Electronic Journal of Differential Equations, Vol. 2010(2010), No. 124, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SQUARE-MEAN ALMOST PERIODIC MILD SOLUTIONS TO SOME NONAUTONOMOUS STOCHASTIC SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we use the well-known Schauder fixed point principle to obtain the existence of square-mean almost periodic solutions to some classes of nonautonomous second order stochastic differential equations on a Hilbert space.

1. Introduction

Let $\mathbb{B}$ be a Banach space. In Goldstein and N'Guérékata [30], the existence of almost automorphic solutions to the evolution

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{R}$$

where $A : D(A) \subset \mathbb{B} \to \mathbb{B}$ is a closed linear operator on a Banach space $\mathbb{B}$ which generates an exponentially stable C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ and the function $F : \mathbb{R} \times \mathbb{B} \to \mathbb{B}$ is given by $F(t, u) = P(t)Q(u)$ with P, Q being some appropriate continuous functions satisfying some additional conditions, was established. The main tools used in [30] are fractional powers of operators and the fixed-point theorem of Schauder.

Recently Diagana [20] generalized the results of [30] to the *nonautonomous* case by obtaining the existence of almost automorphic mild solutions to

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R} \tag{1.1}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators with domains $D(A(t))$ satisfying Acquistapace-Terreni conditions, and the function $f : \mathbb{R} \times \mathbb{B} \mapsto \mathbb{B}$ is almost automorphic in $t \in \mathbb{R}$ uniformly in the second variable. For that, Diagana utilized similar techniques as in [30], dichotomy tools, and the Schauder fixed point theorem.

Let $\mathbb{H}$ be a Hilbert space. Motivated by the above mentioned papers, the present paper is aimed at utilizing Schauder fixed point theorem to study the existence of p-th mean almost periodic solutions to the nonautonomous stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F_1(t, X(t))\,dt + F_2(t, X(t))\,d\mathbb{W}(t), \quad t \in \mathbb{R}, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34K14, 60H10, 35B15, 34F05.
Key words and phrases. Stochastic differential equation; Wiener process.

Submitted May 4, 2010. Published August 30, 2010.

where $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying Acquistapace and Terreni conditions, the functions $F_1 : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ and $F_2 : \mathbb{R}\times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{L}_2^0)$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}$ is a Wiener process.

Then, we utilize our main results to study the existence of square-mean almost periodic solutions to the second order stochastic differential equations

$$\begin{aligned} &dX'(\omega,t) + a(t)\,dX(\omega,t) \\ &= \Big[-b(t)\,\mathcal{A}X(\omega,t) + f_1(t,X(\omega,t))\Big]\,dt \\ &+f_2(t,X(\omega,t))\,d\mathbb{W}(\omega,t), \end{aligned} \tag{1.3}$$

for all $\omega \in \Omega$ and $t \in \mathbb{R}$, where $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is a self-adjoint linear operator whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace, the functions $a, b : \mathbb{R} \to (0,\infty)$ are almost periodic functions, and the function $f_i (i = 1, 2) : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ are jointly continuous functions satisfying some additional conditions and $\mathbb{W}$ is a one dimensional Brownian motion.

It should be mentioned the existence of almost periodic to (1.2) in the case when $A(t)$ is periodic, that is, $A(t + T) = A(t)$ for each $t \in \mathbb{R}$ for some $T > 0$ was established by Da Prato and Tudor in [17]. In the paper by Bezandry and Diagana [9], upon assuming that the operators $A(t)$ satisfy Acquistapace-Terreni conditions and that F_i $(i = 1, 2, 3,)$ satisfy Lipschitz conditions, the Banach fixed point principle was utilized to obtain the existence of a square-mean almost periodic solutions to (1.2). In this paper is goes back to utilizing Schauder fixed theorem to establish the existence of p-th mean almost periodic solutions to (1.2). Next, we make extensive use of those abstract results to deal with the existence of square-mean almost periodic solutions to the second-order stochastic differential equations formulated in (1.3).

2. Preliminaries

In this section, $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ stands for a self-adjoint linear operator whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace.

Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$. Clearly, for each

$$u \in D(\mathcal{A}) := \Big\{x \in \mathbb{H} : \quad \sum_{j=1}^{\infty} \lambda_j^2 \|E_j x\|^2 < \infty\Big\},$$

$$\mathcal{A}x = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle x, e_j^k\rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j x$$

where $E_j x = \sum_{k=1}^{\gamma_j} \langle x, e_j^k\rangle e_j^k$.

Note that $\{E_j\}_{j\geq 1}$ is a sequence of orthogonal projections on $\mathbb{H}$. Moreover, each $x \in \mathbb{H}$ can written as follows:

$$x = \sum_{j=1}^{\infty} E_j x.$$

It should also be mentioned that the operator $-\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $x\in\mathbb{H}$,

$$T(t)x=\sum_{j=1}^{\infty} e^{-\lambda_j t}E_j x.$$

In addition, the fractional powers $\mathcal{A}^r$ $(r\geq 0)$ of $\mathcal{A}$ exist and are given by

$$D(\mathcal{A}^r)=\Big\{x\in\mathbb{H}:\sum_{j=1}^{\infty}\lambda_j^{2r}\|E_j x\|^2<\infty\Big\}$$

and

$$\mathcal{A}^r x=\sum_{j=1}^{\infty}\lambda_j^{2r}E_j x,\quad \forall x\in D(\mathcal{A}^r).$$

Let $\big(\mathbb{B},\|\cdot\|\big)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{B}$, then $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$, $N(L)$, and $R(L)$ stand respectively for the domain, resolvent, spectrum, null space, and the range of L. also, we set $R(\lambda,L):=(\lambda I-L)^{-1}$ for all $\lambda\in\rho(L)$. If P is a projection, we then set $Q=I-P$. If $\mathbb{B}_1$, $\mathbb{B}_2$ are Banach spaces, then the space $B(\mathbb{B}_1,\mathbb{B}_2)$ denotes the collection of all bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$ equipped with its natural topology. This is simply denoted by $B(\mathbb{B}_1)$ when $\mathbb{B}_1=\mathbb{B}_2$.

2.1. **Evolution Families.** Let $\mathbb{B}$ be a Banach space equipped with the norm $\|\cdot\|$. The family of closed linear operators $A(t)$ for $t\in\mathbb{R}$ on $\mathbb{B}$ with domain $D(A(t))$ (possibly not densely defined) is said to satisfy Acquistapace-Terreni conditions if: there exist constants $\omega\geq 0$, $\theta\in\Big(\frac{\pi}{2},\pi\Big)$, $K,L\geq 0$ and $\mu,\nu\in(0,1]$ with $\mu+\nu>1$ such that

$$S_\theta\cup\{0\}\subset\rho\big(A(t)-\omega\big)\ni\lambda,\quad \|R\big(\lambda,A(t)-\omega\big)\|\leq\frac{K}{1+|\lambda|}\tag{2.1}$$

and

$$\|\big(A(t)-\omega\big)R\big(\lambda,A(t)-\omega\big)\Big[R\Big(\omega,A(t)\Big)-R\Big(\omega,A(s)\Big)\Big]\|\leq L|t-s|^{\mu}\,|\lambda|^{-\nu}\tag{2.2}$$

for $t,s\in\mathbb{R}$, $\lambda\in S_\theta:=\big\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|\leq\theta\big\}$.

