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WEIGHTED PSEUDO-ALMOST PERIODIC SOLUTIONS FOR SOME NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

KHALIL EZZINBI, SAMIR FATAJOU, GASTON M. N'GUÉREKATA

Abstract. In this work, we give sufficient conditions for the existence and uniqueness of a weighted pseudo-almost periodic solutions for some neutral partial functional differential equations. Our working tools are based on the variation of constant formula and the spectral decomposition of the phase space. To illustrate our main result, we propose to study the existence and uniqueness of a weighted pseudo-almost periodic solution for some neutral model arising in physical systems.

1. Introduction

The aim of this work is to study the existence and uniqueness of a weighted pseudo-almost periodic solutions for the following neutral partial functional differential equation

$$\frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where A is a linear operator on a Banach space X satisfying the well-known Hille-Yosida condition:

(H0) There exist $\bar{M} \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{\bar{M}}{(\lambda-\omega)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$.

$\mathcal{D} : C \to X$ is a bounded linear operator, where $C = C([-r, 0]; X)$ is the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. For the well posedness of (1.1), we assume that $\mathcal{D}$ has the form

$$\mathcal{D}(\varphi) = \varphi(0) - \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) \quad \text{for } \varphi \in C,$$

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for a mapping $\eta : [-r, 0] \to \mathcal{L}(X)$ of bounded variation and non atomic at zero, which means that there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\Big| \int_{-s}^{0} [d\eta(\theta)]\varphi(\theta) \Big| \leq \delta(s) \sup_{-r \leq \theta \leq 0} |\varphi(\theta)| \quad \text{for } \varphi \in C \text{ and } s \in [0, r],$$

where $\mathcal{L}(X)$ is the space of bounded linear operators from X to X. For every $t \in \mathbb{R}$, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in [-r, 0].$$

L is a bounded linear operator from C to X and the input function f is weighted pseudo-almost periodic from $\mathbb{R}$ to X. Partial neutral functional differential equations becomes now an interesting field in dynamical systems and have many applications in physical systems. In [27] and [28], the authors proposed and studied a system of partial neutral functional differential-difference equations defined on the unit circle S, which models a continuous circular array of resistively coupled transmission lines, the system is given by

$$\frac{\partial}{\partial t}[x(\xi, t) - qx(\xi, t - r)] = k \frac{\partial^2}{\partial \xi^2}[x(\xi, t) - qx(\xi, t - r)] + f(x_t(\xi, .)) \tag{1.2}$$

for $t \geq 0$, where $\xi \in S^1$ $x_t(\xi, \theta) = x_t(\xi, t + \theta)$, $-r \leq \theta \leq 0$, $t \geq 0$, k is a positive constant, f is a continuous function and $0 \leq q \leq 1$. In [22] and [23], the author investigated many interesting properties for the model (1.2). In [1], [2], [3] and [5], the authors considered a more general neutral partial functional differential equation of the form

$$\begin{gathered} \frac{d}{dt} D(x_t) = AD(x_t) + F(x_t) \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C. \end{gathered} \tag{1.3}$$

where A is a nondensely defined linear operator that satisfies the Hile-Yosida condition in a Banach space X, F is a continuous function from C into E, the authors gave many fundamental results on the behavior of solutions.

The principal working tools in this work is the variation of the constants formula and the spectral decomposition of the phase space. In [5], the authors developed a new variation of constants formula for neutral partial functional differential equations and gave many applications. More details about the problem of variation of constants formula in the context of delay differential equations and partial functional differential equations can be found in the following books [18], [24], [25] and [26]. Recall, in [18] the authors developed a new theory about the sun star reflexivity in order to obtain a variation of constants formula for delay differential equations in finite dimensional spaces and gave many important applications in the asymptotic behavior of solutions.

The organization of this work is as follows: In section 2, we give the framework of the pseudo-almost periodic functions.In section 3, we recall the variation of constants formula and the spectral decomposition that will be used in the whole of this work. In section 4, we prove the existence and uniqueness of a pseudo-almost periodic solution for equation (1.1) when the linear homogeneous equation has an exponential dichotomy. Finally, for illustration, we propose to study the existence and uniqueness of a pseudo-almost periodic solution for the model (1.2).

