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SPECTRAL CONCENTRATION IN STURM-LIOUVILLE EQUATIONS WITH LARGE NEGATIVE POTENTIAL

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Abstract. We consider the spectral function, $\rho_\alpha(\lambda)$, associated with the linear second-order question

$$y'' + (\lambda - q(x))y = 0 \quad \text{in } [0, \infty)$$

and the initial condition

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0, \quad \alpha \in [0, \pi).$$

in the case where $q(x) \to -\infty$ as $x \to \infty$. We obtain a representation of $\rho_0(\lambda)$ as a convergent series for $\lambda > \Lambda_0$ where Λ_0 is computable, and a bound for the points of spectral concentration.

1. Introduction

We consider the linear differential equation

$$y'' + (\lambda - q(x))y = 0 \tag{1.1}$$

on the interval $[0, \infty)$ where the potential, q, is a real-valued function of $C^3[0, \infty)$ and $q(x) \to -\infty$ as $x \to \infty$. When augmented with the boundary condition

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0 \quad \alpha \in [0, \pi) \tag{1.2}$$

Equation (1.1) leads to a self-adjoint operator on the Hilbert space $L^2[0, \infty)$ and an associated spectral function $\rho_\alpha(\lambda)$. The function $\rho_\alpha(\lambda)$, in particular $\rho_0(\lambda)$, is our primary concern here. For a detailed account of its definition we refer to [1, 3, 8].

It is known that if q satisfies

$$\int^\infty (q')^2 |q|^{-5/2} dt < \infty, \quad \int^\infty |q''|\, |q|^{-3/2} dt < \infty, \quad \int^\infty |q|^{-1/2} dt = \infty, \tag{1.3}$$

then $\rho_\alpha(\lambda)$ is absolutely continuous on $(-\infty, \infty)$. This condition is fulfilled, for example, when $q(x) = -x^c$ where $0 < c \le 2$. In this article, we derive an expression, in the form of a uniformly absolutely convergent series, for $\rho_0'(\lambda)$ in the case where λ is positive and sufficiently large. Our results hold under conditions that are somewhat more restrictive than those of (1.3). In particular, if $q(x) = -x^c$, they hold in the case $0 < c \le 1$.

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Representations of $\rho_0'(\lambda)$ have been obtained before, notably in [1] in the case when q is m-times differentiable, but these have been asymptotic results whereas ours hold for all λ greater than some Λ_0 which is, in principle, computable.

A secondary goal of this article is to establish bounds for the points of spectral concentration of (1.1), (1.3). For a discussion of spectral concentration in general we refer to [5], but a point of spectral concentration may broadly be defined as a value of λ which is a local maximum of $\rho_\alpha'(\lambda)$ and is thus a point at which $\rho_\alpha(\lambda)$ is increasing relatively rapidly. Specifically we show the existence of a $\Lambda_1 \geq \Lambda_0$ for which $\rho_0''(\lambda)$ is of one sign.

In our analysis we suppose that the parameter λ is positive and that $\lambda - q(x) > 0$ for all $x \in [0, \infty)$ and $\Lambda_0 \leq \lambda$. This can clearly be done if $q(x)$ is bounded above. Our choice of Λ_0 will be increased as necessary throughout the paper.

Our main result concerning spectral concentration is the following.

Theorem 1.1. *If $q \in C^3[0, \infty)$ satisfies*

(i) $q(x) \to -\infty$ *as* $x \to \infty$
(ii) $q(x) < 0$ *for all* $x \in [0, \infty)$
(iii) $q'(x) < 0$, $q''(x) \geq 0$, $q'''(x) \leq 0$ *for all* $x \in [0, \infty)$
(iv) $q''/|q|^{\frac{3}{2}-\varepsilon}$ *and* $(q')^2/|q|^{\frac{5}{2}-\varepsilon} \in L^1[0, \infty)$ *for some* $\varepsilon > 0$
(v) $\int_0^\infty |q(s)|^{-1/2} \int_s^\infty \frac{|q''|}{|q|^{3/2}} + \frac{(q')^2}{|q|^{5/2}} dt\, ds < \infty$
(vi) $\sup_{x \in [0,\infty)} |q'(x)| \lambda - q(x)|^2 \to 0$ *as* $\lambda \to \infty$.
(vii) $\int_0^\infty \frac{q''(t)}{(\lambda - q(t))^2} dt$ *and* $\int_0^\infty \frac{(q'(t))^2}{(\lambda - q(t))^3} dt$ *are* $o(1)$ *as* $\lambda \to \infty$

Then there exists Λ_1 such that $\rho_0(\lambda)$ has no points of spectral concentration in $[\Lambda_1, \infty)$.