It should mentioned that the conditions (2.1) and (2.2) were introduced in the literature by Acquistapace and Terreni in [2, 3] for $\omega=0$. Among other things, it ensures that there exists a unique evolution family $\mathcal{U}=U(t,s)$ on $\mathbb{B}$ associated with $A(t)$ satisfying

(a) $U(t,s)U(s,r)=U(t,r)$;
(b) $U(t,t)=I$ for $t\geq s\geq r$ in $\mathbb{R}$;
(c) $(t,s)\mapsto U(t,s)\in B(\mathbb{B})$ is continuous for $t>s$;
(d) $U(\cdot,s)\in C^1((s,\infty),B(\mathbb{B}))$, $\frac{\partial U}{\partial t}(t,s)=A(t)U(t,s)$ and
$$\|A(t)^kU(t,s)\|\leq K\,(t-s)^{-k}\tag{2.3}$$
for $0<t-s\leq 1$, $k=0,1$; and
(e) $\partial_s^+U(t,s)x=-U(t,s)A(s)x$ for $t>s$ and $x\in D(A(s))$ with $A(s)x\in\overline{D(A(s))}$.

It should also be mentioned that the above-mentioned properties were mainly established in [1, Theorem 2.3] and [50, Theorem 2.1], see also [3, 49]. In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;
(g) the restriction $U_Q(t,s) : Q(s)\mathbb{B} \to Q(t)\mathbb{B}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$); and
(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\widetilde{U}_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the following excellent books [29], and [38] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{B}$ (for that, in (2.1)-(2.2), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{B}^A_\alpha := \Big\{x \in \mathbb{B} : \|x\|^A_\alpha := \sup_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty\Big\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|^A_\alpha$. For convenience we further write

$$\mathbb{B}^A_0 := \mathbb{B}, \quad \|x\|^A_0 := \|x\|, \quad \mathbb{B}^A_1 := D(A)$$

and

$$\|x\|^A_1 := \|(\omega - A)x\|.$$

Moreover, let $\hat{\mathbb{B}}^A := \overline{D(A)}$ of $\mathbb{B}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{B}^A_\beta \hookrightarrow D((\omega-A)^\alpha) \hookrightarrow \mathbb{B}^A_\alpha \hookrightarrow \hat{\mathbb{B}}^A \hookrightarrow \mathbb{B}, \tag{2.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{B}^A_\alpha$ and $\mathbb{B}$. However, we have the following continuous injection

$$\mathbb{B}^A_\beta \to \overline{D(A)}^{\|\cdot\|^A_\alpha} \tag{2.5}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (2.1)-(2.2), we set

$$\mathbb{B}^t_\alpha := \mathbb{B}^{A(t)}_\alpha, \quad \hat{\mathbb{B}}^t := \hat{\mathbb{B}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.4) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ [38, Definition 1.1.1] and hence there is a constant $c(\alpha)$ such that

$$\|y\|^t_\alpha \leq c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.6}$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.1. [7] *Suppose the evolution family $U = U(t,s)$ has exponential dichotomy. For $x \in \mathbb{B}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:*

(i) *There is a constant $c(\alpha)$, such that*

$$\|U(t,s)P(s)x\|^t_\alpha \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.7}$$

(ii) *There is a constant $m(\alpha)$, such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.8}$$

We need the following technical lemma.

Lemma 2.2 ([20, 21, Diagana])**.** *For each $x \in \mathbb{B}$, suppose that the family of operators $A(t)$ $(t \in \mathbb{R})$ satisfy Acquistapce-Terreni conditions, assumption* (H.2) *holds, and that there exist real numbers μ, α, β such that $0 \le \mu < \alpha < \beta < 1$ with $2\alpha > \mu + 1$. Then there is a constant $r(\mu,\alpha) > 0$ such that*

$$\|A(t)U(t,s)x\|_\alpha \le r(\mu,\alpha)e^{-\frac{\delta}{4}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.9}$$

for all $t > s$.

Proof. Let $x \in \mathbb{B}$. First of all, note that $\|A(t)U(t,s)\|_{B(\mathbb{B},\mathbb{B}_\alpha)} \le K(t-s)^{-(1-\alpha)}$ for all t, s such that $0 < t-s \le 1$ and $\alpha \in [0,1]$. Letting $t - s \ge 1$ and using (H2) and the above-mentioned approximate, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\alpha &= \|A(t)U(t,t-1)U(t-1,s)x\|_\alpha \\
&\le \|A(t)U(t,t-1)\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\|U(t-1,s)x\| \\
&\le MKe^{\delta}e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|.
\end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^\alpha \to 0$ as $t \to \infty$ it follows that there exists $c_4(\alpha) > 0$ such that

$$\|A(t)U(t,s)x\|_\alpha \le c_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|.$$

Now, let $0 < t - s \le 1$. Using (2.7) and the fact $2\alpha > \mu + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\alpha &= \|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)x\|_\alpha \\
&\le \|A(t)U(t,\frac{t+s}{2})\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\|U(\frac{t+s}{2},s)x\| \\
&\le k_1\|A(t)U(t,\frac{t+s}{2})\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\|U(\frac{t+s}{2},s)x\|_\mu \\
&\le k_1K\Big(\frac{t-s}{2}\Big)^{\alpha-1}c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&\le c_5(\alpha,\mu)(t-s)^{\alpha-1-\mu}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&\le c_5(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|.
\end{aligned}$$

Therefore there exists $r(\alpha,\mu) > 0$ such that

$$\|A(t)U(t,s)x\|_\alpha \le r(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|$$

for all $t, s \in \mathbb{R}$ with $t \ge s$. □

It should be mentioned that if $U(t,s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbb{R}$. In that case, (2.7) still holds and be rewritten as follows: for all $x \in \mathbb{B}$,

$$\|U(t,s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.10}$$

2.2. **Wiener process and P-th mean almost periodic stochastic processes.** For details of this subsection, we refer the reader to Bezandry and Diagana [9], Corduneanu [14], and the references therein. Throughout this paper, $\mathbb{H}$ and $\mathbb{K}$ will denote real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_{\mathbb{K}}$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. We denote by $L_2(\mathbb{K}, \mathbb{H})$ the space of all Hilbert-Schmidt operators acting between $\mathbb{K}$ and $\mathbb{H}$ equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$.

For a symmetric nonnegative operator $Q \in L_2(\mathbb{K}, \mathbb{H})$ with finite trace we assume that $\{\mathbb{W}(t),\ t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and with values in $\mathbb{K}$. Recall that $\mathbb{W}$ can obtained as follows: let $\{W_i(t),\ t \in \mathbb{R}\}$, $\ i = 1, 2$, be independent $\mathbb{K}$-valued Q-Wiener processes, then

$$\mathbb{W}(t) = \begin{cases} W_1(t) & \text{if } t \geq 0 \\ W_2(-t) & \text{if } t \leq 0 \end{cases}$$

is Q-Wiener process with $\mathbb{R}$ as time parameter. We let $\mathcal{F}_t = \sigma\{\mathbb{W}(s),\ s \leq t\}$.