2. Weighted pseudo-almost periodic functions

In what follows we recall some definitions and notations needed in the sequel. Let X be a Banach space and $L^1_{\rm loc}(\mathbb{R})$ denotes the space of locally integrable scalar functions on $\mathbb{R}$. Let $\mathbb{U}$ be defined by

$$\mathbb{U} := \{\rho \in L^1_{\rm loc}(\mathbb{R}) : \rho(x) > 0 \quad \text{almost everywhere } x \in \mathbb{R}\}.$$

For $\rho \in \mathbb{U}$ and $R > 0$, we set

$$m(R, \rho) := \int_{-R}^{R} \rho(x)\, dx.$$

The space of weighted functions is defined by

$$\mathbb{U}_\infty := \big\{\rho \in \mathbb{U} : \lim_{R\to\infty} m(R,\rho) = \infty\big\}.$$

$$\mathbb{U}_B := \big\{\rho \in \mathbb{U}_\infty : \rho \text{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\big\}.$$

Throughout this work $BC(\mathbb{R}, X)$ stands for the space of all X-valued bounded continuous functions equipped with the sup norm defined by $|\phi|_\infty := \sup_{t\in\mathbb{R}} |\phi(t)|$.

Definition 2.1. [21] A continuous function $f : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$ there exists an $l(\varepsilon) > 0$, such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that $|f(t + \tau) - f(t)| < \varepsilon$ for all $t \in \mathbb{R}$.

This number τ is called ε-translation number of f.

Let $AP(X)$ denote the space of almost periodic functions. Denote by $PAP_0(X)$ the space of ergodic perturbations defined by

$$PAP_0(X) := \big\{f \in BC(\mathbb{R}, X) : \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} |f(t)|\, dt = 0\big\}.$$

Definition 2.2. [14] A function $f : \mathbb{R} \to X$ is called pseudo-almost periodic if $f = g + \phi$, where $g \in AP(X)$ and $\phi \in PAP_0(X)$.

The collection of all pseudo-almost periodic functions from $\mathbb{R}$ into X is denoted by $PAP(X)$.

Definition 2.3. [12] Let $\rho \in \mathbb{U}_\infty$. We define the weighted ergodic space by

$$PAP_0(X, \rho) := \big\{f \in BC(\mathbb{R}, X) : \lim_{R\to\infty} \frac{1}{m(R,\rho)} \int_{-R}^{R} |f(t)|\rho(t)\, dt = 0\big\}.$$

Definition 2.4 ([12])**.** Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \to X$ is called weighted-pseudo-almost periodic (or ρ-pseudo-almost periodic) if it is decomposed as

$$f = g + \phi,$$

where $g \in AP(X)$ and $\phi \in PAP_0(X, \rho)$.

The collection of all weighted-pseudo-almost periodic functions from $\mathbb{R}$ into X is denoted by $PAP(X, \rho)$.

Remark 2.5. [12] The functions g and ϕ appearing in definition 2.4 are respectively called the almost periodic and the weighted ergodic components of f.

Definition 2.6. [12] Let Y be a Banach space. A function $F : \mathbb{R} \times Y \to X$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $y \in Y$ if for each $\varepsilon > 0$ and any compact $K \subset Y$ there exists an $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$|F(t+\tau, y) - F(t, y)| < \varepsilon \quad \text{for each } t \in \mathbb{R} \text{ and } y \in K.$$

The collection of those functions is denoted by $AP(Y, X)$. In the same way, we define $PAP_0(Y, X, \rho)$ as the collection of jointly continuous functions $F : \mathbb{R} \times Y \to X$ such that $F(\cdot, y)$ is bounded and

$$\lim_{R \to \infty} \frac{1}{m(R, \rho)} \int_{-R}^{R} |F(s, y)| \rho(s)\, ds = 0,$$

uniformly with respect to y in each compact subset of Y.

Definition 2.7. [12] A function $F : \mathbb{R} \times Y \to X$ is called weighted pseudo-almost periodic in t with respect to the second argument if

$$F = G + H,$$

where $G \in AP(Y, X)$ and $H \in PAP_0(Y, X, \rho)$.

The class of such functions is denoted by $PAP(Y, X, \rho)$ We give now some properties of a weighted pseudo-almost periodic functions.

Definition 2.8. [12] Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. One says that ρ_1 is equivalent to ρ_2 or $\rho_1 \sim \rho_2$ whenever $\frac{\rho_1}{\rho_2} \in \mathbb{U}_B$.

Theorem 2.9 ([12])**.** *If $\rho_1, \rho_2 \in \mathbb{U}_\infty$, and ρ_1 is equivalent to ρ_2, then $PAP(X, \rho_1)$ $= PAP(X, \rho_2)$.*

An immediate consequence of Theorem 2.9 is the next corollary, which enables us to connect the Zhang's space $PAP(X) = AP(X) \oplus PAP_0(X)$ with a weighted pseudo-almost periodic class $PAP(X, \rho)$.

