Electronic Journal of Differential Equations, Vol. 2010(2010), No. 127, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF PERIODIC SOLUTIONS FOR NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

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Abstract. We use a variation of Krasnoselskii fixed point theorem introduced by Burton to show that the nonlinear neutral differential equation

$$x'(t) = -a(t)x^3(t) + c(t)x'(t-g(t)) + G(t, x^3(t-g(t))$$

has a periodic solution. Since this equation is nonlinear, the variation of parameters can not be applied directly; we add and subtract a linear term to transform the differential into an equivalent integral equation suitable for applying a fixed point theorem. Our result is illustrated with an example.

1. Introduction

We are interested in proving that the retarded scalar neutral non linear differential equation

$$x'(t) = -a(t)x^3(t) + c(t)x'(t-g(t)) + G(t, x^3(t-g(t))) \tag{1.1}$$

possesses a periodic solution. The motivation for studying this problems comes from the problems considered in [1, 3, 4, 5, 6]. Here $a(t)$ is real valued function, $c(t)$ is continuously differentiable, $g(t)$ is twice continuously differentiable, and $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous with respect to its arguments. Clearly, the present problem is nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [1], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, due to Burton [3], to show the existence of a periodic solution. Our result is illustrated with an example at the end of this article.

2000 *Mathematics Subject Classification.* 34K20, 45J05, 45D05.
Key words and phrases. Periodic solution; nonlinear neutral differential equation; large contraction; integral equation.

Submitted April 15, 2010. Published September 7, 2010.

2. Inversion of (1.1)

Let $T>0$ and define $C_T=\{\varphi:\mathbb{R}\to\mathbb{R}:\varphi\in C \text{ and } \varphi(t+T)=\varphi(t)\}$ where C is the space of continuous real valued functions. C_T is a Banach space endowed with the norm

$$\|\varphi\|=\max_{0\le t\le T}|\varphi(t)|\,.$$

We assume that a,c,G,g are continuous functions with $g(t)\ge 0$ and c continuously differentiable such that

$$a(t+T)=a(t),\quad c(t+T)=c(t),\quad G(t+T,x)=G(t,x),\quad g(t+T)=g(t). \tag{2.1}$$

We assume further that $G(t,x)$ is globally Lipschitz continuous in x. That is, there is some positive constant k such that

$$|G(t,x)-G(t,y)|\le k|x-y|. \tag{2.2}$$

Also, we assume that

$$\int_0^T a(s)ds>0, \tag{2.3}$$

and for all t, $0\le t\le T$,and that g is twice continuously differentiable and

$$g'(t)\ne 1. \tag{2.4}$$

Lemma 2.1. *Suppose* (2.1), (2.3) *and* (2.4) *hold. If $x(t)\in C_T$, then $x(t)$ is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t)&=(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\Big[\int_{t-T}^t G(u,x^3(u-g(u)))e^{-\int_u^t a(s)ds}du\\ &\quad+\int_{t-T}^t a(u)(x(u)-x^3(u))e^{-\int_u^t a(s)ds}du\Big]+\frac{c(t)x(t-g(t))}{(1-g'(t))}\\ &\quad-(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t h(u)(x(u-g(u)))e^{-\int_u^t a(s)ds}du, \end{aligned} \tag{2.5}$$

where

$$h(u)=\frac{(c'(u)+a(u)c(u))(1-g'(u))+c(u)g''(u)}{(1-g'(u))^2}.$$

Proof. Let $x(t)$ be a solution of (1.1). Rewrite (1.1) as

$$x'(t)+a(t)x(t)=a(t)x(t)-a(t)x^3(t)+c(t)x'(t-g(t))+G(t,x^3(t-g(t))).$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^t [x(u)e^{\int_0^u a(s)ds}]'du\\ &=\int_{t-T}^t a(u)[x(u)-x^3(u)]e^{\int_0^u a(s)ds}du+\int_{t-T}^t G(u,x^3(u-g(u)))e^{\int_0^u a(s)ds}du\\ &\quad+\int_{t-T}^t c(u)x'(u-g(u))e^{\int_0^u a(s)ds}du. \end{aligned}$$