Let $p \geq 2$. The collection of all strongly measurable, p-th integrable $\mathbb{H}$-valued random variables, denoted by $L^p(\Omega, \mathbb{H})$, is a Banach space equipped with norm

$$\|X\|_{L^p(\Omega,\mathbb{H})} = (\mathbf{E}\|X\|^p)^{1/p} ,$$

where the expectation $\mathbf{E}$ is defined by

$$\mathbf{E}[g] = \int_\Omega g(\omega) d\mathbf{P}(\omega) .$$

Let $\mathbb{K}_0 = Q^{\frac{1}{2}}\mathbb{K}$ and $\mathbb{L}_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$ with respect to the norm

$$\|\Phi\|^2_{\mathbb{L}_2^0} = \|\Phi\, Q^{\frac{1}{2}}\|_2^2 = \mathrm{Tr}(\Phi Q \Phi^*) .$$

Definition 2.3. A stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ is said to be continuous whenever

$$\lim_{t \to s} \mathbf{E}\|X(t) - X(s)\|^p = 0.$$

Definition 2.4. A stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ is said to be stochastically bounded whenever

$$\lim_{N \to \infty} \sup_{t \in \mathbf{R}} \mathbf{P}\Big\{\|X(t)\| > N\Big\} = 0.$$

Definition 2.5. A continuous stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ is said to be p-th mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbf{R}} \mathbf{E}\|X(t+\tau) - X(t)\|^p < \varepsilon . \tag{2.11}$$

A continuous stochastic process X, which is 2-nd mean almost periodic will be called *square-mean almost periodic*.

Like for classical almost periodic functions, the number τ will be called an ε-translation of X.

The collection of all p-th mean almost periodic stochastic processes $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ will be denoted by $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$.

The next lemma provides with some properties of p-th mean almost periodic processes.

Lemma 2.6. *If X belongs to $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$, then*

(i) *the mapping* $t \to \mathbf{E}\|X(t)\|^p$ *is uniformly continuous;*
(ii) *there exists a constant* $M > 0$ *such that* $\mathbf{E}\|X(t)\|^p \leq M$, *for each* $t \in \mathbb{R}$;
(iii) X *is stochastically bounded.*

Lemma 2.7. $AP(\mathbb{R}; L^p(\Omega; \mathbb{B})) \subset BUC(\mathbb{R}; L^p(\Omega; \mathbb{B}))$ *is a closed subspace.*

In view of Lemma 2.7, it follows that the space $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$ of p-th mean almost periodic processes equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1, \|\cdot\|_1)$ and $(\mathbb{B}_2, \|\cdot\|_2)$ be Banach spaces and let $L^p(\Omega; \mathbb{B}_1)$ and $L^p(\Omega; \mathbb{B}_2)$ be their corresponding L^p-spaces, respectively.

Definition 2.8. A function $F : \mathbb{R} \times L^p(\Omega; \mathbb{B}_1) \to L^p(\Omega; \mathbb{B}_2))$, $(t, Y) \mapsto F(t, Y)$, which is jointly continuous, is said to be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in K$ where $K \subset L^p(\Omega; \mathbb{B}_1)$ is a compact if for any $\varepsilon > 0$, there exists $l_\varepsilon(K) > 0$ such that any interval of length $l_\varepsilon(K)$ contains at least a number τ for which

$$\sup_{t \in \mathbf{R}} \mathbf{E}\|F(t+\tau, Y) - F(t, Y)\|_2^p < \varepsilon$$

for each stochastic process $Y : \mathbb{R} \to K$.

We have the following composition result.

Theorem 2.9. *Let* $F : \mathbb{R} \times L^p(\Omega; \mathbb{B}_1) \to L^p(\Omega; \mathbb{B}_2)$, $(t, Y) \mapsto F(t, Y)$ *be a p-th mean almost periodic process in* $t \in \mathbb{R}$ *uniformly in* $Y \in K$, *where* $K \subset L^p(\Omega; \mathbb{B}_1)$ *is any compact subset. Suppose that* $F(t, \cdot)$ *is uniformly continuous on bounded subsets* $K' \subset L^p(\Omega; \mathbb{B}_1)$ *in the following sense: for all* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that* $X, Y \in K'$ *and* $\mathbf{E}\|X - Y\|_1^p < \delta_\varepsilon$, *then*

$$\mathbf{E}\|F(t, Y) - F(t, Z)\|_2^p < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Then for any p-th mean almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega; \mathbb{B}_1)$, *the stochastic process* $t \mapsto F(t, \Phi(t))$ *is p-th mean almost periodic.*

3. Main Results

In this section, we study the existence of p-th mean almost periodic solutions to the class of nonautonomous stochastic differential equations of type (1.2) where $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators on $L^p(\Omega; \mathbb{H})$ satisfying (2.1)-(2.2), and the functions $F_1 : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$, $F_2 : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{L}_2^0)$ are p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in the second variable, and $\mathbb{W}$ is Q-Wiener process taking its values in $\mathbb{K}$ with the real number line as time parameter.

Our method for investigating the existence and uniqueness of a p-th mean almost periodic solution to (1.2) consists of making extensive use of ideas and techniques utilized in [30], [21], and the Schauder fixed-point theorem.

To study the existence of p-th mean almost periodic solutions to (1.2), we suppose that the following assumptions hold:

(H1) The injection $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$ is compact.
(H2) The family of operators $A(t)$ satisfy Acquistapace-Terreni conditions and the evolution family $U(t, s)$ associated with $A(t)$ is exponentially stable; that is, there exist constant M, $\delta > 0$ such that

$$\|U(t, s)\| \leq M e^{-\delta(t-s)}$$

for all $t \geq s$.

(H3) Let μ, α, β be real numbers such that $0 \le \mu < \alpha < \beta < 1$ with $2\alpha > \mu + 1$. Moreover, $\mathbb{H}_\alpha^t = \mathbb{H}_\alpha$ and $\mathbb{H}_\beta^t = \mathbb{H}_\beta$ for all $t \in \mathbb{R}$, with uniform equivalent norms.

(H4) $R(\zeta, A(\cdot)) \in AP(\mathbb{R}, L^p(\Omega; \mathbb{H}))$.

(H5) The function $F_1 : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ is p-th mean almost periodic in the first variable uniformly in the second variable. Furthermore, $X \to F_1(t, X)$ is uniformly continuous on any bounded subset $\mathcal{O}$ of $L^p(\Omega, \mathbb{H})$ for each $t \in \mathbb{R}$. Finally,
$$\sup_{t\in\mathbb{R}} \mathbf{E}\|F_1(t, X)\|^p \le \mathcal{M}_1\big(\|X\|_\infty\big)$$
where $\mathcal{M}_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying
$$\lim_{r\to\infty} \frac{\mathcal{M}_1(r)}{r} = 0\,.$$

(H6) The function $F_2 : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{L}_2^0)$ is p-th mean almost periodic in the first variable uniformly in the second variable. Furthermore, $X \to F_2(t, X)$ is uniformly continuous on any bounded subset $\mathcal{O}'$ of $L^p(\Omega, \mathbb{H})$ for each $t \in \mathbb{R}$. Finally,
$$\sup_{t\in\mathbb{R}} \mathbf{E}\|F_2(t, X)\|^p \le \mathcal{M}_2\big(\|X\|_\infty\big)$$
where $\mathcal{M}_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $\lim_{r\to\infty} \mathcal{M}_2(r)/r = 0$.

In this section, Γ_1 and Γ_2 stand respectively for the nonlinear integral operators defined by
$$\begin{gathered}
(\Gamma_1 X)(t) := \int_{-\infty}^t U(t,s) F_1(s, X(s))\, ds,\\
(\Gamma_2 X)(t) := \int_{-\infty}^t U(t,s) F_2(s, X(s))\, d\mathbb{W}(s)\,.
\end{gathered}$$
In addition to the above-mentioned assumptions, we assume that $\alpha \in \big(0, \frac{1}{2} - \frac{1}{p}\big)$ if $p > 2$ and $\alpha \in \big(0, \frac{1}{2}\big)$ if $p = 2$.