Corllary 2.10 ([12])**.** *If $\rho \in \mathbb{U}_{\mathbb{B}}$, then $PAP(X, \rho) = PAP(X)$.*

3. Variation of constants formula and spectral decomposition

In the sequel, we assume that the operator A satisfies the Hille-Yosida condition (H0). To equation (1.1), we associate the Cauchy problem

$$\begin{gathered} l\frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + L(u_t) + f(t) \text{ for } t \geq \sigma, \\ u_\sigma = \varphi \in C. \end{gathered} \tag{3.1}$$

Definition 3.1. [5] A continuous function $u : [-r+\sigma, +\infty) \to X$ is called an integral solution of (3.1), if

(i) $\int_\sigma^t \mathcal{D}(u_s) ds \in D(A)$ for $t \geq \sigma$,
(ii) $\mathcal{D}(u_t) = \mathcal{D}(\varphi) + A \int_\sigma^t \mathcal{D}(u_s) ds + \int_\sigma^t [L(u_s) + f(s)] ds$ for $t \geq \sigma$,
(iii) $u_\sigma = \varphi$.

If u is an integral solution of (3.1), then from the continuity of u, we have $\mathcal{D}(u_t) \in \overline{D(A)}$ for all $t \geq \sigma$. In particular, $\mathcal{D}(\varphi) \in \overline{D(A)}$.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ defined by

$$\begin{gathered} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax \text{ for } x \in D(A_0). \end{gathered}$$

Lemma 3.2. [5] *Assume that* $(\mathbf{H}_0)$ *holds. Then* A_0 *generates a strongly continuous semigroup* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

For the existence of the integral solutions, one has the following results.

Theorem 3.3 ([5])**.** *Assume that (H0) holds. Then, for all* $\varphi \in C$ *such that* $\mathcal{D}(\varphi) \in \overline{D(A)}$, *equation* (3.1) *has a unique integral solution* u *on* $[-r+\sigma, +\infty)$. *Moreover,* u *is given by*

$$\mathcal{D}(u_t) = T_0(t-\sigma)\mathcal{D}(\varphi) + \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)B_\lambda[L(u_s)+f(s)]ds \quad \textit{for } t \geq \sigma,$$

where $B_\lambda = \lambda R(\lambda, A)$ *for* $\lambda > \omega$.

In the sequel, $u(., \sigma, \varphi, f)$ denotes the integral solution of (3.1). The phase space C_0 of equation (3.1) is given by

$$C_0 = \{\varphi \in C : \mathcal{D}(\varphi) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $U(t)$ on C_0 by

$$U(t)\varphi = v_t(., \varphi),$$

where $v(., \varphi)$ is the integral solution of the homogeneous equation

$$\begin{gathered} \frac{d}{dt}\mathcal{D}(v_t) = A\mathcal{D}(v_t) + L(v_t) \quad \text{for } t \geq 0, \\ v_0 = \varphi \in C. \end{gathered} \tag{3.2}$$

We have the following result.

Proposition 3.4 ([5])**.** *Assume that* (H0) *holds. Then* $(U(t))_{t\geq 0}$ *is a strongly continuous semigroup on* C_0. *Moreover, the operator* $\mathcal{A}$ *defined on* C_0 *by*

$$\begin{gathered} D(\mathcal{A}) = \big\{\varphi \in C^1([-r,0];X) : \mathcal{D}(\varphi) \in D(A), \mathcal{D}(\varphi') \in \overline{D(A)} \\ \textit{and } \mathcal{D}(\varphi') = A\mathcal{D}(\varphi) + L(\varphi)\big\}, \\ \mathcal{A}\varphi = \varphi', \end{gathered}$$

is the infinitesimal generator of $(U(t))_{t\geq 0}$ *on* C_0.

To determine the asymptotic behavior of the semigroup $(U(t))_{t\geq 0}$, we need to introduce some preliminary results. In neutral system, many fundamental properties depend essentially on the choice of the difference operator $\mathcal{D}$. Here we suppose that $\mathcal{D}$ is stable in the sense given in literature, more details can be found in [22, 25, 26].

Definition 3.5 ([22, 25])**.** The operator $\mathcal{D}$ is said to be stable if there exist positive constants η and μ such that the solution of the following homogenous difference equation

$$\begin{gathered} \mathcal{D}(u_t) = 0 \quad \text{for } t \geq 0, \\ u_0 = \varphi, \end{gathered}$$

where $\varphi \in \{\psi \in C : \mathcal{D}(\psi) = 0\}$, satisfies

$$|u_t(., \varphi)| \leq \mu e^{-\eta t}|\varphi| \quad \text{for } t \geq 0.$$

As an example, the operator $\mathcal{D}$ defined by

$$\mathcal{D}(\varphi) = \varphi(0) - q\varphi(-r)$$

is stable if and only if $|q| < 1$.

In the following, we assume

- (H1) The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$ whenever $t > 0$.
- (H2) The operator $\mathcal{D}$ is stable.

Then, we have the following fundamental result on the semigroup $(U(t))_{t\geq 0}$.