Rewrite the last term as

$$\int_{t-T}^t c(u)x'(u-g(u))e^{\int_0^u a(s)ds}du=\int_{t-T}^t \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))}e^{\int_0^u a(s)ds}du.$$

Using integration by parts, and that c, g, x are periodic we obtain

$$
\begin{aligned}
&\int_{t-T}^{t} e^{\int_0^u a(s)ds} c(u)x'(u-g(u))du \\
&= \frac{c(t)}{(1-g'(t))} x(t-g(t)) e^{\int_0^t a(s)ds}(1-e^{-\int_{t-T}^t a(s)ds}) \\
&\quad - \int_{t-T}^{t} h(u)(x(u-g(u))) e^{\int_0^u a(s)ds} du.
\end{aligned} \tag{2.6}
$$

We arrive at

$$
\begin{aligned}
&x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} \\
&= \int_{t-T}^{t} a(u)[x(u)-x^3(u)] e^{\int_0^u a(s)ds} du + \int_{t-T}^{t} G(u, x^3(u-g(u))) e^{\int_0^u a(s)ds} du \\
&\quad + \frac{c(t)}{(1-g'(t))} x(t-g(t)) e^{\int_0^t a(s)ds}(1-e^{-\int_{t-T}^t a(s)ds}) \\
&\quad - \int_{t-T}^{t} h(u)(x(u-g(u))) e^{\int_0^u a(s)ds} du.
\end{aligned}
$$

Now, the lemma follows by dividing both sides of the above equation by $e^{\int_0^t a(s)ds}$ and using the fact that $x(t) = x(t-T)$. □

Krasnoselskii [2, 7] combined the contraction mapping theorem and Shauder's theorem and formulated the following hybrid result.

Theorem 2.2. *Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that the following conditions hold*

(i) *$Ax + By \in M$, for all $x, y \in M$;*
(ii) *A is continuous and AM is contained in a compact set;*
(iii) *B is a contraction with $\alpha < 1$.*

Then there is a $z \in M$, with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent year much attention has been paid to this theorem. Burton [2] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 2.4 below (see [3] for the proof).

Let (M, d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$ and if for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in M,\ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2.3. *Let (M, d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 2.4. *Let M be a bounded convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map M into M and that*

(i) *for all $x, y \in M \Rightarrow Ax + By \in M$,*
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in M$ with $z = Az + Bz$.

We will use this theorem to prove the existence of periodic solutions for (1.1).

3. Existence of periodic solutions

To apply Theorem 2.4, we need to define a Banach space S, a bounded convex subset M of S and construct two mappings, one is a large contraction and the other is completely continuous. So, we let $(S, \|\cdot\|) = (C_T, \|\cdot\|)$ and $M = \{\varphi \in S : \|\varphi\| \leq L, \varphi' \text{ is bounded}\}$, where $L = \sqrt{3}/3$. We express (1.1) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (H\varphi)(t),$$

where $A, B : S \to S$ are defined by

$$(B\varphi)(t) := (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} a(u)(\varphi(u) - \varphi^3(u)) e^{-\int_u^t a(s)ds} du, \tag{3.1}$$

and

$$\begin{aligned}(A\varphi)(t) := &(1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} G(u, \varphi^3(u - g(u))) e^{-\int_u^t a(s)ds} du \\ &+ \frac{c(t)\varphi(t - g(t))}{(1 - g'(t))} \\ &- (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} h(u)(\varphi(u - g(u))) e^{-\int_u^t a(s)ds} du.\end{aligned} \tag{3.2}$$

We need the following assumptions

$$(kL^3 + |G(t, 0)|) \leq \beta L a(t), \tag{3.3}$$

$$|h(t)| \leq \delta a(t), \tag{3.4}$$

$$\max_{t \in [0,T]} \left|\frac{c(t)}{(1 - g'(t))}\right| = \alpha, \tag{3.5}$$

$$J(\beta + \alpha + \delta) \leq 1, \tag{3.6}$$

where α, β, δ and J are constants with $J \geq 3$.