Lemma 3.1. *Under assumptions* (H2)–(H6)*, the mappings* $\Gamma_i : BC(\mathbb{R}, L^p(\Omega, \mathbb{H})) \to BC(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ $(i = 1, 2)$ *are well defined and continuous.*

Proof. We first show that $\Gamma_i\big(BC\big(\mathbb{R}, L^p(\Omega, \mathbb{H})\big)\big) \subset BC\big(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)\big)$ $(i = 1, 2)$. Let us start with $\Gamma_1 X$. Using (2.10) it follows that for all $X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$,
$$\begin{aligned}
&\mathbf{E}\|\Gamma_1 X(t)\|_\alpha^p\\
&\le \mathbf{E}\Big[\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|F_1(s, X(s))\|\, ds\Big]^p\\
&\le c(\alpha)^p \Big(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha} e^{-\frac{\delta}{2}(t-s)}\, ds\Big)^{p-1} \Big(\int_{-\infty}^t e^{-\frac{\delta}{2}(t-s)} \mathbf{E}\|F_1(s, X(s))\|^p\, ds\Big)\\
&\le c(\alpha)^p \Big(\Gamma\big(1 - \frac{p}{p-1}\alpha\big) \Big(\frac{2}{\delta}\Big)^{1-\frac{p}{p-1}\alpha} \Big(\frac{2}{\delta}\Big)^{p-1} \mathcal{M}_1\big(\|X\|_\infty\big)\\
&\le c(\alpha)^p \Big(\Gamma\big(1 - \frac{p}{p-1}\alpha\big)\Big)^{p-1} \Big(\frac{2}{\delta}\Big)^{p(1-\alpha)} \mathcal{M}_1\big(\|X\|_\infty\big)\,,
\end{aligned}$$

and hence

$$\|\Gamma_1 X\|^p_{\alpha,\infty} := \sup_{t\in\mathbb{R}} \mathbf{E}\|\Gamma_1 X(t)\|^p_\alpha \le l(\alpha,\delta,p)\mathcal{M}_1\big(\|X\|_\infty\big)\,,$$

where $l(\alpha,\delta,p) = c(\alpha)^p\Big(\Gamma\big(1-\frac{p}{p-1}\alpha\big)\Big)^{p-1}\Big(\frac{2}{\delta}\Big)^{p(1-\alpha)}$.

As to $\Gamma_2 X$, we proceed into two steps. For $p>2$, we need the following estimates.

Lemma 3.2. *Let $p>2$, $0<\alpha<1$, $\alpha+\frac{1}{p}<\xi<1/2$, and $\Psi:\Omega\times\mathbb{R}\to\mathbb{L}_2^0$ be an $(\mathcal{F}_t)$-adapted measurable stochastic process such that*

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0} < \infty\,.$$

Then

(i) $\mathbf{E}\|\int_{-\infty}^t (t-s)^{-\xi}U(t,s)\Psi(s)\,d\mathbb{W}(s)\|^p \le s(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}$;

(ii) $\mathbf{E}\|\int_{-\infty}^t U(t,s)\Psi(s)\,d\mathbb{W}(s)\|^p_\alpha \le k(\Gamma,\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}$

where $s(\Gamma,\xi,\delta,p)$ and $k(\Gamma,\alpha,\xi,\delta,p)$ are positive constants with Γ a classical Gamma function.

Proof. (i) A direct application of a Proposition due to De Prato and Zabczyk [18] and Holder's inequality allows us to write

$$\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^t (t-\sigma)^{-\xi}U(t,\sigma)\Psi(\sigma)\,d\mathbb{W}(\sigma)\|^p\\
&\le C_p\mathbf{E}\Big[\int_{-\infty}^t (t-\sigma)^{-2\xi}\|U(t,\sigma)\Psi(\sigma)\|^2\,d\sigma\Big]^{p/2}\\
&\le C_pN^p\mathbf{E}\Big[\int_{-\infty}^t (t-\sigma)^{-2\xi}e^{-2\delta(t-\sigma)}\|\Psi(\sigma)\|^2_{\mathbb{L}_2^0}\,d\sigma\Big]^{p/2}\\
&\le C_pN^p\Big(\int_{-\infty}^t (t-\sigma)^{-2\xi}e^{-2\delta(t-\sigma)}\,d\sigma\Big)^{p-1}\Big(\int_{-\infty}^t e^{-2\delta(t-\sigma)}\mathbf{E}\|\Psi(\sigma)\|^p_{\mathbb{L}_2^0}\,d\sigma\Big)\\
&\le C_pN^p\Big(\Gamma(1-\frac{2p\xi}{p-2})(2\delta)^{\frac{2p\xi}{p-2}-1}\Big)^{\frac{p-2}{2}}\Big(\frac{1}{2\delta}\Big)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}\\
&\le s(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}\,.
\end{aligned}$$

To prove (ii), we use the factorization method of the stochastic convolution integral.

$$\int_{-\infty}^t U(t,s)\Psi(s)\,d\mathbb{W}(s) = \frac{\sin\pi\xi}{\pi}(R_\xi\mathbb{S}_\Psi)(t)\quad\text{a.s.}\tag{3.1}$$

where

$$(R_\xi\mathbb{S}_\Psi)(t) = \int_{-\infty}^t (t-s)^{\xi-1}U(t,s)\mathbb{S}_\Psi(s)\,ds$$

with

$$\mathbb{S}_\Psi(s) = \int_{-\infty}^s (s-\sigma)^{-\xi}U(s,\sigma)\Psi(\sigma)\,d\mathbb{W}(\sigma)\,,$$

and ξ satisfying $\alpha+\frac{1}{p}<\xi<1/2$. We can now evaluate

$$\mathbf{E}\|\int_{-\infty}^t U(t,s)\Psi(s)\,d\mathbb{W}(s)\|^p_\alpha$$

$$
\begin{aligned}
&\le \Big|\frac{\sin(\pi\xi)}{\pi}\Big|^p \mathbf{E}\Big[\int_{-\infty}^t (t-s)^{-\xi}\|U(t,s)\mathbb{S}_\Psi(s)\|_\alpha\,ds\Big]^p \\
&\le M(\alpha)^p\Big|\frac{\sin(\pi\xi)}{\pi}\Big|^p \mathbf{E}\Big[\int_{-\infty}^t (t-s)^{\xi-\alpha-1}e^{-\delta(t-s)}\|\mathbb{S}_\Psi(s)\|_\alpha\,ds\Big]^p \\
&\le M(\alpha)^p\Big|\frac{\sin(\pi\xi)}{\pi}\Big|^p \Big(\int_{-\infty}^t (t-s)^{\frac{p}{p-1}(\xi-\alpha-1)}e^{-\delta(t-s)}\,ds\Big)^{p-1}\times \\
&\quad\times\Big(\int_{-\infty}^t e^{-\delta(t-s)}\mathbf{E}\|\mathbb{S}_\Psi(s)\|^p\,ds\Big) \\
&\le r(\Gamma,\alpha,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\Psi(s)\|^p\,.
\end{aligned}
$$

On the other hand, it follows from part (i) that

$$
\mathbf{E}\|\mathbb{S}_\Psi(t)\|^p \le s(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}\,. \tag{3.2}
$$

Thus,

$$
\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^t U(t,s)\Psi(s)\,d\mathbb{W}(s)\|_\alpha^p \\
&\le r(\Gamma,\alpha,\xi,\delta,p)s(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0} \\
&\le k(\Gamma,\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi(t)\|^p_{\mathbb{L}_2^0}\,.
\end{aligned}
$$