Theorem 3.6 ([5, Lemma 10]). *Assume that* (H0)–H(2) *hold. Then the semigroup* $(U(t))_{t\geq 0}$ *is decomposed on* C_0 *as follows*

$$U(t) = U_1(t) + U_2(t) \quad \text{for } t \geq 0,$$

where $(U_1(t))_{t\geq 0}$ *is an exponentially stable semigroup on* C_0*, which means that there are positive constants* γ_0 *and* N_0 *such that*

$$|U_1(t)\varphi| \leq N_0 e^{-\gamma_0 t}|\varphi| \quad \text{for } t \geq 0 \text{ and } \varphi \in C_0.$$

Moreover $U_2(t)$ *is compact for every* $t > 0$.

We introduce the Kuratowski's measure of noncompactness $\alpha(\cdot)$ of bounded sets K in a Banach space Y by

$$\alpha(K) = \inf\{\varepsilon > 0 : K \text{ has a finite cover of balls of diameter} < \varepsilon\}.$$

For a bounded linear operator B on Y, $|B|_\alpha$ is defined by

$$|B|_\alpha = \inf\{\varepsilon > 0 : \alpha(B(K)) \leq \varepsilon\alpha(K) \text{ for any bounded set } K \text{ of } Y\}.$$

The essential growth bound $\omega_{\text{ess}}(U)$ of the semigroup $(U(t))_{t\geq 0}$ is defined by

$$\omega_{\text{ess}}(U) = \lim_{t\to+\infty} \frac{1}{t}\log|U(t)|_\alpha = \inf_{t>0}\frac{1}{t}\log|U(t)|_\alpha.$$

Consequently form Theorem 3.6, we obtain the following interesting results that will be used for the spectral decomposition.

Corllary 3.7. *Assume that* (H0)–(H2) *hold. Then* $\omega_{\text{ess}}(U) < 0$.

Definition 3.8. Let $\mathcal{C}$ be a densely defined operator on a Banach space Y. The essential spectrum $\sigma_{\text{ess}}(\mathcal{C})$ of $\mathcal{C}$ is the set of λ in the spectrum $\sigma(\mathcal{C})$ of $\mathcal{C}$, such that one of the following conditions holds:

- (i) $\operatorname{Im}(\lambda I - \mathcal{C})$ is not closed,
- (ii) the generalized eigenspace $M_\lambda(\mathcal{C}) = \cup_{k\geq 1}\ker(\lambda I - \mathcal{C})^k$ is of infinite dimension,
- (iii) λ is a limit point of $\sigma(\mathcal{C}) \setminus \{\lambda\}$.

The essential radius of any bounded operator $\mathcal{T}$ in Y is defined by

$$r_{\text{ess}}(\mathcal{T}) = \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(\mathcal{T})\}.$$

Lemma 3.9 ([4]). *Assume that* (H0)–(H2) *hold. Then*

$$\sigma^+(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re}(\lambda) \geq 0\}$$

is a finite set of the eigenvalues of $\mathcal{A}$ *which are not in the essential spectrum. More precisely,* $\lambda \in \sigma^+(\mathcal{A})$ *if and only if there exists* $a \in D(A)\setminus\{0\}$ *solving the characteristic equation*

$$\Delta(\lambda)a = \lambda\mathcal{D}(e^{\lambda\cdot}a) - A\mathcal{D}(e^{\lambda\cdot}a) - L(e^{\lambda\cdot}a) = 0.$$

Definition 3.10. The semigroup $(U(t))_{t\geq 0}$ is said to have an exponential dichotomy if
$$\sigma(\mathcal{A})\cap i\mathbb{R}=\emptyset.$$

Since (H0), (H1) and (H2) hold, by Corollary 3.7 we have $\omega_{\rm ess}(U)<0$. Consequently, we get the following result on the spectral decomposition of C_0.

Theorem 3.11 ([4])**.** *Assume that* (H0)–(H2) *hold. If the semigroup $(U(t))_{t\geq 0}$ has an exponential dichotomy, then the space C_0 is decomposed as a direct sum $C_0=\mathcal{S}\oplus\mathcal{U}$ of two $U(t)$ invariant closed subspaces $\mathcal{S}$ and $\mathcal{U}$ such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants M and c such that*
$$|U(t)\varphi|\leq Me^{-ct}|\varphi|\quad \text{for } t\geq 0 \text{ and } \varphi\in\mathcal{S}$$
$$|U(t)\varphi|\leq Me^{ct}|\varphi|\quad \text{for } t\leq 0 \text{ and } \varphi\in\mathcal{U}.$$
Here $\mathcal{S}$ and $\mathcal{U}$ are the stable and unstable spaces. [4]