We note that by writing $\lambda = (\lambda - \lambda_0) + \lambda_0$ in (1.1) condition (ii) effectively requires that q be bounded above.

Our principal tool, as in [2, 4, 6] is the connection between (1.1) and the Riccati equation

$$v' + v^2 + (\lambda - q) = 0. \tag{1.4}$$

Let $v(x, \lambda)$ be the unique complex-valued solution of (1.4) which exists for all $x \in [0, \infty)$. For $\alpha = 0$ and $\xi \in \mathbb{C}^+$, $v(x, \xi)$ is the logarithmic derivative with respect to x of the Weyl solution $u(x, \xi)$ of $y'' + (\xi - q(x))y = 0$. That is:

$$v(x, \xi) = u'(x, \xi)/u(x, \xi).$$

It follows that $v(0, \xi) = m(\xi, 0)$ where $m(\xi, 0)$ is the Dirichlet Titchmarsh-Weyl m-function. For the class of potentials considered, the solution $v(x, \xi)$ of (1.4) is continuously extendable onto the real λ-axis as $\xi = \lambda + i\varepsilon \downarrow \lambda$. It then follows that

$$\rho_0'(\lambda) = \frac{1}{\pi} \operatorname{Im}\{v(0, \lambda)\}. \tag{1.5}$$

Our strategy then is to identify a suitable solution of (1.4) which is complex-valued for λ real and suitably large. Consequently we have from (1.5) that

$$\rho_0''(\lambda) = \frac{1}{\pi} \frac{\partial}{\partial \lambda} \{\operatorname{Im} v(0, \lambda)\}. \tag{1.6}$$

2. Preliminaries

To derive our main result it is convenient to show that the conditions imposed on the potential, q, imply the existence of a function $I(x,\lambda)$ which satisfies the conclusion of the following lemma.

Lemma 2.1. *If $q(x)$ satisfies* (i)–(iv), (vi) *of Theorem 1.1, then there exists a real-valued function $I(x,\lambda)$ such that $I(x,\lambda)>0$ for $x\in[0,\infty)$, $\lambda>0$ and*

(i) $I(\cdot,\lambda)\in L^1[0,\infty)$
(ii) $\frac{I(x,\lambda)}{(\lambda-q(x))^{1/2}}$ *is a decreasing function of x for each $\lambda>0$*
(iii) $\int_0^\infty I(x,\lambda)dx\to 0$ *as $\lambda\to\infty$.*
(iv) $(\lambda-q(x))^{1/2}\big|\int_x^\infty e^{2i\int_x^t(\lambda-q(s))\,ds}\{\frac{q''(t)}{4(\lambda-q(t)^{3/2}}+\frac{5(q'(t)^2}{16(\lambda-q(t)^{5/2}}\}dt\big|\le I(x,\lambda)$.

Proof. We set

$$I(x,\lambda):=\frac{q''(x)}{2(\lambda-q(x))^{3/2}}+\frac{5(q'(x))^2}{8(\lambda-q(x))^{5/2}}$$

and show that this choice of $I(x,\lambda)$ satisfies (i)–(iv) if q satisfies the conditions of Theorem 1.1. Part (i) follows from Theorem 1.1 (iv).

Differentiation with respect to x of $(\lambda-q(x))^{-1/2}I(x,y)$ and Theorem 1.1(iii) shows each of the terms is decreasing (in x) for each λ which establishes (ii).

To see (iii) we rewrite the terms of $\int_0^\infty I(x,\lambda)\,dx$ as

$$\frac12\int_0^\infty\frac{q''(x)}{(\lambda-q(x))^{\varepsilon}(\lambda-q(x))^{3/2-\varepsilon}}dx\le\frac12\lambda^{-\varepsilon}\int_0^\infty\frac{q''(x)}{|q(x)|^{3/2-\varepsilon}}\,dx.$$

The other terms in the sum is treated similarly.