□

We now use the estimates obtained in Lemma 3.2 (ii) to obtain

$$
\begin{aligned}
\mathbf{E}\|\Gamma_2X(t)\|_\alpha^p &\le k(\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\|F_2(s,X(s))\|^p_{\mathbb{L}_2^0} \\
&\le k(\alpha,\xi,\delta,p)\mathcal{M}_2\big(\|X\|_\infty\big)\,,
\end{aligned}
$$

and hence

$$
\|\Gamma_2X\|^p_{\alpha,\infty}\le k(\alpha,\xi,\delta,p)\mathcal{M}_2\big(\|X\|_\infty\big)\,,
$$

where $k(\alpha,\xi,\delta,p)$ is a positive constant. For $p=2$, we have

$$
\begin{aligned}
\mathbf{E}\|\Gamma_2X(t)\|_\alpha^2 &= \mathbf{E}\|\int_{-\infty}^t U(t,s)F_2(s,X(s))\,d\mathbb{W}(s)\|_\alpha^2 \\
&\le c(\alpha)^2\int_{-\infty}^t (t-s)^{-2\alpha}e^{-\delta(t-s)}\mathbf{E}\|F_2(s,X(s))\|^2_{\mathbb{L}_2^0} \\
&\le c(\alpha)^2\Gamma\big(1-2\alpha\big)\delta^{1-2\alpha}\mathcal{M}_2\big(\|X\|_\infty\big)\,,
\end{aligned}
$$

and hence

$$
\|\Gamma_2X\|^2_{\alpha,\infty}\le s(\alpha,\delta)\mathcal{M}_2\big(\|X\|_\infty\big)\,,
$$

where $s(\alpha,\delta)=c(\alpha)^2\Gamma\big(1-2\alpha\big)\delta^{1-2\alpha}$.

For the continuity, let $X^n\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$ be a sequence which converges to some $X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$; that is, $\|X^n-X\|_\infty\to 0$ as $n\to\infty$. It follows from the estimates in Proposition 2.1 that

$$
\mathbf{E}\|\int_{-\infty}^t U(t,s)[F_1(s,X^n(s))-F_1(s,X(s))]\,ds\|_\alpha^p
$$

$$\leq \mathbf{E}\Big[\int_{-\infty}^{t} c(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|F_1(s,X^n(s)) - F_1(s,X(s))\|\,ds\Big]^p .$$

Now, using the continuity of F_1 and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\|\int_{-\infty}^{t} U(t,s)[F_1(s,X^n(s)) - F_1(s,X(s))]\,ds\|_\alpha^p \to 0 \quad \text{as } n\to\infty .$$

Therefore,

$$\|\Gamma_1 X^n - \Gamma_1 X\|_{\infty,\alpha} \to 0 \quad \text{as } n\to\infty.$$

For the term containing the Wiener process $\mathbb{W}$, we use the estimates in Lemma 3.2 to obtain

$$\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^{t} U(t,s)[F_2(s,X^n(s)) - F_2(s,X(s))]\,d\mathbb{W}(s)\|_\alpha^p\\
&\leq k(\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}} \mathbf{E}\|F_2(t,X^n(t)) - F_2(t,X(t))\|^p
\end{aligned}$$

for $p>2$ and

$$\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^{t} U(t,s)[F_2(s,X^n(s)) - F_2(s,X(s))]\,d\mathbb{W}(s)\|_\alpha^2\\
&\leq n(\alpha)^2 \int_{-\infty}^{t} (t-s)^{-2\alpha} e^{-\delta(t-s)} \mathbf{E}\|F_2(s,X(s)^n) - F_2(s,X(s))\|^2\,ds
\end{aligned}$$

for $p=2$.

Now, using the continuity of G and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\|\int_{-\infty}^{t} U(t,s)[F_2(s,X^n(s)) - F_2(s,X(s))]\,d\mathbb{W}(s)\|_\alpha^p \to 0 \quad \text{as } n\to\infty .$$

Therefore,

$$\|\Gamma_2 X^n - \Gamma_2 X\|_{\infty,\alpha} \to 0 \quad \text{as } n\to\infty.$$

□

Lemma 3.3. *Under assumptions* (H2)–(H6)*, the integral operator* Γ_i $(i=1,2)$ *maps* $AP\big(\mathbb{R},L^p(\Omega,\mathbb{H})\big)$ *into itself.*

Proof. Let us first show that $\Gamma_1 X(\cdot)$ is p-th mean almost periodic et let $f_1(t) = F_1(t,X(t))$. Indeed, assuming that X is p-th mean almost periodic and using assumption (H5), Theorem 2.9, and [39, Proposition 4.4], given $\varepsilon>0$, one can find $l_\varepsilon>0$ such that any interval of length l_ε contains at least τ with the property that

$$\|U(t+\tau,s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\geq\varepsilon$, and

$$\mathbf{E}\|f_1(\sigma+\tau) - f_1(\sigma)\|^p < \eta$$

for each $\sigma\in\mathbb{R}$, where $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$. Moreover, it follows from Lemma 2.6 (ii) that there exists a positive constant K_1 such that

$$\sup_{\sigma\in\mathbb{R}} \mathbf{E}\|f_1(\sigma)\|^p \leq K_1 .$$

Now, using assumption (H2) and Holder's inequality, we obtain

$$\mathbf{E}\|\Gamma_1 X(t+\tau) - \Gamma_1 X(t)\|^p$$

$$
\begin{aligned}
&\le 3^{p-1}\mathbf{E}\Big[\int_0^\infty \|U(t+\tau,t+\tau-s)\|\|f_1(t+\tau-s)-f_1(t-s)\|\,ds\Big]^p\\
&\quad + 3^{p-1}\,\mathbf{E}\Big[\int_\varepsilon^\infty \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\|f_1(t-s)\|\,ds\Big]^p\\
&\quad + 3^{p-1}\,\mathbf{E}\Big[\int_0^\varepsilon \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\|f_1(t-s)\|\,ds\Big]^p\\
&\le 3^{p-1}M^p\mathbf{E}\Big[\int_0^\infty e^{-\delta s}\|f_1(t+\tau-s)-f_1(t-s)\|\,ds\Big]^p\\
&\quad + 3^{p-1}\varepsilon^p\,\mathbf{E}\Big[\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\|f_1(t-s)\|\,ds\Big]^p + 3^{p-1}M^p\,\mathbf{E}\Big[\int_0^\varepsilon 2e^{-\delta s}\|f_1(t-s)\|\,ds\Big]^p\\
&\le 3^{p-1}M^p\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^\infty e^{-\delta s}\mathbf{E}\|f_1(t+\tau-s)-f_1(t-s)\|^p\,ds\Big)\\
&\quad + 3^{p-1}\varepsilon^p\,\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_\varepsilon^\infty e^{-\frac{\delta p s}{2}}\mathbf{E}\|f_1(t-s)\|^p\,ds\Big)\\
&\quad + 6^{p-1}M^p\,\Big(\int_0^\varepsilon e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^\varepsilon e^{-\frac{\delta p s}{2}}\mathbf{E}\|f_1(t-s)\|^p\,ds\Big)\\
&\le 3^{p-1}M^p\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\|f_1(t+\tau-s)-f_1(t-s)\|^p\\
&\quad + 3^{p-1}\varepsilon^p\,\Big(\int_\varepsilon^\infty e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\|f_1(t-s)\|^p\\
&\quad + 6^{p-1}M^p\,\Big(\int_0^\varepsilon e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\|f_1(t-s)\|^p\\
&\le 3^{p-1}M^p\Big(\frac{1}{\delta^p}\Big)\eta + 3^{p-1}M^pK_1\Big(\frac{1}{\delta^p}\Big)\varepsilon^p + 6^{p-1}M^p\varepsilon^pK_1\varepsilon^p.
\end{aligned}
$$

As for $\Gamma_2X(\cdot)$, we split the proof in two cases: $p>2$ and $p=2$. To this end, we let $f_2(t)=F_2(t,X(t))$. Let us start with the case where $p>2$. Assuming that X is p-th mean almost periodic and using assumption (H6), Theorem 2.9, and [39, Proposition 4.4], given $\varepsilon>0$, one can find $l_\varepsilon>0$ such that any interval of length l_ε contains at least τ with the property that

$$\|U(t+\tau,s+\tau)-U(t,s)\|\le\varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\ge\varepsilon$, and

$$\mathbf{E}\|f_2(\sigma+\tau)-f_2(\sigma)\|^p<\eta$$

for each $\sigma\in\mathbb{R}$, where $\eta(\varepsilon)\to0$ as $\varepsilon\to0$.