To give a variation of constants formula associated to equation (3.1), we need to extend the semigroup $(U(t))_{t\geq 0}$ to the space $C_0\oplus\langle X_0\rangle$ where $\langle X_0\rangle$ is the space defined by
$$\langle X_0\rangle=\{X_0y: y\in X\},$$
the function X_0y is given, for $y\in X$, by
$$(X_0y)(\theta)=\begin{cases}0 & \text{if } \theta\in[-r,0),\\ y & \text{if } \theta=0.\end{cases}$$
The space $C_0\oplus\langle X_0\rangle$ equipped with the norm $|\phi+X_0y|=|\phi|+|y|$ for $(\phi,y)\in C_0\times X$, is a Banach space. Consider the extension $\widetilde{\mathcal{A}}$ of the operator $\mathcal{A}$ on $C_0\oplus\langle X_0\rangle$ defined by
$$D(\widetilde{\mathcal{A}})=\{\varphi\in C^1([-r,0];X):\mathcal{D}(\varphi)\in D(A)\text{ and }\mathcal{D}(\varphi')\in\overline{D(A)}\},$$
$$\widetilde{\mathcal{A}}\varphi=\varphi'+X_0(A\mathcal{D}(\varphi)+L(\varphi)-\mathcal{D}(\varphi')).$$
To compute the resolvent operator $R(\lambda,\widetilde{\mathcal{A}})=(\lambda-\widetilde{\mathcal{A}})^{-1}$, we introduce the assumption

(H3) $\mathcal{D}(e^{\lambda\cdot}y)\in D(A)$, for all $y\in D(A)$ and all complex λ.

Lemma 3.12 ([5, Theorem 13])**.** *Assume that* (H0), (H3) *hold. Then $\widetilde{\mathcal{A}}$ satisfies the Hille-Yosida condition on $C_0\oplus\langle X_0\rangle$: there exist $\widetilde{M}\geq 0$ and $\widetilde{\omega}\in\mathbb{R}$ such that $(\widetilde{\omega},+\infty)\subset\rho(\widetilde{\mathcal{A}})$ and*
$$\left|R(\lambda,\widetilde{\mathcal{A}})^n\right|\leq\frac{\widetilde{M}}{(\lambda-\widetilde{\omega})^n}\quad \text{for } n\in\mathbb{N} \text{ and } \lambda>\widetilde{\omega}.$$

Now, we can state the variation of constants formula associated with (3.1).

Theorem 3.13 ([5, Theorem 16])**.** *Assume that* (H0), (H3) *hold. Then, for all $\varphi\in C_0$, the integral solution $u(.,\sigma,\varphi,f)$ of* (3.1) *is given by the variation of constants formula*
$$u_t(.,\sigma,\varphi,f)=U(t-\sigma)\varphi+\lim_{n\to+\infty}\int_\sigma^t U(t-s)(\widetilde{B}_nX_0f(s))ds\quad \text{for } t\geq\sigma,$$
where $\widetilde{B}_nX_0y=nR(n,\widetilde{\mathcal{A}})(X_0y)$ for $n>\widetilde{\omega}$ and $y\in X$.

Theorem 3.14 ([5]). *Assume that* (H0)–(H3) *hold and the semigroup* $(T(t))_{t\geq 0}$ *has an exponential dichotomy. If f is bounded on $\mathbb{R}$, then equation* (1.1) *has a unique bounded integral solution on $\mathbb{R}$ which is given by*

$$x_t = \lim_{n\to+\infty}\int_{-\infty}^{t} T^s(t-\tau)\Pi^s(\widetilde{B}_n X_0 f(\tau))\,d\tau + \lim_{n\to+\infty}\int_{+\infty}^{t} T^u(t-\tau)\Pi^u(\widetilde{B}_n X_0 f(\tau))\,d\tau,$$

where Π^s and Π^u are the projections of C onto the stable and unstable subspaces, respectively, T^s and T^u are the restrictions of $T(t)$ respectively on S and U.

4. Weighted pseudo-almost periodic solutions

In this section we study the existence and uniqueness of a weighted-pseudo-almost periodic solution to equation (1.1). We give the main result of this work, which shows the existence and uniqueness of $\rho-$pseudo-almost periodic solution if the input function f is $\rho-$pseudo-almost periodic.

Theorem 4.1. *Fix $\rho \in \mathbb{U}_\infty$. Assume that* (H0)–(H3) *hold and the semigroup $(T(t))_{t\geq 0}$ has an exponential dichotomy. If f is ρ-pseudo-almost periodic in $t \in \mathbb{R}$ and ρ is non increasing with*

$$P(c) := \sup_{R>0}\Big(\int_{-R}^{R} e^{-c(t+R)}\rho(t)\,dt\Big) < \infty, \tag{4.1}$$

where c is the positive constant given in Theorem 3.11. Then (1.1) *has one and only one ρ-pseudo-almost periodic integral solution.*

Proof. Equation (1.1) has one and only one bounded solution on $\mathbb{R}$ which is given by

$$x_t = \lim_{n\to+\infty}\int_{-\infty}^{t} T^s(t-\tau)\Pi^s(\widetilde{B}_n X_0 f(\tau))\,d\tau + \lim_{n\to+\infty}\int_{+\infty}^{t} T^u(t-\tau)\Pi^u(\widetilde{B}_n X_0 f(\tau))\,d\tau$$

