As announced in the introduction we focus our study to the case of radial solutions of (3.1) with $r = d$. We will assume Ω is the open ball of $\mathbb{R}^n$ with radius $d/2$ centered at zero. We set

$$L^2_{\rm rad}(\Omega) = \{u \in L^2(\Omega) : \exists \tilde{u} \in L^2(]0, d/2[) \text{ such that } u(x) = \tilde{u}(\|x\|)\},$$

and assume that

$$\begin{gathered} f \in L^2_{\rm rad}(\Omega), \quad g \in L^2_{\rm rad}(\Omega), \\ a \in W^{1,\infty}(\mathbb{R}), \quad \inf_{\mathbb{R}} a > 0, \\ f \geq 0 \quad \text{a.e in} \quad \Omega, \quad g \geq 0 \quad \text{a.e. in } \Omega. \end{gathered} \tag{3.3}$$

We start by giving in some sense in a linear case a result that will be used later to explain the asymptotic behavior.

Proposition 3.3. *Let $A, B \in C(\overline{\Omega})$ be positive radial functions such that $A \leq B$ in $\overline{\Omega}$ and also $f, h \in L^2(\Omega)$ two positive radial functions. Let $u \in H^1_0(\Omega)$ the radial solution to*

$$-\operatorname{div}(A(x)\nabla u) = f \quad \text{in } \Omega, \tag{3.4}$$

$$-\operatorname{div}(B(x)\nabla u) = h \quad \text{in } \Omega. \tag{3.5}$$

Then $f \leq h$ a.e. in Ω.

Proof. We proved in [3] that if u is a the radial solution of (3.4) then for a.e. t in $[0, d/2]$,

$$\tilde{u}'(t) = -\frac{1}{\tilde{A}(t)} \int_0^t \big(\frac{s}{t}\big)^{n-1} \tilde{f}(s)\,ds. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$\frac{\tilde{B}(t)}{\tilde{A}(t)} \int_0^t \big(\frac{s}{t}\big)^{n-1} \tilde{f}(s)\,ds = \int_0^t \big(\frac{s}{t}\big)^{n-1} \tilde{h}(s)\,ds.$$

Since $A \leq B$ in $\overline{\Omega}$ and $f, h \geq 0$ with $f \not\equiv 0$, $h \not\equiv 0$ hence $f \leq g$. $\square$

In a nonlocal case, some results of existence of radial solutions and comparison principle between u_r, u_d and u_0 have been demonstrated in [3]. It is also proved when for $r \in [0, d]$ we set

$$I_r := [\inf_{\Omega} l_r(\phi), \sup_{\Omega} l_r(\phi)]. \tag{3.7}$$

Here ϕ denotes the solution of

$$\begin{gathered} -\Delta\phi = f \quad \text{in } \Omega \\ \phi \in H^1_0(\Omega). \end{gathered} \tag{3.8}$$

By the inclusion or not of I_r at an interval of $\mathbb{R}$ we somehow generalize the theorem 3.1.

Lemma 3.4. *Let $r \in [0, d]$. Assume that (3.3) holds and there exist $0 \leq m_1 \leq m_2$ such that*

$$a(m_1) = \max_{[m_1, m_2]} a \quad a(m_2) = \min_{[m_1, m_2]} a, \tag{3.9}$$

$$I_r \subset [m_1 a(m_1), m_2 a(m_2)]. \tag{3.10}$$

Then (3.1) admits a radial solution u, and

$$m_1 \leq l_r(u) \leq m_2 \quad \text{a.e. in } \Omega. \tag{3.11}$$

For the proof of the above lemma, we refer the reader to [3]. Generalizing this construction type of the diffusion coefficient a we obtain

Proposition 3.5. *Let $r \in [0,d]$. Assume that (3.3) holds and there exist an odd integer n_1 and n_1+1 positive real numbers $\{m_i\}_{i=0\dots n_1}$, with $m_0=0$ and for all $i \in \{0,\dots,n_1-1\}$ we have $m_i < m_{i+1}$. Moreover*

$$
\begin{gathered}
a(m_i) = \max_{[m_i,m_{i+1}]} a, \quad a(m_{i+1}) = \min_{[m_i,m_{i+1}]} a \quad \forall i \in \{0,2,\dots,n_1-3,n_1-1\} \\
I_r \subset \cap_{i=0,2,\dots,n_1-3,n_1-1}[m_i a(m_i), m_{i+1}a(m_{i+1})]
\end{gathered} \tag{3.12}
$$

Then (3.1) admits at least $(n_1+1)/2$ radial solutions $\{u_i\}_{i\in\{0,2,\dots,n_1-1\}}$ such that

$$
m_i \le l_r(u_i) \le m_{i+1} \quad \forall i \in \{0,2,\dots,n_1-3,n_1-1\}.
$$