To prove (iv) we note that $\frac{q''}{4(\lambda-q)^2}+\frac{5(q')^2}{16(\lambda-q)^3}$ is decreasing so, by the Second Mean Value Theorem,

$$\begin{aligned}
&(\lambda-q(x))^{1/2}\Big|\int_x^\infty e^{2i\int_x^t(\lambda-q(s))^{1/2}ds}\big\{\frac{q''}{4(\lambda-q)^{3/2}}+\frac{5(q')^2}{16(\lambda-q)^{5/2}}\big\}\,dx\Big|\\
&=(\lambda-q(x))^{1/2}\frac12\Big|\int_x^\infty\big\{2(\lambda-q(t))^{1/2}cis\Big(2\int_x^t(\lambda-q(s))^{1/2}ds\Big)\big\}\\
&\quad\times\big\{\frac{q''}{4(\lambda-q)^2}+\frac{5(q')^2}{16(\lambda-q)^3}\big\}dt\Big|\\
&=\frac12\big\{\frac{q''(x)}{4(\lambda-q(x))^{3/2}}+\frac{5}{16}\frac{(q'(x))^2}{(\lambda-q(x))^{5/2}}\big\}\\
&\quad\times\Big|\int_{\xi_1}^\infty 2(\lambda-q)^{1/2}\cos(2\int_x^t(\lambda-q(s))^{1/2}ds\,dt\\
&\quad+i\int_{\xi_2}^\infty 2(\lambda-q)^{1/2}\sin\big(2\int_x^t\big(\lambda-q(s)\big)^{1/2}ds\,dt\Big|\\
&\le 2\big\{\frac{q''(x)}{4(\lambda-q(x))^{3/2}}+\frac{5}{16}\frac{q'(x))^2}{(\lambda-q(x))^{5/2}}\big\}=I(x,\lambda).
\end{aligned}$$

The proof is complete. □

To obtain the required complex-valued solution of the Riccati equation (1.4), we proceed as in [2, 4]. Based on the asymptotic representation established in [6], we seek a solution in the form of

$$v(x,\lambda) = i(\lambda - q(x))^{1/2} + \frac{1}{4}q'(x)(\lambda - q(x))^{-1} + \sum_{n=1}^{\infty} v_n(x,\lambda). \tag{2.1}$$

Substitution of (2.1) into (1.4) gives

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(v_n' + 2\big\{i(\lambda - q)^{1/2} + \frac{1}{4}q'(\lambda - 2)^{-1}\big\}v_n\Big) \\
&= -Q - v_1^2 - \sum_{n=3}^{\infty}\Big(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\Big),
\end{aligned}$$

where

$$Q := \frac{q''}{4(\lambda - q)} + \frac{5(q')^2}{16(\lambda - q)^2}.$$

We choose $v_1, v_2 \dots$ so that

$$\begin{gathered}
v_1' + \Big(2i(\lambda - q)^{1/2} + \frac{q'}{2(\lambda - q)}\Big)v_1 = -Q \\
v_2' + \Big(2i(\lambda - q)^{1/2} + \frac{q'}{2(\lambda - q)}\Big)v_2 = -v_1^2 \\
v_n' + \Big(2i(\lambda - q)^{1/2} + \frac{q'}{2(\lambda - q)}\Big)v_n = -v_{n-1}^2 - 2v_{n-1}\sum_{m=1}^{n-2} v_m
\end{gathered} \tag{2.2}$$

for $n = 3, 4, \dots$. The required solution to (2.1) is

$$\begin{aligned}
v_1(x,\lambda) &= (\lambda - q(x))^{1/2}\int_x^{\infty}(\lambda - q(t))^{-1/2}e^{2i\int_x^t(\lambda - q)^{1/2}ds}Q(t,\lambda)dt \\
v_2(x,\lambda) &= (\lambda - q(x))^{1/2}\int_x^{\infty}(\lambda - q(t))^{-1/2}e^{2i\int_x^t(\lambda - q)^{1/2}ds}v_1(t,\lambda)^2\,dt \\
v_n(x,\lambda) &= (\lambda - q(x))^{1/2}\int_x^{\infty}(\lambda - q(t))^{-1/2}e^{2i\int_x^t(\lambda - q)^{1/2}ds} \\
&\quad\times\Big(v_{n-1}^2 + 2\sum_{m=1}^{n-2} v_m v_{n-1}\Big)\,dt
\end{aligned} \tag{2.3}$$

for $n = 3, 4, \dots$.