We will show that both of the above terms are ρ-pseudo-almost periodic. Since f is ρ-pseudo-almost periodic then, $f = g + \phi$ where g is almost periodic and $\phi \in PAP_0(X,\rho)$. Recall that $\phi \in PAP_0\ (X,\rho)$ if and only if

$$\phi \in BC(\mathbb{R},X) \quad\text{and}\quad \lim_{R\to\infty}\frac{1}{m(R,\rho)}\int_{-R}^{R}|\phi(t)|\rho(t)dt = 0.$$

Note that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{-\infty}^{t} T^s(t-\tau)\Pi^s(\widetilde{B}_n X_0 f(\tau))\,d\tau\\
&= \lim_{n\to+\infty}\int_{-\infty}^{t} T^s(t-\tau)\Pi^s(\widetilde{B}_n X_0 g(\tau))\,d\tau + \lim_{n\to+\infty}\int_{-\infty}^{t} T^s(t-\tau)\Pi^s(\widetilde{B}_n X_0 \phi(\tau))\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{+\infty}^{t} T^u(t-\tau)\Pi^u(\widetilde{B}_n X_0 f(\tau))\,d\tau\\
&= \lim_{n\to+\infty}\int_{+\infty}^{t} T^u(t-\tau)\Pi^u(\widetilde{B}_n X_0 g(\tau))\,d\tau\\
&\quad + \lim_{n\to+\infty}\int_{+\infty}^{t} T^u(t-\tau)\Pi^u(\widetilde{B}_n X_0 \phi(\tau))\,d\tau\,.
\end{aligned}$$

Using the dominated convergence theorem, we show that the two integrals on the right=hand side are almost periodic. It remains to show that

$$\lim_{R\to\infty}\frac{1}{m(R,\rho)}\int_{-R}^{R}\Big|\lim_{n\to+\infty}\int_{-\infty}^{t}T^s(t-\tau)\Pi^s(\widetilde{B}_nX_0\phi(\tau))\,d\tau\Big|\rho(t)\,dt=0$$

and

$$\lim_{R\to\infty}\frac{1}{m(R,\rho)}\int_{-R}^{R}\Big|\lim_{n\to+\infty}\int_{+\infty}^{t}T^u(t-\tau)\Pi^u(\widetilde{B}_nX_0\phi(\tau))\,d\tau\Big|\rho(t)\,dt=0\,.$$

Let us put

$$\begin{aligned}
I(t)&=\lim_{n\to+\infty}\int_{-\infty}^{t}T^s(t-\tau)\Pi^s(\widetilde{B}_nX_0\phi(\tau))\,d\tau,\\
J(t)&=\lim_{n\to+\infty}\int_{+\infty}^{t}T^u(t-\tau)\Pi^u(\widetilde{B}_nX_0\phi(\tau))\,d\tau\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\lim_{R\to\infty}\frac{1}{m(R,\rho)}\int_{-R}^{R}|I(t)|\rho(t)\,dt\\
&\le\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}\Big[\int_{-\infty}^{t}e^{-c(t-\tau)}|\phi(\tau)|\,d\tau\Big]\rho(t)\,dt\\
&=\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}\Big[\int_{-R}^{t}e^{-c(t-\tau)}|\phi(\tau)|\,d\tau\Big]\rho(t)\,dt\\
&\quad+\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}\Big[\int_{-\infty}^{-R}e^{-c(t-\tau)}|\phi(\tau)|\,d\tau\Big]\rho(t)\,dt\\
&=I_1(\rho)+I_2(\rho)
\end{aligned}$$

Using the Fubini's theorem, we obtain

$$\begin{aligned}
I_1(\rho)&=\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}\Big[\int_{-R}^{t}e^{-c(t-\tau)}|\phi(\tau)|\,d\tau\Big]\rho(t)\,dt\\
&=\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}|\phi(\tau)|\Big[\int_{\tau}^{R}e^{-c(t-\tau)}\rho(t)\,dt\Big]\,d\tau
\end{aligned}$$

Since ρ is a decreasing function then we have

$$I_1(\rho)\le\lim_{R\to\infty}\frac{M\widetilde{M}}{m(R,\rho)}\int_{-R}^{R}|\phi(\tau)|\rho(\tau)\Big[\frac{1}{c}\Big(1-e^{-c(R-\tau)}\Big)\Big]\,d\tau.$$

Furthermore, $-R\le t\le R$ and $c>0$ then $\frac{1}{c}\big(1-e^{-c(R-\tau)}\big)$ is bounded uniformly in τ.