Proof. By induction, we set

$$
\mathcal{P}_{n_1} = \{\text{If condition (3.12) is satisfied then (3.1) admits at least } \frac{n_1+1}{2} \text{ solutions.}\}
$$

By using lemma 3.4 with $m_1=0$ and $m_2=m_1$, it is easy to prove that for $n_1=1$, $\mathcal{P}_{n_1}$ is true. For $n_1>1$, This procedure can be repeated to prove that if $\mathcal{P}_{n_1-2}$ holds true then $\mathcal{P}_{n_1}$ holds too. □

Example 3.6. Let us see a function a satisfying proposition 3.5. For this, we consider the case $n_1=3$ and $r\in(0,d]$. Considering (3.3) and the strong maximum principle we get $\min I_r > 0$. Taking

$$
m_1 := 2\,\frac{\max I_r}{a(0)}, \quad a(m_1) := \frac{a(0)}{2}
$$

with $a(0)>0$ and also a decreasing on $[0,m_1]$ then we prove the conditions of lemma 3.4.

By repeating this process with $m_2 > m_1$ and setting

$$
a(m_2) := \frac{\min I_r}{m_2}, \quad m_3 := 2\frac{\max I_r}{a(m_2)}
$$

with $a(m_3) := \frac{a(m_2)}{2}$ and also a is decreasing on $[m_2,m_3]$. This shows the existence of such a.

In the representation of a in Figure 1, we have deliberately left, on solid line parts of the curve satisfying the conditions of proposition 3.5, and dotted line one without constraints.

Remark 3.7. As previously announced, proposition 3.5 generalizes a point of view Theorem 3.1. However it does not accurately determine the exact number of solutions of (3.1) and the bifurcation points of branch of solutions. We have shown in [2] a way to solve this problem using the linearized problem, the principle of comparisons obtained in [3] and the Krein-Rutman theorem.

3.3. Stable solutions of (3.1).

Definition 3.8. Given a domain $\Omega \subset \mathbb{R}^n$, a solution $u_r \in H_0^1(\Omega)$ of (3.1) is stable if for all $\phi \in H_0^1(\Omega)$,

$$
G_{u_r}(\phi) := \int_\Omega a(l_r(u_r))|\nabla\phi|^2 - \int_\Omega a'(l_r(u_r))l_r(\phi)\nabla u_r \nabla\phi \ge 0. \tag{3.13}
$$

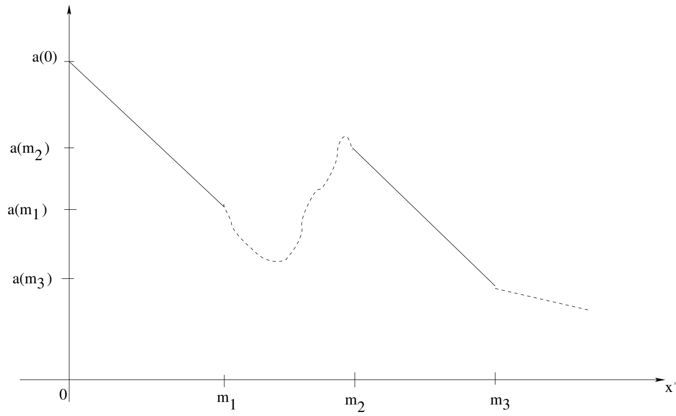

FIGURE 1. The case $n_1 = 3$

Definition 3.9. Given $u : [0, d] \to H_0^1(\Omega)$, the graph of u is called a (global) branch of solutions if

(i) $u \in C([0, d], H_0^1(\Omega))$,
(ii) $u(r)$ is solution to (3.1) for all r in $[0, d]$.

The function u is called a local branch if it is defined only on a subinterval of $[0, d]$ with positive measure.

Before concluding this section, we will focus on the case a non-increasing, to prove the stability of the global branch of solutions. Assume for all $r \in [0, d]$, u_r is a solution to (3.1) and

$$0 \le l_r(u_r)(x) \le \mu_d \quad \text{for a.e. } x \in \Omega. \tag{3.14}$$

Assume that there exists a solution μ_d to (3.2) such that

$$a(\mu_d) = \min_{[0,\mu_d]} a \quad \text{and} \quad a(0) = \max_{[0,\mu_d]} a. \tag{3.15}$$

Theorem 3.10 ([3]). *Assume* (3.3), (3.14), (3.15) *and* (3.2) *holds. Assume in addition that* $a \in W^{1,\infty}(\mathbb{R})$ *and for some positive constant* ϵ, *it holds that*

$$C_1 |g|_2 |f|_2 |a'|_{\infty,[-\epsilon,\mu_d+\epsilon]} \frac{1}{a(\mu_d)^2} < 1, \tag{3.16}$$

where C_1 *is a constant dependent to* Ω. *Then*

(i) *For all* r *in* $[0, d]$, (3.1) *possesses a unique radial solution* u_r *in* $[u_0, u_d]$;
(ii) $\{(r, u_r) : r \in [0, d]\}$ *is a branch of solutions without bifurcation point;*
(iii) *it is only global branch of solutions;*
(iv) *if in addition, a is non-increasing on* $[0, \mu_d]$ *then* $r \mapsto u_r$ *is nondecreasing.*

Remark 3.11. It is very difficult to obtain property (iv) for any a. However when a is non-increasing provide us important information for studying the stability of this branch of solutions.