Moreover, it follows from Lemma 2.6 (ii) that there exists a positive constant K_2 such that

$$\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|f_2(\sigma)\|^p\le K_2\,.$$

Now

$$
\begin{aligned}
&\mathbf{E}\|f_2(t+\tau)-f_2(t)\|^p\\
&\le 3^{p-1}\mathbf{E}\Big\|\int_0^\infty U(t+\tau,t+\tau-s)\Big[f_2(t+\tau-s)-f_2(t-s)\Big]\,d\mathbb{W}(s)\|^p\\
&\quad + 3^{p-1}\,\mathbf{E}\Big\|\int_\varepsilon^\infty\Big[U(t+\tau,t+\tau-s)-U(t,t-s)\Big]f_2(t-s)\,d\mathbb{W}(s)\|^p
\end{aligned}
$$

$$+ 3^{p-1}\,\mathbf{E}\Big\|\int_0^{\varepsilon}\Big[U(t+\tau,t+\tau-s)-U(t,t-s)\Big]f_2(t-s)\,d\mathbb{W}(s)\|^p.$$

We then have

$$\begin{aligned}
&\mathbf{E}\|\Gamma_2X(t+\tau)-\Gamma_2X(t)\|^p\\
&\le 3^{p-1}C_p\mathbf{E}\Big[\int_0^{\infty}\|U(t+\tau,t+\tau-s)\|^2\|f_2(t+\tau-s)-f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\quad+3^{p-1}C_p\,\mathbf{E}\Big[\int_{\varepsilon}^{\infty}\|U(t+\tau,t+\tau-s)-U(t,t-s)\|^2\|f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\quad+3^{p-1}C_p\,\mathbf{E}\Big[\int_0^{\varepsilon}\|U(t+\tau,t+\tau-s)-U(t,t-s)\|^2\|f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\le 3^{p-1}C_pM^p\mathbf{E}\Big[\int_0^{\infty}e^{-2\delta s}\|f_2(t+\tau-s)-f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\quad+3^{p-1}C_p\varepsilon^p\mathbf{E}\Big[\int_{\varepsilon}^{\infty}e^{-\delta s}\|f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\quad+3^{p-1}2^{p/2}C_p\,\mathbf{E}\Big[\int_0^{\varepsilon}e^{-2\delta s}\|f_2(t-s)\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\le 3^{p-1}C_pM^p\Big(\int_0^{\infty}e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\infty}e^{-\frac{p\delta s}{2}}\,\|f_2(t+\tau-s)-f_2(t-s)\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&\quad+3^{p-1}C_p\varepsilon^p\Big(\int_{\varepsilon}^{\infty}e^{-\frac{p\delta s}{2(p-2)}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_{\varepsilon}^{\infty}e^{-\frac{p\delta s}{4}}\,\mathbf{E}\|f_2(t-s)\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&\quad+3^{p-1}2^{p/2}C_p\,M^p\Big(\int_0^{\varepsilon}e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\varepsilon}e^{-\frac{p\delta s}{2}}\,\mathbf{E}\|f_2(t-s)\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&\le 3^{p-1}C_pM^p\eta\Big(\int_0^{\infty}e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\infty}e^{-\frac{p\delta s}{2}}\,ds\Big)\\
&\quad+3^{p-1}C_p\varepsilon^pK_2\Big(\int_{\varepsilon}^{\infty}e^{-\frac{p\delta s}{2(p-2)}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_{\varepsilon}^{\infty}e^{-\frac{p\delta s}{4}}\,ds\Big)\\
&\quad+3^{p-1}2^{p/2}C_p\,M^pK_2\Big(\int_0^{\varepsilon}e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\varepsilon}e^{-\frac{p\delta s}{2}}\,ds\Big)\\
&\le 3^{p-1}C_pM^p\eta\Big(\frac{p-2}{p\delta}\Big)^{p-2}\Big(\frac{2}{p\delta}\Big)\\
&\quad+3^{p-1}C_p\varepsilon^pK_2\Big(\frac{2(p-2)}{p\delta}\Big)^{\frac{p-2}{2}}\Big(\frac{4}{p\delta}\Big)+3^{p-1}2^{p/2}C_p\,M^pK_2\varepsilon^p.
\end{aligned}$$

As to the case $p=2$, we proceed in the same way an using isometry inequality to obtain

$$\begin{aligned}
&\mathbf{E}\|\Gamma_2X(t+\tau)-\Gamma_2X(t)\|^2\\
&\le 3\,M^2\Big(\int_0^{\infty}e^{-2\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|f_2(\sigma+\tau)-f-2(\sigma)\|^2_{\mathbb{L}_2^0}\\
&\quad+3\varepsilon^2\Big(\int_{\varepsilon}^{\infty}e^{-\delta\,s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|f_2(\sigma)\|^2_{\mathbb{L}_2^0}+6M^2\Big(\int_0^{\varepsilon}e^{-2\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|f_2(\sigma)\|^2_{\mathbb{L}_2^0}\\
&\le 3\Big[\eta\frac{M^2}{2\delta}+\varepsilon\frac{K_2}{\delta}+2\varepsilon K_2\Big].
\end{aligned}$$

Hence, $\Gamma_2 X(\cdot)$ is p-th mean almost periodic. $\square$

Let $\gamma \in (0,1]$ and let

$$BC^\gamma\big(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)\big) = \Big\{X \in BC\big(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)\big) : \|X\|_{\alpha,\gamma} < \infty\Big\},$$

where

$$\|X\|_{\alpha,\gamma} = \sup_{t\in\mathbb{R}} \Big[\mathbf{E}\|X(t)\|_\alpha^p\Big]^{1/p} + \gamma \sup_{t,s\in\mathbb{R}, s\neq t} \frac{\Big[\mathbf{E}\|X(t)-X(s)\|_\alpha^p\Big]^{1/p}}{\big|t-s\big|^\gamma}\,.$$

Clearly, the space $BC^\gamma\big(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)\big)$ equipped with the norm $\|\cdot\|_{\alpha,\gamma}$ is a Banach space, which is in fact the Banach space of all bounded continuous Holder functions from $\mathbb{R}$ to $L^p(\Omega,\mathbb{H}_\alpha)$ whose Holder exponent is γ.

Lemma 3.4. *Under assumptions* (H1)–(H6)*, the mapping* Γ_1 *defined previously maps bounded sets of* $BC\big(\mathbb{R}, L^p(\Omega,\mathbb{H})\big)$ *into bounded sets of* $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ *for some* $0<\gamma<1$.