We begin with the following proposition (see [1, 2]).

Proposition 3.1. *Let $\|\cdot\|$ be the supremum norm,*

$$M = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C,\ \|\varphi\| \leq \sqrt{3}/3, \|\varphi'\| \leq L'\},$$

and define $(B\varphi)(t) := \varphi(t) - \varphi^3(t)$. Then B is a large contraction of the set M.

Proof. For each $t \in \mathbb{R}$ we have, for φ, ψ real functions,

$$\begin{aligned}|(B\varphi)(t) - (B\psi)(t)| &= |\varphi(t) - \varphi^3(t) - \psi(t) + \psi^3(t)| \\ &= |\varphi(t) - \psi(t)||1 - (\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t))|.\end{aligned}$$

Then for

$$|\varphi(t) - \psi(t)|^2 = \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \leq 2(\varphi^2(t) + \psi^2(t))$$

and for $\varphi^2(t) + \psi^2(t) < 1$, we have

$$\begin{aligned}|(B\varphi)(t) - (B\psi)(t)| &= |\varphi(t) - \psi(t)|[1 - (\varphi^2(t) + \psi^2(t)) + |\varphi(t)\psi(t)|] \\ &\leq |\varphi(t) - \psi(t)|[1 - (\varphi^2(t) + \psi^2(t)) + \frac{\varphi^2(t) + \psi^2(t)}{2}]\end{aligned}$$

$$\le |\varphi(t)-\psi(t)|[1-\frac{\varphi^2(t)+\psi^2(t)}{2}].$$

Thus, B is pointwise a large contraction. But application B is still a large contraction for the supremum norm. For, let $\varepsilon \in (0,1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \ge \varepsilon$.

(a) Suppose that for some t we have $\varepsilon/2 \le |\varphi(t)-\psi(t)|$. Then

$$(\varepsilon/2)^2 \le |\varphi(t)-\psi(t)|^2 \le 2(\varphi^2(t)+\psi^2(t));$$

that is, $\varphi^2(t)+\psi^2(t) \ge \varepsilon^2/8$. For such t we have

$$|(B\varphi)(t)-(B\psi)(t)| \le |\varphi(t)-\psi(t)|[1-\frac{\varepsilon^2}{8}] \le \|\varphi-\psi\|[1-\frac{\varepsilon^2}{8}].$$

(b) Suppose that for some t, $|\varphi(t)-\psi(t)| \le \varepsilon/2$. Then

$$|(B\varphi)(t)-(B\psi)(t)| \le |\varphi(t)-\psi(t)| \le (1/2)\|\varphi-\psi\|.$$

Consequently,

$$\|B\varphi - B\psi\| \le \min[1/2, 1-\frac{\varepsilon^2}{8}]\|\varphi-\psi\|.$$

□

We shall prove that the mapping H has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 3.2. *For A defined in* (3.2)*, suppose that* (2.1)-(2.3) *and* (3.3)-(3.6) *hold. Then $A : M \to M$ is continuous in the supremum norm and maps M into a compact subset of M.*

Proof. Clearly, if φ is continuous then $A\varphi$ is. A change of variable in (3.2) shows that $(A\varphi)(t+T) = \varphi(t)$. Observe that

$$|G(t,x)| \le |G(t,x)-G(t,0)| + |G(t,0)| \le k|x| + |G(t,0)|.$$

So, for any $\varphi \in M$, we have

$$\begin{aligned}
|(A\varphi)(t)| &\le (1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t |G(u,\varphi^3(u-g(u)))|e^{-\int_u^t a(s)ds}du \\
&\quad + |\frac{c(t)\varphi(t-g(t))}{(1-g'(t))}| \\
&\quad + (1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t |h(u)(\varphi(u-g(u)))|e^{-\int_u^t a(s)ds}du \\
&\le (1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t (kL^3+|G(u,0)|)e^{-\int_u^t a(s)ds}du + \alpha L \\
&\quad + (1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t \delta a(u)Le^{-\int_u^t a(s)ds}du \\
&\le \beta L(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t a(u)e^{-\int_u^t a(s)ds}du + \alpha L \\
&\quad + \delta L(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t a(u)e^{-\int_u^t a(s)ds}du \\
&\le (\beta+\alpha+\delta)L \le \frac{L}{J} < L.
\end{aligned}$$

That is $A\varphi \in M$.