Lemma 2.2. *If Λ_0 is so large that for all $\lambda \ge \Lambda_0$,*

$$9\int_0^{\infty} I(t,\lambda)\,dt \le 1$$

then for $n = 1, 2, 3, \dots$,

$$|v_n(x,\lambda)| \le I(x,\lambda)/2^{n-1} \quad \textit{for all } x \in [0,\infty).$$

Proof. We use induction on n. When $n = 1$ this is Lemma 2.1 (iv). For $n = 2$,

$$\begin{aligned}
|v_2(x,\lambda)| &\le (\lambda - q(x))^{1/2}\int_x^{\infty}(\lambda - q(t))^{-1/2}I(t,\lambda)^2\,dt \\
&\le I(x,\lambda)\int_0^{\infty} I(t,\lambda)\,dt,
\end{aligned}$$

by Lemma 2.1 (ii). If $n \geq 3$ then, by the induction hypothesis:

$$\begin{aligned}
|v_n(x,\lambda)| &\leq (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} \Big[\frac{I(t,\lambda)^2}{2^{n-2}} \sum_{m-1}^{\infty} \frac{1}{2^{m-1}}\Big]\, dt \\
&\leq (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} \frac{I(t,\lambda)}{2^{n-1}} \big[\frac{1}{2^{n-2}} + 8\big] I(t,\lambda)\, dt \\
&\leq \frac{I(x,\lambda)}{2^{n-1}} \cdot 9 \int_0^\infty I(t,\lambda)\, dt
\end{aligned}$$

and the result follows. □

The uniform, absolute convergence of $\sum_{n=1}^\infty v_n(x,\lambda)$ follows from Lemma 2.2. The uniform absolute convergence of $\sum_{n=1}^\infty v_n'(x,\lambda)$ which justifies the term differentiation used to derive the series solution, follows from the bound for the v_n obtained in Lemma 2.2 and the representation of the v_n' in (2.2). Since, for example,

$$\begin{aligned}
|v_n'(x,\lambda)| &\leq \Big(|2(\lambda - q(x)^{1/2}| + \big|\frac{q'(x)}{2(\lambda - q(x))}\big|\Big)|v_n(x,\lambda)| \\
&\quad + |v_{n-1}(x,\lambda)|^2 + 2|v_{n-1}| \sum_{m=1}^{n-2} |v_m|. \\
&\leq \Big(2|\lambda - q|^{1/2} + \big|\frac{q'}{2(\lambda - q)}\big|\Big) I(x,\lambda) \cdot \frac{1}{2^{n-1}} \\
&\quad + \Big(\frac{I(x,\lambda)}{2^{n-2}}\Big)^2 + 2\frac{I(x)^2}{2^{n-2}} \sum_{m=1}^\infty \frac{1}{2^{m-1}} \quad \text{for } n = 3, 4, \dots
\end{aligned}$$

It follows readily that $\sum |v_n'(x,\lambda)|$ is uniformly absolutely convergent for $x \in [0,\infty)$ and $\lambda > \Lambda_0$. We have proved the following result.

Theorem 2.3. *Let q satisfy the conditions of Theorem 1.1. If Λ_0 is so large that for all $\lambda \geq \Lambda_0 > 0$, $9\int_0^\infty I(t,\lambda)\,dt \leq 1$ and $(\lambda - q(x) > 0$ for all $x \in [0,\infty)$ then*

$$\rho_0'(\lambda) = \frac{1}{\pi}(\lambda - q(0))^{1/2} + \frac{1}{\pi} \sum_{n=1}^\infty Im(v_n(0,\lambda)).$$

for all $\lambda > \Lambda_0$.

3. Spectral Concentration

We seek the second derivative of $\rho_0(\lambda)$. Our strategy is to differentiate the equations of (2.2) with respect to λ, justify the equality of the mixed second order partial derivatives and derive expressions for $\frac{\partial v_n}{\partial \lambda}$ akin to (2.3) which we then bound as in Lemma 2.2.