Corollary 3.12. *Let* u_d^1 *the smallest solution to* (3.1). *Assume* (3.3) *and* (3.2) *holds true and there exists a solution* μ_d *to* (3.2) *satisfied* (3.15). *Assume in addition* *that $a \in W^{1,\infty}(\mathbb{R})$, u_d^1 satisfied (3.14) and for some positive constant ϵ, it holds that*

$$C_1|g|_2|f|_2|a'|_{\infty,[-\epsilon,\mu_d+\epsilon]}\frac{1}{a(\mu_d)^2} < 1, \tag{3.17}$$

where C_1 is a constant dependent to Ω. Then $\{(r,u_r) : r \in [0,d]\}$ is the only global branch of solutions starting to u_d^1.

Proof. The fact that $\{(r,u_r) : r \in [0,d]\}$ is the only global branch of solutions results from theorem 3.10. We will now show that this unique branch of solutions is stable and start at $r = d$ by u_d^1. For this we consider without loss of generality (3.1) admits two solutions u_d^1 and u_d^2 such that $u_d^1 \le u_d^2$. We denote by μ_1 and μ_2 respectively solutions of (3.2) corresponding to u_d^1 and u_d^2 (see figure 2). It is easy to see that μ_1 and μ_2 satisfied (3.15).

Assume $\{(r,u_r) : r \in [0,d]\}$ is the only global branch of solutions starting to u_d^2. Then we get $C_1|g|_2|f|_2|a'|_{\infty,[-\epsilon,\mu_2+\epsilon]}\frac{1}{a(\mu_2)^2} < 1$. In this case, using theorem 3.10 we get (3.1) possesses a unique radial solution u_r in $[u_0, u_d^2]$ and the mapping $r \mapsto u_r$ is nondecreasing. By continuity of this mapping, we can find a $r_0 \in]0,d[$ such that $u_{r_0} = u_d^1$ for a.e $x \in \Omega$. This means that u_d^1 is a solution of (P_{r_0}). This gives us a contradiction and concludes the proof. □

We are now able to prove the following result.

Proposition 3.13. *Under assumptions and notation of corollary 3.12, the global branch of solutions described in theorem 3.10 is composed by stable solutions.*

Proof. For all $r \in [0,d]$, let u_r be a solution belonging to the global branch of solutions described in theorem 3.10. By using the linearized problem of (3.1), we get for all $\phi \in H_0^1(\Omega)$,

$$\begin{aligned} &\int_\Omega a(l_r(u_r))|\nabla\phi|^2 - \int_\Omega a'(l_r(u_r))l_r(\phi)\nabla u_r\nabla\phi \\ &\ge \inf_\Omega a(l_r(u_r))|\nabla\phi|_2^2 - C|g|_2|a'|_{\infty,[-\epsilon,\mu_1+\epsilon]}|\nabla u_r|_2|\nabla\phi|_2^2. \end{aligned} \tag{3.18}$$

Taking into account that $|\nabla u_r|_2 \le C(\Omega)\frac{|f|_2}{\inf_\Omega a(l_r(u_r))}$ where $C(\Omega)$ designed the Poincaré Sobolev constant. We obtain

$$\begin{aligned} &\int_\Omega a(l_r(u_r))|\nabla\phi|^2 - \int_\Omega a'(l_r(u_r))l_r(\phi)\nabla u_r\nabla\phi \\ &\ge |\nabla\phi|_2^2\Big(\inf_\Omega a(l_r(u_r)) - C_1|g|_2|a'|_{\infty,[-\epsilon,\mu_1+\epsilon]}\frac{|f|_2}{\inf_\Omega a(l_r(u_r))}\Big). \end{aligned} \tag{3.19}$$

Moreover by assumptions (3.14) and (3.15) we get $a(\mu_1) \le \inf_\Omega a(l_r(u_r))$. Thus (3.17) becomes

$$C_1|g|_2|f|_2|a'|_{\infty,[-\epsilon,\mu_d+\epsilon]}\frac{1}{\inf_\Omega a(l_r(u_r))^2} < 1. \tag{3.20}$$

We obtain

$$\int_\Omega a(l_r(u_r))|\nabla\phi|^2 - \int_\Omega a'(l_r(u_r))l_r(\phi)\nabla u_r\nabla\phi \ge 0.$$