$$I_1(\rho)\le\lim_{R\to\infty}\frac{M\widetilde{M}}{cm(R,\rho)}\int_{-R}^{R}|\phi(\tau)|\rho(\tau)\,d\tau=0.$$

By (4.1) we have

$$
\begin{aligned}
I_2(\rho) &= \lim_{R\to\infty} \frac{M\widetilde{M}}{m(R,\rho)} \int_{-R}^{R} \Big[\int_{-\infty}^{-R} e^{-c(t-\tau)} |\phi(\tau)|\, d\tau \Big] \rho(t)\, dt \\
&= \lim_{R\to\infty} \frac{M\widetilde{M}}{m(R,\rho)} \int_{-R}^{R} e^{-ct} \Big[\int_{-\infty}^{-R} e^{c\tau} |\phi(\tau)|\, d\tau \Big] \rho(t)\, dt \\
&= \lim_{R\to\infty} \frac{M\widetilde{M}}{m(R,\rho)} \int_{-R}^{R} e^{-ct} \rho(t)\, dt \Big[\int_{-\infty}^{-R} e^{c\tau} |\phi(\tau)|\, d\tau \Big] \\
&= \lim_{R\to\infty} \frac{M\widetilde{M}}{cm(R,\rho)e^{cR}} \sup_{\tau\in\mathbb{R}} |\phi(\tau)| \int_{-R}^{R} e^{-ct} \rho(t)\, dt = 0.
\end{aligned}
$$

By a similar argument we show that

$$
\lim_{R\to\infty} \frac{1}{m(R,\rho)} \int_{-R}^{R} |J(t)| \rho(t)\, dt = 0.
$$

This completes the proof of the theorem. □

5. Application

To apply the abstract results of the previous section, we consider the following model (1.2) proposed in [27, 28]:

$$
\begin{gathered}
\frac{\partial}{\partial t}[w(t,\xi) - qw(t-r,\xi)] \\
= \frac{\partial^2}{\partial x^2}[w(t,\xi) - qw(t-r,\xi)] + \int_{-r}^{0} \gamma(\theta) w(t+\theta,\xi) d\theta + \kappa(t)\mu(\xi) \\
\text{for } t \in \mathbb{R},\ \xi \in [0,\pi], \\
w(t,\xi) - qw(t-r,\xi) = 0 \quad \text{for } \xi = 0,\ t \in \mathbb{R}
\end{gathered}
\tag{5.1}
$$

where $\gamma : [-r,0] \to \mathbb{R}$, $\mu : [0,\pi] \to \mathbb{R}$ are continuous function, and $q \in (0,1)$. The function $\kappa : \mathbb{R} \to \mathbb{R}$ is given by

$$
\kappa(t) = \sin t + \sin \sqrt{2} t + e^{\alpha t} \quad \text{for } t \in \mathbb{R},
$$

where $\alpha > 0$. To rewrite equation (5.1) in the abstract form (1.1), we introduce $X = C([0,\pi];\mathbb{R})$ the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology and we define the operator $A : D(A) \subset X \to X$ by

$$
\begin{gathered}
D(A) = \{ y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = 0 \}, \\
Ay = y''.
\end{gathered}
$$

Lemma 5.1 ([6])**.** *The operator A satisfies the Hille-Yosida condition on the space X, $(0,+\infty) \subset \rho(A)$, and*

$$
|(\lambda I - A)^{-1}| \le \frac{1}{\lambda} \quad \textit{for } \lambda > 0.
$$

This lemma implies that condition (H0) is satisfied. Let A_0 be the part of the operator A in $\overline{D(A)}$. Then A_0 is given by

$$
\begin{gathered}
[c]lD(A_0) = \{ y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \}, \\
A_0 y = y'' \quad \text{for } y \in D(A_0).
\end{gathered}
$$

Then it is well-known that A_0 generates a strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. This implies that (H1) holds. On the other hand, we can see that

$$\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\}.$$

Let us introduce the bounded linear operator $\mathcal{D} : C := C([-r, 0]; X) \to X$ by

$$\mathcal{D}(\phi) = \phi(0) - q\phi(-r).$$

Since $0 < q < 1$, then $\mathcal{D}$ is stable and Condition (H2) holds. Moreover, by definitions of the operators A and $\mathcal{D}$, it follows that Condition (H3) is satisfied. Let $L : C \to X$ be the operator defined by

$$L(\phi)(\xi) = \int_{-r}^{0} \gamma(\theta)\phi(\theta)(\xi)d\theta \quad \text{for } \xi \in [0, \pi] \text{ and } \phi \in C.$$

Let $f : \mathbb{R} \to X$ be defined by

$$f(t)(\xi) = \kappa(t)\mu(\xi) \quad \text{for } y \in X \text{ and } t \in \mathbb{R}, \xi \in [0, \pi].$$