We show that A is continuous in the supremum norm. Let $\varphi, \psi \in M$, and let

$$\begin{gathered}\alpha' = \max_{t\in[0,T]}(1-e^{-\int_{t-T}^t a(s)ds})^{-1}, \quad \beta' = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds},\\ \sigma = \max_{t\in[0,T]}\{a(t)\}, \quad \rho = \max_{t\in[0,T]}|G(t,0)|,\\ \mu = \max_{t\in[0,T]}|\frac{c'(t)}{(1-g'(t))}|, \quad \upsilon = \max_{t\in[0,T]}|\frac{g''(t)c(t)}{(1-g'(t))^2}|.\end{gathered} \tag{3.7}$$

Then

$$\begin{aligned}&|(A\varphi)(t)-(A\psi)(t)|\\ &\le (1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t |G(u,\varphi^3(u-g(u)))-G(u,\psi^3(u-g(u)))|\\ &\quad\times e^{-\int_u^t a(s)ds}du + |\frac{c(t)\varphi(t-g(t))}{(1-g'(t))}-\frac{c(t)\psi(t-g(t))}{(1-g'(t))}|\\ &\quad+(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_0^t |h(u)||(\varphi(u-g(u))-\psi(u-g(u))|\\ &\quad\times e^{-\int_u^t a(s)ds}du\\ &\le k(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\|\varphi^3-\psi^3\|\int_{t-T}^t e^{-\int_u^t a(s)ds}+\alpha\|\varphi-\psi\|+\delta\|\varphi-\psi\|\\ &\le (3kT\alpha'\beta'L^2+\alpha+\delta)\|\varphi-\psi\|.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \frac{\varepsilon}{K}$, with $K = 3kT\alpha'\beta'L^2+\alpha+\delta$, where k is given by (2.2). Then, for $\|\varphi-\psi\| < \eta$, we obtain

$$\|A\varphi - A\psi\| \le K\|\varphi-\psi\| < \varepsilon.$$

It is left to show that A is compact. Let $\varphi_n \in M$, where n is a positive integer. Then, as above, we see that

$$\|A\varphi_n\| \le L. \tag{3.8}$$

Moreover, a direct calculation shows that

$$\begin{aligned}&(A\varphi_n)'(t)\\ &= (G(t,\varphi_n^3(t-g(t)))-h(t)\varphi_n(t-g(t))-a(t)(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\\ &\quad\times\int_{t-T}^t [G(u,\varphi_n^3(u-g(u)))-h(u)\varphi_n(u-g(u))]e^{-\int_u^t a(s)ds}du\\ &\quad+\frac{c'(t)\varphi_n(t-g(t))+c(t){\varphi_n}'(t-g(t))}{1-g'(t)}+\frac{g''(t)c(t)\varphi_n(t-g(t))}{(1-g'(t))^2}.\end{aligned}$$

By invoking the conditions (2.2), (3.3)-(3.5), (3.7) and (3.8) we obtain

$$\begin{aligned}|(A\varphi_n)'(t)| &\le kL^3+\rho+\delta a(t)L+a(t)L+\alpha L+\mu L+\alpha L'+\upsilon L\\ &\le kL^3+\rho+(\delta+1)\sigma L+(\alpha+\mu+\upsilon)L+\alpha L' \le D,\end{aligned}$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that the subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous $T-$periodic function. Thus, A is continuous and AM is a compact set. $\square$

Lemma 3.3. *Suppose* (2.1)-(2.3) *and* (3.3) *hold. For A, B defined in* (3.2) *and* (3.1), *if $\varphi, \psi \in M$ are arbitrary, then*

$$A\varphi + B\psi : M \to M.$$

Moreover, B is a large contraction on M with a unique fixed point in M.