This concluded the proof. □

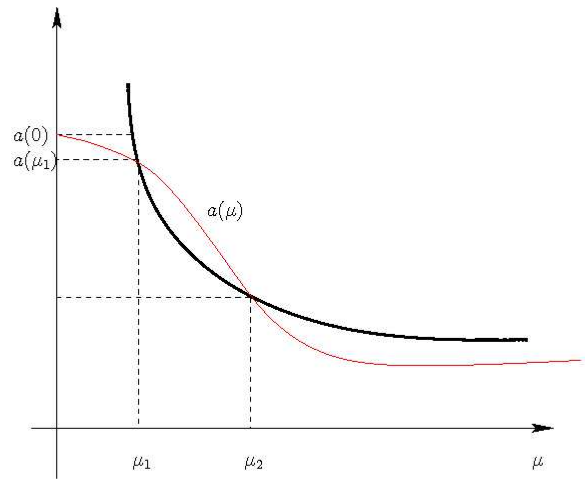

Figure 2. case of 2 solutions

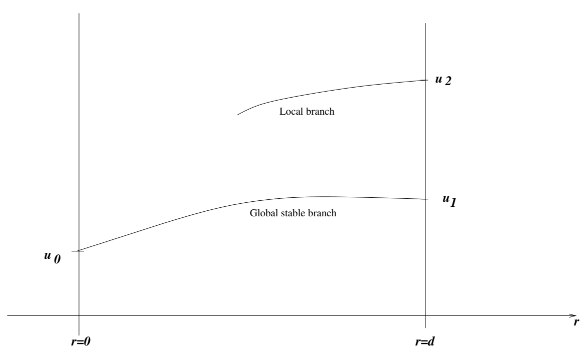

Figure 3. Branch of solutions

4. Parabolic problem

4.1. L^∞ **estimate.** In what follows we obtain L^∞ estimate of the solution (1.1) from L^q estimate. The method we use is based on iterations Moser type, for more details on the method see [14].

Theorem 4.1. *Let $n \geq 3$ and u a classical solution of (1.1) defined on $[0,T)$. Assume that $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose further that $U_q = \sup_{t<T} |u(t)|_q < \infty$, $f \in L^\infty(0,\infty, L^q(\Omega))$. If $p < \frac{n}{n-2}$ then $U_\infty < \infty$.*

To prove this theorem we need the following result.

Lemma 4.2. *Consider u a classical solution of (1.1) on $[0,T)$, $r \geq 1$ and $p > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$ with $p < \frac{n}{n-2}$. We take $\tilde{U}_r = \max\{1, |u_0|_\infty, U_r = sup_{t<T}|u(t)|_r\}$ and let*

$$\sigma(r) = \frac{p(n+2)}{2[r(2p - pn + n) + np]}.$$

Then there exists a constant $C_2 = C_2(\Omega, m)$ *such that*

$$\tilde{U}_{2r} \le [C_2 \, \|f\|_{L^\infty(0,\infty,L^q(\Omega))}]^{\sigma(r)} r^{\sigma(r)} \tilde{U}_r.$$

Proof. Multiplying (1.1) by u^{2r-1} and then using the Hölder inequality yields

$$\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r}dx + m\frac{2r-1}{r^2}\int_\Omega |\nabla(u^r)|^2 dx \le |f|_q |u^{2r-1}|_p. \tag{4.1}$$

As

$$|u^{2r-1}|_p = |u^r|_{p\frac{2r-1}{r}}^{\frac{2r-1}{r}}, \tag{4.2}$$

by taking $w = u^r$ in (4.1) and (4.2), we get easily

$$\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{2r-1}{r^2}|\nabla w|_2^2 \le |f|_q |w|_{\alpha p}^\alpha, \tag{4.3}$$

with $\alpha = (2r-1)/r$. Let β be such that

$$\frac{1}{\alpha p} = \beta + \frac{1-\beta}{2^\star}, \tag{4.4}$$

with $2^\star = \frac{2n}{n-2}$. We claim that $\beta \in (0,1)$. In fact

$$\beta = \frac{2nr - (n-2)(2r-1)p}{(n+2)(2r-1)p}.$$

Since $p < \frac{2r}{2r-1}\frac{n}{n-2}$, it follows that $\beta > 0$. Also $(n+2)(2r-1)p > 2nr-(n-2)(2r-1)p$ implies $\beta < 1$. This prove that $\beta \in (0,1)$. Using an interpolation inequality in (4.3) and (4.4) (see [14]), we obtain

$$\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{2r-1}{r^2}|\nabla w|_2^2 \le |f|_q \Big(|w|_1^\beta |w|_{2^\star}^{1-\beta}\Big)^\alpha. \tag{4.5}$$