Then L is a bounded linear operator from C to X. Let $u(t) = w(t, .)$ for $t \in \mathbb{R}$. Then equation (5.1) takes the abstract form

$$\frac{d}{dt}\mathcal{D}(u_t) = A\mathcal{D}(u_t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Proposition 5.2. *Assume the above conditions and that*

$$\int_{-r}^{0} |\gamma(\theta)|d\theta < 1 - q. \tag{5.3}$$

Then the semigroup $(T(t))_{t\geq 0}$ is exponentially stable: there exist $M \geq 1$ and $\omega > 0$ such that

$$|T(t)| \leq Me^{-\omega t} \quad \textit{for all } t \geq 0.$$

Proof. It is sufficient to show that $\sigma^+(\mathcal{A}) = \emptyset$. We proceed by contradiction and suppose that there exists $\lambda \in \sigma^+(\mathcal{A})$, then by Lemma 3.9, there exists $y \in D(A)\backslash\{0\}$ such that $\Delta(\lambda)y = 0$, which is equivalent to say that

$$Ay = \Big(\lambda - \frac{1}{1 - qe^{-\lambda r}} \int_{-r}^{0} \gamma(\theta)e^{\lambda\theta}d\theta\Big)y = 0. \tag{5.4}$$

This implies

$$\lambda - \frac{1}{1 - qe^{-\lambda r}} \int_{-r}^{0} \gamma(\theta)e^{\lambda\theta}d\theta \in \sigma_p(A).$$

Since the spectrum $\sigma(A)$ is reduced to the point spectrum $\sigma_p(A)$ and $\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Then λ is a solution of the characteristic equation (5.4) with $\mathrm{Re}(\lambda) \geq 0$ if and only if λ satisfies

$$\lambda - \frac{1}{1 - qe^{-\lambda r}} \int_{-r}^{0} \gamma(\theta)e^{\lambda\theta}d\theta = -n^2 \quad \text{for some } n \in \mathbb{N}^*. \tag{5.5}$$

It follows that

$$\mathrm{Re}(\lambda) \leq \frac{1}{|1 - qe^{-\lambda r}|} \int_{-r}^{0} |\gamma(\theta)|e^{\mathrm{Re}(\lambda)\theta}d\theta - 1,$$

$$\mathrm{Re}(\lambda) \leq \frac{1}{1 - q} \int_{-r}^{0} |\gamma(\theta)|e^{\mathrm{Re}(\lambda)\theta}d\theta - 1 < 0.$$

Then a contradiction is obtained with the fact that $\mathrm{Re}(\lambda) \geq 0$. Consequently $\sigma^+(\mathcal{A}) = \emptyset$ and Lemma 3.9 implies that the semigroup $(T(t))_{t\geq 0}$ solution of the homogeneous linear equation

$$\frac{d}{dt}\mathcal{D}(v_t) = A\mathcal{D}(v_t) + L(v_t) \quad \text{for } t \geq 0,$$
$$v_0 = \varphi \in C$$

has an exponential dichotomy. For $\beta > 0$, set the weighted function

$$\rho(t) = \begin{cases} 1 & \text{if } t < 0, \\ e^{-\beta t} & \text{if } t \geq 0. \end{cases}$$

Then, $\lim_{R\to+\infty} m(R, \rho) = +\infty$ and hence $\rho \in \mathbb{U}_\infty$. □

Proposition 5.3. *Assume that* $0 < \alpha < \beta$. *Then,* (5.2) *has a unique* ρ*-pseudo-almost periodic solution.*

Proof. If $\alpha < \beta$, then condition (4.1) is satisfied, namely

$$P(\omega) := \sup_{R>0} \Big(\int_{-R}^{R} e^{-\omega(R+t)} \rho(t)\, dt \Big) < \infty.$$

The function ψ does not belong to $PAP(\mathbb{R})$ since

$$\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} e^{\alpha t}\, dt = \infty.$$

and $\psi \in PAP(\mathbb{R}, \rho)$ with $\sin t + \sin(\sqrt{2}t)$ is the almost periodic component and $e^{\alpha t}$ is the ρ-ergodic component which satisfies

$$\lim_{R\to\infty} \frac{1}{m(R,\rho)} \int_{-R}^{R} e^{\alpha t} \rho(t)\, dt = 0.$$

By Theorem 4.1, we deduce that (5.2) has a unique ρ-pseudo-almost periodic integral solution. □

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Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390, Marrakech, Morocco
E-mail address: ezzinbi@ucam.ac.ma

Samir Fatajou
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390, Marrakech, Morocco
E-mail address: s.fatajou@ucam.ac.ma

Gaston M. N'Guérekata
Department of Mathematics, Morgan State University, 1700 East Cold Spring Lane, Baltimore, MD 21251, USA
E-mail address: Gaston.N'Guerekata@morgan.edu, nguerekata@aol.com