Proof. Let $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and let $f_1(t) = F_1(t, X(t))$ for each $t\in\mathbb{R}$. Proceeding as before, we have

$$\mathbf{E}\|\Gamma_1 X(t)\|_\alpha^p \leq c\mathbf{E}\|\Gamma_1 X(t)\|_\beta^p \leq c\cdot l(\beta,\delta,p)\mathcal{M}_1\big(\|X\|_\infty\big)\,.$$

Let $t_1 < t_2$. Clearly, we have

$$\begin{aligned}
&\mathbf{E}\|(\Gamma_1 X)(t_2) - (\Gamma_1 X)(t_1)\|_\alpha^p \\
&\leq 2^{p-1}\mathbf{E}\|\int_{t_1}^{t_2} U(t_2,s)f_1(s)\,ds\|_\alpha^p + 2^{p-1}\mathbf{E}\|\int_{-\infty}^{t_1}[U(t_2,s)-U(t_1,s)]f_1(s)\,ds\|_\alpha^p \\
&= 2^{p-1}\mathbf{E}\|\int_{t_1}^{t_2} U(t_2,s)f_1(s)\,ds\|_\alpha^p + 2^{p-1}\mathbf{E}\|\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\frac{\partial U(\tau,s)}{\partial\tau}d\tau\Big)f_1(s)\,ds\|_\alpha^p \\
&= 2^{p-1}\mathbf{E}\|\int_{t_1}^{t_2} U(t_2,s)f_1(s)\,ds\|_\alpha^p + 2^{p-1}\mathbf{E}\|\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}A(\tau)U(\tau,s)f_1(s)\,d\tau\Big)\,ds\|_\alpha^p \\
&= N_1 + N_2.
\end{aligned}$$

Clearly,

$$\begin{aligned}
N_1 &\leq \mathbf{E}\Big\{\int_{t_1}^{t_2}\|U(t_2,s)f_1(s)\|_\alpha\,ds\Big\}^p \\
&\leq c(\alpha)^p\mathbf{E}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\frac{\delta}{2}(t_2-s)}\|f_1(s)\|\,ds\Big\}^p \\
&\leq c(\alpha)^p\Big(\mathcal{M}_1\big(\|X\|\big)\Big)\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}e^{-\frac{\delta}{2}(t_2-s)}\Big)^{p-1}\Big(\int_{t_1}^{t_2}e^{-\frac{\delta}{2}(t_2-s)}\,ds\Big) \\
&\leq c(\alpha)^p\Big(\mathcal{M}_1\big(\|X\|\big)\Big)\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}\Big)^{p-1}\Big(t_2-t_1\Big) \\
&\leq c(\alpha)^p\mathcal{M}_1\big(\|X\|\big)\Big(1-\frac{p}{p-1}\alpha\Big)^{-(p-1)}(t_2-t_1)^{p(1-\alpha)}\,.
\end{aligned}$$

Similarly, using estimates in Lemma 2.2

$$N_2 \leq \mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\|A(\tau)U(\tau,s)f_1(s)\|_\alpha\,d\tau\Big)\,ds\Big\}^p$$

$$
\begin{aligned}
&\le r(\mu,\alpha)^p \mathbf{E}\Big\{ \int_{-\infty}^{t_1} \Big(\int_{t_1}^{t_2} (\tau-s)^{-\alpha} e^{-\frac{\delta}{4}(\tau-s)} \|f_1(s)\| \, d\tau \Big) \, ds \Big\}^p \\
&\le r(\mu,\alpha)^p \mathbf{E}\Big[\int_{t_1}^{t_2} \Big(\Big(\int_{-\infty}^{t_1} (\tau-s)^{-\frac{p}{p-1}\alpha} e^{-\frac{\delta}{4}(\tau-s)} \, ds \Big)^{\frac{p-1}{p}} \Big) \\
&\quad \times \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(\tau-s)} \|f_1(s)\|^p \, ds \Big)^{1/p} d\tau \Big]^p \\
&\le r(\mu,\alpha)^p \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(t_1-s)} \mathbf{E}\|f_1(s)\|^p \, ds \Big) \\
&\quad \times \Big[\int_{t_1}^{t_2} \Big(\Big(\int_{-\infty}^{t_1} (\tau-s)^{-\frac{p}{p-1}\alpha} e^{-\frac{\delta}{4}(\tau-s)} \, ds \Big)^{\frac{p-1}{p}} \Big) d\tau \Big]^p \\
&\le r(\mu,\alpha)^p \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(t_1-s)} \mathbf{E}\|f_1(s)\|^p \, ds \Big) \\
&\quad \times \Big[\int_{t_1}^{t_2} (\tau-t_1)^{-\alpha} \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(\tau-s)} \, ds \Big)^{\frac{p-1}{p}} \Big) d\tau \Big]^p \\
&\le r(\mu,\alpha)^p \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(t_1-s)} \mathbf{E}\|f_1(s)\|^p \, ds \Big) \\
&\quad \times \Big[\int_{t_1}^{t_2} (\tau-t_1)^{-\alpha} \Big(\int_{\tau-t_1}^{\infty} e^{-\frac{\delta}{4}r} \, dr \Big)^{\frac{p-1}{p}} \Big) d\tau \Big]^p \\
&\le r(\mu,\alpha)^p \mathcal{M}_1\big(\|X\|\big) \Big(\frac{2}{p}\Big)^p (1-\beta)^{-p} (t_2-t_1)^{p(1-\alpha)} \,.
\end{aligned}
$$

For $\gamma = 1-\alpha$, one has

$$
\mathbf{E}\|(\Gamma_1 X)(t_2) - (\Gamma_1 X)(t_1)\|_\alpha^p \le s(\alpha,\beta,\delta)\mathcal{M}_1\big(\|X\|\big)\big|t_2-t_1\big|^{p\gamma}
$$

where $s(\alpha,\beta,\delta)$ is a positive constant. $\square$

Lemma 3.5. *Let $\alpha,\beta \in \big(0,\frac{1}{2}\big)$ with $\alpha < \beta$. Under assumptions* (H1)-(H6)*, the mapping Γ_2 defined previously maps bounded sets of $BC\big(\mathbb{R}, L^p(\Omega,\mathbb{H})\big)$ into bounded sets of $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ for some $0<\gamma<1$.*

Proof. Let $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and let $f_2(t) = F_2(t, X(t))$ for each $t \in \mathbb{R}$. We break down the computations in two cases: $p>2$ and $p=2$.