Applying Sobolev injections in (4.5), we have

$$\begin{aligned} &\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{2r-1}{r^2}|\nabla w|_2^2 \\ &\le \Big[|f|_q \big(\frac{2r}{m}\big)^{\frac{\alpha(1-\beta)}{2}} |w|_1^{\beta\alpha} C^{(1-\beta)\alpha}\Big]\Big[\big(\frac{m}{2r}\big)^{\frac{\alpha(1-\beta)}{2}} |\nabla w|_2^{(1-\beta)\alpha}\Big], \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{2r-1}{r^2}|\nabla w|_2^2 \\ &\le \Big[|f|_q \big(\frac{2r}{m}\big)^{\frac{\alpha(1-\beta)}{2}} |w|_1^{\beta\alpha} C^{(1-\beta)\alpha}\Big]\Big[\big(\frac{m}{2r}\big)|\nabla w|_2^2\Big]^{\frac{\alpha(1-\beta)}{2}}. \end{aligned} \tag{4.7}$$

Since $\beta \in (0,1)$ and $\frac{\alpha}{2} \in (0,1)$ it is clear that $\frac{\alpha(1-\beta)}{2} \in (0,1)$. Applying Young's inequality to (4.7) with $\frac{\alpha(1-\beta)}{2} + 1 - \frac{\alpha(1-\beta)}{2} = 1$. We obtain

$$\begin{aligned} &\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{2r-1}{r^2}|\nabla w|_2^2 \\ &\le \delta\Big[|f|_q^{\frac{1}{\delta}} \big(\frac{2r}{m}\big)^{\frac{\alpha(1-\beta)}{2\delta}} |w|_1^{\frac{\beta\alpha}{\delta}} C^{2/\delta}\Big] + \frac{\alpha(1-\beta)}{2}\Big[\big(\frac{m}{2r}\big)|\nabla w|_2^2\Big], \end{aligned} \tag{4.8}$$

with $\delta = 1 - \frac{\alpha(1-\beta)}{2}$. Joining the fact that $\frac{\alpha(1-\beta)}{2} \in (0,1)$ and $\delta < 1$ to (4.8), we deduce

$$\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{3r-2}{2r^2}|\nabla w|_2^2 \le |f|_q^{1/\delta} \big(\frac{2r}{m}\big)^{\frac{\alpha(1-\beta)}{2\delta}} |w|_1^{\frac{\beta\alpha}{\delta}} C^{2/\delta}. \tag{4.9}$$

We set

$$2r\sigma(r) - 1 = \frac{\alpha(1-\beta)}{2\delta} \quad \text{and} \quad 2\rho(r) = \frac{\beta\alpha}{\delta}.$$

Then (4.9) becomes

$$\frac{1}{2r}\frac{d}{dt}|w|_2^2 + m\frac{3r-2}{2r^2}|\nabla w|_2^2 \le |f|_q^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)-1}|w|_1^{2\rho(r)}\,C^{2/\delta}. \tag{4.10}$$

Taking into account that $\frac{3r-2}{r} > 1$, this gives us

$$\frac{d}{dt}|w|_2^2 + m|\nabla w|_2^2 \le |f|_q^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)}|w|_1^{2\rho(r)} mC^{2/\delta}. \tag{4.11}$$

By a calculation we can verify that

$$\rho(r) = \frac{2nr - (n-2)(2r-1)p}{2r(p(n+2)+n) - 2n(2r-1)p},$$

and that $\rho(r) \in (0,1)$.

Using the Poincaré Sobolev inequality and that $\rho(r) < 1$ in (4.11) yields

$$\frac{d}{dt}|w|_2^2 + \frac{m}{C_1(\Omega)}|w|_2^2 \le |f|_q^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)}|w|_1^2 mC^{2/\delta}, \tag{4.12}$$

where $C_1(\Omega)$ designed the Poincaré Sobolev constant. Noticing that

$$\begin{aligned} e^{-\frac{m}{C_1(\Omega)}t}\frac{d}{dt}\Big(e^{\frac{m}{C_1(\Omega)}t}|w|_2^2\Big) &= \frac{d}{dt}|w|_2^2 + \frac{m}{C_1(\Omega)}|w|_2^2 \\ &\le |f|_q^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)}|w|_1^2 mC^{2/\delta}. \end{aligned} \tag{4.13}$$

and integrating (4.13) on $[0,t)$ we obtain

$$|w(t)|_2^2 \le |w(0)|_2^2 + \|f\|_{L^\infty(0,\infty,L^q(\Omega))}^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)} m\,C^{2/\delta}|w|_1^2. \tag{4.14}$$

Since

$$|w(0)|_2^2 = \int_\Omega w(0)^2 dx = \int_\Omega u(0)^{2r} dx \le |\Omega||u(0)|_\infty^{2r} \le |\Omega|\tilde{U}_r^{2r}, \tag{4.15}$$