Differentiating the first equation of (2.2) with respect to λ gives

$$\frac{\partial^2 v_1}{\partial\lambda\partial x} + \Big(2i(\lambda - q)^{1/2} + \frac{q'}{2(\lambda - q)}\Big)\frac{\partial v_1}{\partial \lambda} = -\frac{\partial Q}{\partial \lambda} - i(\lambda - q)^{-1/2} v_1 + \frac{1}{2} q'(\lambda - q)^{-2} v_1 \quad (3.1)$$

We note from (2.3) that $v_1(x,\lambda)$ is continuous and so, by (2.2), is $\frac{\partial v_1}{\partial x}$. It remains to show that $\frac{\partial v_1}{\partial \lambda}$ is continuous. We do this by differentiating the first equation of

(2.3) under the integral sign to obtain

$$
\begin{aligned}
\frac{\partial v_1}{\partial \lambda} &= \frac{1}{2}(\lambda - q(x))^{-1/2} \int_x^\infty e^{2i\int_x^t (\lambda-q(s))^{1/2}ds} \big\{ \frac{q''}{4(\lambda-q)^{3/2}} + \frac{5(q')^2}{16(\lambda-q)^{5/2}} \big\} dt \\
&\quad + (\lambda - q(x))^{1/2} \int_x^\infty 2i \Big(\int_x^t (\lambda - q(s))^{-1/2}\, ds \Big) e^{2i\int_x^t (\lambda-q(s))^{1/2}ds} \\
&\quad \times \Big\{ \frac{q''}{4(\lambda-q)^{3/2}} + \frac{5(q')^2}{16(\lambda-q)^{5/2}} \Big\}\, dt \\
&\quad + (\lambda - q(x))^{1/2} \int_x^\infty e^{2i\int_x^t (\lambda-q(s))^{1/2}\, ds} \\
&\quad \times \big\{ -\frac{3q''}{8}(\lambda-q)^{-5/2} - \frac{25}{32}(q')^2(\lambda-q)^{-7/2} \big\} dt
\end{aligned}
\tag{3.2}
$$

providing that the differentiation under the integral sign is justified. To ensure that it is, we note that under the conditions of Theorem 1.1, the integrand in the expression for $v_1(x,\lambda)$ in (2.3) is continuously differentiable with respect to λ, and that each term of the integrand in (3.2) is integrable with respect to $t \in \mathbb{R}^+$; to see this in the case of the second term, note that by a change in the order of integration

$$
\begin{aligned}
&\Big| \int_x^\infty \Big(\int_x^t (\lambda - q(s))^{-1/2}\, ds \Big) e^{2i\int_x^t (\lambda-q(s))^{1/2}ds} \big\{ \frac{q''}{4(\lambda-q)^{3/2}} + \frac{5(q')^2}{16(\lambda-q)^{5/2}} \big\} dt \Big| \\
&\le \int_x^\infty (\lambda - q(s))^{-1/2} \int_s^\infty \frac{q''}{4(\lambda-q)^{3/2}} + \frac{5(q')^2}{16(\lambda-q)^{5/2}}\, dt\, ds.
\end{aligned}
$$

It now follows from (3.2) that $\frac{\partial v_1}{\partial \lambda}$ is continuous in x and λ, so the equality of the mixed partial derivatives is established. We may therefore replace $\frac{\partial^2 v_1}{\partial\lambda\partial x}$ by $\frac{\partial^2 v_1}{\partial x\partial\lambda}$ in (3.1), then integrate with respect to x to obtain a more suitable representation of $\frac{\partial v_1}{\partial \lambda}$. This yields

$$
\begin{aligned}
\frac{\partial v_1}{\partial \lambda}(x,\lambda) &= (\lambda - q(x))^{1/2} \int_x^\infty (\lambda-q)^{-1/2} e^{2i\int_x^t (\lambda-q(s))^{1/2}ds} \big\{ \big[-\frac{q''}{4}(\lambda-q)^{-2} \\
&\quad - 5/8(q')^2(\lambda-q)^{-3} \big] + [-i(\lambda-q)^{-1/2} v_1] + [\frac{q'}{2}(\lambda-q)^{-2} v_1] \big\} dt \\
&=: I_1 + I_2 + I_3
\end{aligned}
\tag{3.3}
$$