Proof. Let $\varphi, \psi \in M$ be arbitrary. Note first that $|\psi(t)| \leq \sqrt{3}/3$ implies

$$|\psi(t) - \psi^3(t)| \leq (2\sqrt{3})/9.$$

Using the definition of B, and the result of Lemma 3.2, we obtain

$$\begin{aligned}
&|(A\varphi)(t) + (B\psi)(t)| \\
&\leq |(1 - e^{-\int_{t-T}^t a(s)ds})^{-1} \int_{t-T}^t G(u, \varphi^3(u - r(u)))e^{-\int_u^t a(s)ds} du \\
&\quad + \frac{c(t)\varphi(t - g(t))}{(1 - g'(t))} - (1 - e^{-\int_{t-T}^t a(s)ds})^{-1} \int_{t-T}^t h(u)(\varphi(u - g(u)))e^{-\int_u^t a(s)ds} du| \\
&\quad + |(1 - e^{-\int_{t-T}^t a(s)ds})^{-1} \int_{t-T}^t a(u)|\psi(u) - \psi^3(u)|e^{-\int_u^t a(s)ds} du| \\
&\leq \frac{\sqrt{3}}{3J} + \frac{2\sqrt{3}}{9} \leq L.
\end{aligned}$$

Thus, $A\varphi + B\psi \in M$. Left to show that B is a large contraction with a unique fixed point in M. Proposition 3.1 shows that $\psi - \psi^3$ is a large contraction in the supremum norm. For any ε, from the proof of that proposition, we have found a $\delta < 1$, such that

$$\begin{aligned}
|(B\psi)(t) - (B\varphi)(t)| &\leq (1 - e^{-\int_{t-T}^t a(s)ds})^{-1} \int_{t-T}^t a(u)\delta\|\psi - \varphi\|e^{-\int_u^t a(s)ds} du \\
&\leq \delta\|\psi - \varphi\|.
\end{aligned}$$

Further, since $0 \in M$ the above inequality shows, when $\varphi = 0$, we see that $B : M \to M$. This completes the proof. $\square$

Theorem 3.4. *Let $(S, \|\cdot\|)$ be the Banach space of continuous T-periodic real functions and $M = \{\varphi \in S : \|\varphi\| \leq L, \varphi'$ is bounded$\}$, where $L = \sqrt{3}/3$. Suppose* (2.1)- (2.3) *and* (3.3)-(3.6) *hold. Then equation* (1.1) *possesses a periodic solution φ in the subset M.*

Proof. By Lemma 2.1, φ is a solution of (1.1) if

$$\varphi = A\varphi + B\varphi,$$

where A and B are given by (3.2), (3.1) respectively. By Lemma 3.2, $A : M \to M$ is continuous and AM is contained in compact subset of M. By Lemma 3.3, $A\varphi + B\psi \in M$ whenever $\varphi, \psi \in M$. Moreover, $B : M \to M$ is a large contraction. Clearly, all the hypotheses of Theorem 2.4 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. Hence (1.1) has a T-periodic solution. $\square$

Example 3.5. Let S and M as in Theorem 3.4 with $T = 2\pi$ and consider the neutral nonlinear equation

$$x'(t) = -4.10^{-1}x^3(t) + 10^{-3}\sin t.x'(t - 1) + 10^{-3}(\cos t + x^3(t - 1)).$$

Then

$$T = 2\pi, \quad a(t) = 4.10^{-1}, \quad c(t) = 10^{-3}\sin t, \quad G(t,x) = 10^{-3}(\cos t + x), \quad g(t) = 1.$$

Doing straightforward computations, we obtain

$$k = \rho = \alpha = \mu = 10^{-3}, \quad \upsilon = 0, \quad \alpha' = (1 - e^{-0.8\pi})^{-1}, \quad \beta' = 1.$$

By replacing, in (3.3) and (3.4), $\beta = \delta = 10^{-2}$, then any $J \in [3, 47]$ satisfies (3.6). All hypotheses of Theorem 3.4 are fulfilled and so the equation have at least a 2π-periodic solution belonging to M.

Acknowledgments. The authors would like to thank the anonymous referee for his/her valuable remarks.

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