(4.14) and (4.15) gives us

$$\tilde{U}_{2r}^{2r} \le |\Omega|\tilde{U}_r^{2r} + \|f\|_{L^\infty(0,\infty,L^q(\Omega))}^{1/\delta}\big(\frac{2r}{m}\big)^{2r\sigma(r)} mC^{2/\delta}\tilde{U}_r^{2r}. \tag{4.16}$$

Whereas $1/\delta > 1$, $2r\sigma(r) > 0$ and $\sigma(r) = 1/(2r\delta)$ it follows that

$$\tilde{U}_{2r} \le C_2^{\sigma(r)}\|f\|_{L^\infty(0,\infty,L^q(\Omega))}^{\sigma(r)} r^{\sigma(r)}\tilde{U}_r, \tag{4.17}$$

with $C_2 = C_2(\Omega, m)$. This completes the proof of Lemma. □

Lemma 4.3. *Let $r > 1$, $n \ge 3$, $p < \frac{n}{n-2}$ and $\sigma(r) = \frac{p(n+2)}{2[r(2p-pn+n)+np]}$ then we have*

$$\sigma(2^k r) \le \theta^k \sigma(r) \quad \forall k \in \mathbb{N},$$

with $\theta \in (0,1)$.

Proof. Setting $c_1 = \frac{p(n+2)}{2}$, $c_2 = (2p - pn + n)$ and $c_3 = np$ yields $\sigma(r) = \frac{c_1}{rc_2 + c_3}$ with $c_1, c_2, c_3 \in \mathbb{R}_+^\star$. By taking $\theta = 1 - \frac{c_2}{2c_2 + c_3}$ the proof of this lemma is deduced by reasoning by induction. □

Returning now to the proof of the theorem.

Proof of Theorem 4.1. Using lemma 4.2 we have

$$\tilde{U}_{2r} \leq [C_2 \|f\|_{L^\infty(0,\infty,L^q(\Omega))}]^{\sigma(r)} r^{\sigma(r)} \tilde{U}_r.$$

By iterating this equation and taking $r = h, r = 2h, r = 2^2 h, etc$, we obtain

$$\tilde{U}_{2^{k+1}h} \leq [C_2 \|f\|_{L^\infty(0,\infty,L^q(\Omega))}]^{\lambda 1} 2^{\lambda 2} h^{\lambda 1} \tilde{U}_h,$$

with

$$\lambda_1 := \sigma(h) + \sigma(2h) + \sigma(2^2 h) + \cdots + \sigma(2^{k-1} h) + \sigma(2^k r),$$
$$\lambda_2 := \sigma(2h) + 2\sigma(2^2 h) + 3\sigma(2^3 h) + \cdots + (k-1)\sigma(2^{k-1} h) + k\sigma(2^k r).$$

To complete the proof we just need to show that $\lambda_1, \lambda_2 < +\infty$. Indeed by lemma 4.3

$$\lambda_1 \leq \sum_{\mu=0}^{k} \alpha^\mu \sigma(h) \leq \sum_{\mu=0}^{\infty} \alpha^\mu \sigma(h) = \frac{\sigma(h)}{(1-\alpha)} < \infty.$$

Noting also that

$$\sigma(2^k h) \leq \theta^{k-1} \sigma(2h) \quad \forall k \in \mathbb{N}^\star,$$

it follows that

$$\lambda_2 \leq \sum_{\mu=1}^{k} \mu \alpha^{\mu-1} \sigma(2h) \leq \sum_{\mu=1}^{\infty} \mu \alpha^{\mu-1} \sigma(2h) = \frac{\sigma(2h)}{(1-\alpha)^2} < \infty.$$

This completes the proof of the theorem. □

4.2. **Uniform estimate in time.** We prove an estimate for u in $L^\infty(\mathbb{R}^+, H_0^1(\Omega))$.

Theorem 4.4. *Assume that $f \in L^2(\Omega)$, $g \in H^1(\Omega)$, $u_0 \in H_0^1(\Omega)$ and $a \in W^{1,\infty}(\mathbb{R})$ with $\inf_{\mathbb{R}} a > 0$. Then a solution u of (1.1) is such that $u \in L^\infty(\mathbb{R}^+, H_0^1(\Omega))$.*

Proof. Taking a spectral basis related to the Laplace operator in the Galerkin approximation (see [16]) we find that $-\Delta u$ can be regarded as test function in $L^2(0, T, L^2(\Omega))$ for all $T > 0$. By multiplying (1.1) by $-\Delta u(t)$ and integrating over Ω,

$$(u_t, -\Delta u) + (-\operatorname{div}(a(l_r(u))\nabla u), -\Delta u) = (f, -\Delta u), \tag{4.18}$$

and

$$\frac{1}{2}\frac{d}{dt}\|u\|_V^2 + (-a(l_r(u))\Delta u, -\Delta u) + (-a'(l_r(u))\nabla l_r(u).\nabla u, -\Delta u) = (f, -\Delta u). \tag{4.19}$$