This provides a convenient first step for an iterative scheme to establish upper bounds on $\big|\frac{\partial v_n}{\partial \lambda}\big|$ for $x \ge 0$ and λ sufficiently large. To this end we note that

$$
\begin{gathered}
\big| \frac{q''}{4}(\lambda-q)^{-2} + 5/8(q')^2(\lambda-q)^{-3} \big| \le (\lambda-q)^{-1/2} I(x,\lambda) \text{ so} \\
|I_1| \le \Big(\sup_{x\in[0,\infty)} (\lambda - q(x))^{-1/2} \Big) (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} I(t,\lambda) dt
\end{gathered}
$$

Also,

$$
\begin{aligned}
|I_2| &\le (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1} I(t,\lambda)\, dt \\
&\le \Big(\sup_{x\in[0,\infty)} (\lambda - q(x))^{-1/2} \Big) (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt
\end{aligned}
$$

and

$$|I_3| \le \frac{1}{2}\Big(\sup_{x\in[0,\infty)} |q''(x)|(\lambda - q(x))^{-2}\Big)(\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)dt.$$

It follows that

$$\begin{aligned} \big|\frac{\partial v_1}{\partial\lambda}(x,\lambda)\big| &\le \big\{2 \sup_{x\in[0,\infty)} (\lambda - q(x))^{-1/2} + \frac{1}{2} \sup_{x\in[0,\infty)} |q'(x)(\lambda - q(x))^{-2}|\big\} \\ &\quad \times (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt. \end{aligned} \tag{3.4}$$

Lemma 3.1. *If $\Lambda_1 \ge \Lambda_0 > 0$ is so large that for all $\lambda \ge \Lambda_1$,*

$$16 \int_0^\infty I(t,\lambda)dt + 2 \sup_{x\in[0,\infty)} (\lambda - q(x))^{-1/2} + \frac{1}{2} \sup_{x\in[0,\infty} |q'(x)(\lambda - q(x))^{-2}| \le 1,$$

then for $x \in [0,\infty), \lambda > \Lambda$, and $n = 1,2,3,\dots$,

$$\big|\frac{\partial v_n}{\partial\lambda}(x,\lambda)\big| \le \frac{1}{2^{n-1}}(\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt\,. \tag{3.5}$$

Proof. We use induction on n to prove the hypothesis: $\frac{\partial v_n}{\partial\lambda}(x,\lambda)$ is continuous in x and λ, for $x \in [0,\infty)$, $\lambda > \Lambda$ and inequality (3.4) holds.

The case $n = 1$ follows from (3.4) since the hypothesis of the lemma implies the asserted bound. The case $n = 2$ will follow from the general case, the difference being that some of the series terms are vacuous.

In the general case, suppose the induction hypothesis holds for $\frac{\partial v_1}{\partial\lambda}, \dots, \frac{\partial v_{n-1}}{\partial\lambda}$. As in the case for $\frac{\partial v_1}{\partial\lambda}$, we differentiate (2.2) with respect to λ, show the equality of the mixed second order derivatives and obtain an integral representation for $\frac{\partial v_n}{\partial\lambda}$ which we bound.

The function $\frac{\partial v_n}{\partial\lambda}$ is continuous from (2.2) and, differentiating (2.2) with respect to λ shows that $\frac{\partial^2 v_n}{\partial\lambda\partial x}$ is continuous if $\frac{\partial v_n}{\partial\lambda}$ is. This we now show by differentiating (2.3) with respect to λ under the integral.

$$\begin{aligned} \frac{\partial v_n}{\partial\lambda} &= \frac{1}{2}(\lambda - q(x))^{-1} v_n - \frac{1}{2}(\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q)^{-3/2} e^{ei\int_x^t (\lambda-q)^{1/2} ds} \\ &\quad \times \Big(v_{n-1}^2 + 2\sum_{m=1}^{n-2} v_m v_{n-1}\Big)\, dt \\ &\quad + (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q)^{-1/2} \big\{i \int_x^t (\lambda - q(s))^{-1/2} ds\big\} e^{2i\int_x^t (\lambda-q)^{1/2} ds} \\ &\quad \times \Big(v_{n-1}^2 + 2\sum_{m=1}^{n-2} v_m v_{n-1}\Big) dt \\ &\quad + (\lambda - q(x))^{1/2} \int_x^\infty (x - q)^{-1/2} e^{2i\int_x^t (\lambda-q)^{1/2} ds} \\ &\quad \times \frac{\partial}{\partial\lambda}\Big(v_{n-1}^2 + 2\sum_{m=1}^{n-2} v_m v_{n-1}\Big)\, dt. \end{aligned} \tag{3.6}$$