Here $(.,.)$ is the usual scalar product on $L^2(\Omega)$. Taking into account

$$|\nabla l_r(u)|_2 \leq K \|g\|_{H^1(\Omega)} |\nabla u|_2, \tag{4.20}$$

where K is a constant depending of Ω. the above equality yields

$$|(-a'(l_r(u))\nabla l_r(u).\nabla u, -\Delta u)| \leq K\|g\|_{H^1(\Omega)} |a'|_\infty \|u\|_V^2 |\Delta u|_2 \tag{4.21}$$

Now from (4.21) and (4.19), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_V^2 + m|\Delta u|_2^2 - K\|g\|_{H^1(\Omega)} |a'|_\infty \|u\|_V^2 |\Delta u|_2 \leq |f|_2 |\Delta u|_2. \tag{4.22}$$

Using Young's inequality $ab \leq \frac{1}{2m} a^2 + \frac{m}{2} b^2$, we obtain

$$\frac{d}{dt}\|u\|_V^2 \leq \frac{1}{m}|f|_2^2 + \frac{1}{m}(K \|g\|_{H^1(\Omega)})^2 \|a'\|_\infty^2 \|u\|^4. \tag{4.23}$$

To apply the uniform Gronwall lemma to (4.23), we start with a small estimate. Recall that

$$\frac{d}{dt}|u|_2^2 + m\|u\|_V^2 \le \frac{1}{\lambda\, m}|f|_2^2, \tag{4.24}$$

where λ is the principal eigenvalue of the Laplacian operator with Dirichlet boundary conditions.

By integrating on $[t, t_0)$ we have

$$|u(t+t_0)|_2^2 + m\int_t^{t+t_0} \|u\|_V^2\, ds \le \int_t^{t+t_0} \frac{1}{\lambda\, m}|f|_2^2\, ds + |u(t)|_2^2, \tag{4.25}$$

and

$$\int_t^{t+t_0} \|u\|_V^2\, ds \le \frac{t_0}{\lambda\, m^2}|f|_2^2\, ds + \frac{1}{m}|u(t)|_2^2. \tag{4.26}$$

Let $\rho_0 > 0$ such that $|u(t)|_2^2 \le \rho_0^2$. By setting

$$a_1 = \frac{1}{m}c_1(\Omega)^2|a'|_\infty^2 a_3, \quad a_2 = \frac{t_0}{m}|f|_2^2, \quad a_3 = \frac{t_0\lambda}{m^2}|f|_2^2 + \frac{1}{m}\rho_0^2,$$

and using uniform Gronwall lemma to (4.23), we obtain

$$\|u(t+t_0)\|_V \le (\frac{a_3}{t_0} + a_2)\exp(a_1) \quad \forall t \ge 0, \quad t_0 > 0. \tag{4.27}$$

Hence $u \in L^\infty(t_0, +\infty, H_0^1(\Omega))$. Using (4.23) and the classical Gronwall lemma it is easy to see that $u \in L^\infty(0, t_0, H_0^1(\Omega))$. This completes the proof of the theorem. $\square$

Remark 4.5. This theorem shows us the existence of absorbing set in $H_0^1(\Omega)$. By considering $S(t)$ the semigroup associated with the equation (1.1) defined by

$$S(t) : L^2(\Omega) \to L^2(\Omega), \quad u_0 \mapsto u(t),$$

with $u(t)$ a solution of (1.1). As a result of the theorem 4.4 and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ we deduce that the semigroup $S(t)$ possesses a global attractor. Indeed it is easy to show the existence of absorbing set in $L^2(\Omega)$, the main difficulty here is to show that $S(t)$ is such that for all $B \subset L^2(\Omega)$ bounded, there exists $t_0 = t_0(B)$ such that

$$\cap_{t\ge t_0} \cup S(t)B \quad \text{is relatively compact in } L^2(\Omega). \tag{4.28}$$

This property known and that $S(t)$ is uniformly compact for t large can be proved by using theorem 4.4 and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.

4.3. **Asymptotic behaviour.** In this part we are interested in the asymptotic behaviour of a weak solutions of (1.1). Our main interest here is radial solutions. By radial solutions we mean $\tilde{u}(|x|, t) = u(x, t)$. As in the stationary case Ω is a open ball of $\mathbb{R}^n$. Remember that

$$L^2_{\rm rad}(\Omega) = \{v \in L^2(\Omega) : \exists \tilde{v} \in L^2(]0, d/2[) \text{ such that } v(x) = \tilde{v}(\|x\|)\}.$$

Not to confuse u_0, the solution to (3.1) with $r = 0$, and the initial value of (1.1), we will take u^0 the initial value of (1.1).