The continuity of all but the third term is clear. This consists of a sum of terms which are

$$\begin{aligned}
&O\Big((\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(t))^{-1/2}\big\{\int_x^t(\lambda-q(s))^{-1/2}\,ds\big\}I(t,\lambda)^2\,dt\Big)\\
&=O\Big((\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(s))^{-1/2}\int_s^\infty(\lambda-q(t))^{-1/2}I(t,\lambda)^2\,dt\,ds\Big)\\
&=O\Big((\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(s))^{-1}I(s,\lambda)\int_0^\infty I(t,\lambda)\,dt\,ds\Big).
\end{aligned}$$

By Lemma 2.1 (i) and (ii), the continuity of the third term follows.

By the induction hypothesis the fourth term consists of a sum of terms each of which is

$$O\Big((\lambda-q(x))^{1/2}\int_x^\infty I(t,\lambda)\int_t^\infty(\lambda-q(s))^{-1/2}I(s,\lambda)\,ds\,dt\Big)$$

and so is bounded by Lemma 2.1 (i).

The continuity of $\frac{\partial v_n}{\partial\lambda}$, and hence of $\frac{\partial^2 v_n}{\partial\lambda\partial x}$ now follows and, by the equality of the second order mixed partial derivatives, we have from (2.2):

$$\begin{aligned}
\frac{\partial v_n}{\partial\lambda}&=(\lambda-q(x))^{1/2}\int_x^\infty e^{2i\int_x^t(\lambda-q(s))^{1/2}\,ds}(\lambda-q(t))^{-1/2}\\
&\quad\times\Big\{i(\lambda-q)^{-1/2}v_n-\frac{q'}{2(\lambda-q)^2}v_n+2\sum_{m=1}^{n-1}\frac{\partial v_{n-1}}{\partial\lambda}v_m+2\sum_{m=1}^{n-2}v_{n-1}\frac{\partial v_m}{\partial\lambda}\Big\}\,dt\\
&=:I_1+\dots+I_4
\end{aligned}\tag{3.7}$$

From Lemma 2.2,

$$\begin{aligned}
|I_1|&\le\frac{1}{2^{n-1}}(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(t))^{-1}I(t,\lambda)\,dt\\
&\le\frac{1}{2^{n-1}}\big\{\sup_{0\le x<\infty}(\lambda-q(x))^{-1/2}\big\}(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(t))^{-1/2}I(t,\lambda)\,dt.
\end{aligned}$$

$$\begin{aligned}
|I_2|&\le\frac{1}{2^{n-1}}(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(t))^{-1/2}\Big|\frac{q'(t)}{2(\lambda-q(t))^2}\Big|I(t,\lambda)\,dt\\
&\le\frac{1}{2^{n-1}}\sup_{0\le x<\infty}\Big|\frac{q'(x)}{2(\lambda-q(x))^2}\Big|(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(t))^{-1/2}I(t,\lambda)\,dt.
\end{aligned}$$

$$\begin{aligned}
|I_3|&\le 2(\lambda-q(x))^{1/2}\int_x^\infty\Big(\int_t^\infty(\lambda-q(s))^{-1/2}\frac{I(s,\lambda)}{2^{n-2}}\,ds\Big)I(t,\lambda)\sum_{m=1}^{n-1}\frac{1}{2^{m-1}}\,dt\\
&\le\frac{1}{2^{n-1}}(\lambda-q(x))^{1/2}8\int_x^\infty I(t,\lambda)\int_t^\infty(\lambda-q(s))^{-1/2}I(s,\lambda)\,ds\,dt\\
&=\frac{8}{2^{n-1}}(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(s))^{-1/2}I(s,\lambda)\int_x^s I(t,\lambda)\,dt\,ds\\
&\le\frac{1}{2^{n-1}}(\lambda-q(x))^{1/2}\int_x^\infty(\lambda-q(s))^{-1/2}I(s,\lambda)\,ds\big\{8\int_0^\infty I(t,\lambda)\,dt\big\}.
\end{aligned}$$