Theorem 4.6. *Assume that $f, g \in L^2_{\rm rad}(\Omega)$, a is a continuous function and the assumption* (1.3) *checked then* (1.1) *admits a radial solution.*

Proof. Let $w \in L^2(0, t, L^2_{\rm rad}(\Omega))$ it is clear that $l_r(w)$ is radial and also $a(l_r(w))$. Thus by (2.4) F_r maps $L^2(0, t, L^2_{\rm rad}(\Omega))$ into itself. The proof now follows by using arguments similar to those used in theorem 2.1. $\square$

Assume now

$$f, g \geq 0 \quad \text{in } \Omega, \tag{4.29}$$

$$u_0 \leq u^0 \leq u_d, \tag{4.30}$$

with u^0 the initial value to (1.1) and u_0 and u_d respectively the solution of (3.1) with $r = 0$ and of (3.1).

We can now give a stability result assuming that (1.1) admits a unique solution.

Theorem 4.7. *Assume* (4.29) *and* $f, g \in L^2_{\rm rad}(\Omega)$. *Let* u, u_d *and* u_0 *respectively the solution of* (1.1), (P_d) *and* (P_0). *If* $u_0 \leq u^0 \leq u_d$, *then*

$$u_0 \leq u \leq u_d \quad \forall t.$$

Proof. Let

$$\mathcal{S} = \{t : l(u(s)) \in [0, l_d(u_d)],\ \forall s \leq t\}. \tag{4.31}$$

It is easy to prove that $\mathcal{S}$ contains 0 (see 4.30). By setting

$$t^\star = \sup\{t : t \in \mathcal{S}\}. \tag{4.32}$$

By continuity of the mapping $t \mapsto l_d(u(t))$, we have

$$l_d(u(t^\star)) \in [0, l_d(u_d)]. \tag{4.33}$$

By using (1.1) and (3.1) we get in $\mathcal{D}(0, t^\star)$

$$\frac{d}{dt}(u_d - u, \phi) + \int_\Omega a(l_d(u))\nabla(u_d - u)\nabla\phi = -\int_\Omega (a(l_d(u_d)) - a(l_d(u)))\nabla u_d \nabla\phi \tag{4.34}$$

for all $\phi \in H^1_0(\Omega)$. Choosing $\phi = (u_d - u)^-$, (4.34) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|(u_d - u)^-|_2^2 + \int_\Omega a(l_d(u_d))|\nabla(u_d - u)^-|^2 \\ &= \int_\Omega (a(l_d(u)) - a(l_d(u_d)))\nabla u_d \nabla(u_d - u)^-. \end{aligned} \tag{4.35}$$

Since a is non-increasing $(a(l_d(u)) - a(l_d(u_d)) \geq 0$ for all $t \leq t^\star)$ hence proposition 3.3 yields

$$\int_\Omega (a(l_d(u)) - a(l_d(u_d)))\nabla u_d \nabla(u_d - u)^- \leq 0. \tag{4.36}$$

Thus

$$\frac{1}{2}\frac{d}{dt}|(u_d - u)^-|_2^2 + a(l_d(u_d))|\nabla(u_d - u)^-|_2^2 \leq 0 \tag{4.37}$$

Applying Poincarr Sobolev inequality, we have

$$\frac{1}{2}\frac{d}{dt}|(u_d - u)^-|_2^2 + C_2|(u_d - u)^-|_2^2 \leq 0, \tag{4.38}$$

this proves

$$\frac{d}{dt}\{e^{2t\,C_2}|(u_d - u)^-|_2^2\} \leq 0.$$

Moreover, $(u_d - u)^-(0) = (u_d - u^0)^- = 0$ it follows that $u_d \geq u \quad \forall t \in [0, t^\star]$. In the same way we can also prove $u_0 \leq u$ for all $t \in [0, t^\star]$. It follows that

$$u_0 \leq u \leq u_d \quad \forall t \in [0, t^\star] \tag{4.39}$$

To finish we just need to prove that $t^\star$ is very large, this is typically the case. Indeed if $t^\star < \infty$ then

$$l(u(t^\star)) = 0 \quad \text{or} \quad l_d(u_d). \tag{4.40}$$

From (4.29) and (4.39) we deduce

$$u(t^\star) = u_0 \quad \text{or} \quad u(t^\star) = u_d. \tag{4.41}$$

Due to the uniqueness of (1.1), we deduce that $t = \infty$. This shows that

$$u_0 \le u \le u_d \quad \forall t,$$

and completes the proof. □

Remark 4.8. The fact that $|u(t)|_2^2$ is not a Lyapunov function that is to say decreases in time, makes very complex the study of certain asymptotic properties of our problem. Indeed under our study it is tempting to show that for r fixed $r \in]0, d[$

$$u(t) \to u_r^1 \quad \text{in} \quad L^2(\Omega),$$

where u is the solution of (1.1) and u_r^1 the solution belonging to the stable global branch described previously. A numerical study would be a great contribution to try to carry out some of our theoretical intuitions.

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