$$
\begin{aligned}
|I_4| &\le 2(\lambda - q(x))^{1/2} \int_x^\infty \frac{I(t,\lambda)}{2^{n-2}} \int_t^\infty (\lambda - q(s))^{-1/2} I(s,\lambda) \sum_{m=1}^{n-2} \frac{1}{2^{m-1}}\, ds\, dt \\
&\le \frac{8}{2^{n-1}} (\lambda - q(x))^{1/2} \int_x^\infty I(t,\lambda) \int_t^\infty (\lambda - q(s))^{-1/2} I(s,\lambda)\, ds\, dt \\
&\le \frac{1}{2^{n-1}} (\lambda - q(x))^{1/2} \int_x^\infty (\lambda - q(s))^{-1/2} I(s,\lambda)\, ds \big\{ 8 \int_0^\infty I(t,\lambda)\, dt \big\}.
\end{aligned}
$$

The result now follows since for $\lambda \ge \Lambda_1$,

$$
16 \int_0^\infty I(t,\lambda)\, dt + \sup_{0 \le x < \infty} \big| \frac{q'(x)}{2(\lambda - q(x))^2} \big| + \sup_{0 \le x < \infty} |\lambda - q(x)|^{-1/2} \le 1.
$$

□

4. Proof of Theorem 1.1

If q satisfies the conditions of Theorem 1.1 then there exists a function $I(x,\lambda)$ satisfying the conclusions of Lemma 2.1 and hence Lemmas 2.2 and 3.1. Thus, for $\lambda > \Lambda_1$ the following representation of $\rho''(\lambda)$ holds

$$
\rho''(\lambda) = \frac{1}{2\pi} (\lambda - q(0))^{-1/2} + \frac{1}{\pi} \sum_{n=1}^\infty \frac{\partial}{\partial \lambda} Im(v_n(0,\lambda))
$$

and

$$
\begin{aligned}
\big| \rho''(\lambda) - \frac{1}{2\pi} (\lambda - q(0))^{-1/2} \big| &\le \frac{1}{\pi} \sum_{n=1}^\infty \big| \frac{\partial}{\partial \lambda} v_n(0,\lambda) \big| \\
&\le \frac{2}{\pi} (\lambda - q(0))^{1/2} \int_0^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt.
\end{aligned}
$$

Thus $\rho''(\lambda) > 0$ if λ is so large that

$$
4(\lambda - q(0)) \int_0^\infty (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt < 1.
$$

With the function $I(t,\lambda)$ from Lemma 2.1 this is satisfied if, in addition to the requirements of Lemmas 2.2 and 3.1, λ is so large that

$$
2(\lambda - q(0)) \int_0^\infty \frac{q''(t)}{(\lambda - q(t))^2} + \frac{5(q'(t))^2}{4(\lambda - q(t))^3}\, dt < 1. \tag{4.1}
$$

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References

[1] M. S. P. Eastham; *The asymptotic form of the spectral function in Sturm-Liouville problems with a large potential like $-x^c$ ($c \le 2$).* Proc. Royal Soc. Edinburgh 128A (1998), 37-45.

[2] D. J. Gilbert, B. J. Harris; *Bounds for the points of spectral concentration of Sturm-Liouville problems.* Mathematika, 47 (2000), 327-337.

[3] D. J. Gilbert, B. J. Harris; *Connection formulae for spectral functions associated with singular Sturm-Liouville expressions.* Proc. Royal Soc. Edinburgh, 130A (2000), 25-34.

[4] D. J. Gilbert, B. J. Harris, S. M. Riehl; *The spectral function for Sturm-Liouville equations where the potential is of Wigner-von Neumann type or slowly decaying*. Journal of Differential Equations 201, (2004), 139-159.
[5] D. J. Gilbert, B. J. Harris, S. M. Riehl; *Higher derivatives of spectral functions associated with one-dimensional Schrödinger operators*. Operator Theory: Advances and Applications. 186 (2008), 217-228.
[6] B. J. Harris; *The form of the spectral functions associated with Sturm-Liouville equations with large negative potential*. J. Math Analysis and Appl. 222 (1998), 569-584.
[7] J. C. Kallenbach; *Ph. D. Dissertation* Northern Illinois University 2007.
[8] E. C. Titchmarsh; *Eigenfunction Expansions, part 1*. Clarendon Press: Oxford, 2nd ed. 1962.

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