For $p>2$, we have

$$
\mathbf{E}\|\Gamma_2 X(t)\|_\alpha^p \le c\mathbf{E}\|\Gamma_2 X(t)\|_\beta^p \le c\cdot k(\beta,\xi,\delta,p)\mathcal{M}_2\big(\|X\|_\infty\big)\,.
$$

Let $t_1 < t_2$. Clearly,

$$
\begin{aligned}
&\mathbf{E}\|(\Gamma_2 X)(t_2) - (\Gamma_2 X)(t_1)\|_\alpha^p \\
&\le 2^{p-1}\mathbf{E}\|\int_{t_1}^{t_2} U(t_2,s) f_2(s)\, d\mathbb{W}(s)\|_\alpha^p \\
&\quad + 2^{p-1}\mathbf{E}\|\int_{-\infty}^{t_1} [U(t_2,s) - U(t_1,s)] f_2(s)\, d\mathbb{W}(s)\|_\alpha^p \\
&= N_1' + N_2'.
\end{aligned}
$$

We use the factorization method (3.1) to obtain

$$\begin{aligned}
N_1' &= \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\| \int_{t_1}^{t_2} (t_2-s)^{\xi-1} U(t_2,s)\mathbb{S}_{f_2}(s)\,ds\|_\alpha^p \\
&\le \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\Big[\int_{t_1}^{t_2} (t_2-s)^{\xi-1} \|U(t_2,s)\mathbb{S}_{f_2}(s)\|_\alpha\,ds\Big]^p \\
&\le M(\alpha)^p \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\Big[\int_{t_1}^{t_2} (t_2-s)^{\xi-1}(t_2-s)^{\alpha} e^{-\frac{\delta}{2}(t_2-s)} \|\mathbb{S}_{f_2}(s)\|\,ds\Big]^p \\
&\le M(\alpha)^p \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \Big(\int_{t_1}^{t_2} (t_2-s)^{-\frac{p}{p-1}\alpha}\,ds\Big)^{p-1} \\
&\quad \times \Big(\int_{t_1}^{t_2} (t_2-s)^{-p(1-\xi)} e^{-p\frac{\delta}{2}(t_2-s)} \mathbf{E}\|\mathbb{S}_{f_2}(s)\|^p\,ds\Big) \\
&\le M(\alpha)^p \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \Big(\int_{t_1}^{t_2} (t_2-s)^{-\frac{p}{p-1}\alpha}\,ds\Big)^{p-1} \times \\
&\quad \times \Big(\int_{t_1}^{t_2} (t_2-s)^{-p(1-\xi)} e^{-p\frac{\delta}{2}(t_2-s)}\,ds\Big) \sup_{t\in\mathbb{R}} \mathbf{E}\|\mathbb{S}_{f_2}(t)\|^p \\
&\le s(\xi,\delta,\Gamma,p)\Big(1-\frac{p}{p-1}\alpha\Big)^{-(p-1)} \mathcal{M}_2\big(\|X\|_\infty\big)(t_2-t_1)^{p(1-\alpha)}
\end{aligned}$$

where $s(\xi,\delta,\Gamma,p)$ is a positive constant. Similarly,

$$\begin{aligned}
N_2' &= \mathbf{E}\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} \frac{\partial}{\partial\tau} U(\tau,s)\,d\tau\Big] f_2(s)\,d\mathbb{W}(s)\|_\alpha^p \\
&= \mathbf{E}\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} A(\tau)U(\tau,s)\,d\tau\Big] f_2(s)\,d\mathbb{W}(s)\|_\alpha^p\,.
\end{aligned}$$

Now, using the representation (3.1) together with a stochastic version of the Fubini theorem with the help of Lemma 2.2 gives us

$$\begin{aligned}
N_2' &= \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\| \int_{t_1}^{t_2} \Big(A(\tau)U(\tau,t_1)\int_{-\infty}^{t_1} (t_1-s)^{\xi-1} U(t_1,s)\mathbb{S}_{f_2}(s)\,ds\Big)\,d\tau\|_\alpha^p \\
&\le \big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\Big[\int_{t_1}^{t_2} \Big(\int_{-\infty}^{t_1} (t_1-s)^{\xi-1} \|A(\tau)U(\tau,s)\mathbb{S}_{f_2}(s)\|_\alpha\,ds\Big)\,d\tau\Big]^p \\
&\le r(\mu,\alpha)\big|\frac{\sin(\pi\xi)}{\pi}\big|^p \mathbf{E}\Big[\int_{t_1}^{t_2} \Big(\int_{-\infty}^{t_1} (t_1-s)^{\xi-1}(\tau-s)^{-\alpha} e^{\frac{\delta}{4}(\tau-s)} \|\mathbb{S}_{f_2}(s)\|\,ds\Big)\,d\tau\Big]^p
\end{aligned}$$

where ξ satisfies $\beta+\frac{1}{p}<\xi<1/2$. Since $\tau>t_1$, it follows from Holder's inequality that

$$\begin{aligned}
&N_2' \\
&\le r(\mu,\alpha)\big|\frac{\sin(\pi\xi}{\pi}\big|^p \mathbf{E}\Big[\int_{t_1}^{t_2} (\tau-t_1)^{-\alpha}\Big(\int_{-\infty}^{t_1} (t_1-s)^{\xi-1} e^{-\frac{\delta}{4}(\tau-s)} \|\mathbb{S}_{f_2}(s)\|\,ds\Big)\,d\tau\Big]^p \\
&\le r(\mu,\alpha)\big|\frac{\sin(\pi\xi}{\pi}\big|^p \mathbf{E}\Big[\Big(\int_{t_1}^{t_2} (\tau-t_1)^{-\alpha}\,d\tau\Big)^p
\end{aligned}$$

$$\begin{aligned}
&\times \Big(\int_{-\infty}^{t_1} (t_1 - s)^{\xi-1} e^{-\frac{\delta}{4}(t_1-s)} \|\mathbb{S}_{f_2}(s)\|\, ds\Big)^p \Big] \\
&\le r(\mu,\alpha) \Big|\frac{\sin(\pi\xi}{\pi}\Big|^p (t_2 - t_1)^{p(1-\alpha)} \Big(\int_{-\infty}^{t_1} (t_1 - s)^{\frac{p}{p-1}(\xi-\alpha-1)} e^{\frac{\delta}{4}(t_1-s)}\, ds\Big)^{p-1} \\
&\times \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{4}(t_1-s)}\, ds\Big) \sup_{s\in\mathbb{R}} \mathbf{E}\|\mathbb{S}_{f_2}(s)\|^p \\
&\le r(\xi,\beta,\delta,\Gamma,p)(1-\alpha)^{-p} \mathcal{M}_2\big(\|X\|_\infty\big)(t_2 - t_1)^{p(1-\alpha)}\,.
\end{aligned}$$

For $\gamma = 1-\alpha$, one has

$$\begin{aligned}
&\Big[\mathbf{E}\|(\Gamma_2 X)(t_2) - (\Gamma_2 X)(t_1)\|_\alpha^p\Big]^{1/p} \\
&\le r(\xi,\beta,\delta,\Gamma,p)(1-\alpha)^{-1}\Big[\mathcal{M}_2\big(\|X\|_\infty\big)\Big]^{1/p}(t_2 - t_1)^\gamma\,.
\end{aligned}$$

As for $p=2$, we have

$$\mathbf{E}\|\Gamma_2 X(t)\|_\alpha^2 \le c\mathbf{E}\|\Gamma_2 X(t)\|_\beta^2 \le c\cdot s(\beta,\delta)\mathcal{M}_2\big(\|X\|_\infty\big)\,.$$

For $t_1 < t_2$, let us start with the first term. By Ito isometry identity, we have

$$\begin{aligned}
N_1' &\le c(\alpha)^2 \Big\{ \int_{t_1}^{t_2} (t_2 - s)^{-2\alpha} e^{-\delta(t_2-s)} \mathbf{E}\|f_2(s)\|_{\mathbb{L}_2^0}^2\, ds \\
&\le c(\alpha)^2 \Big(\int_{t_1}^{t_2} (t_2 - s)^{-2\alpha}\, ds\Big) \sup_{s\in\mathbb{R}} \mathbf{E}\|f_2(s)\|_{\mathbb{L}_2^0}^2 \\
&\le c(\alpha)(1-2\alpha)^{-1}\mathcal{M}_2\big(\|X\|_\infty\big)(t_2 - t_1)^{1-2\alpha}\,.
\end{aligned}$$

Similarly, using the estimates in Lemma 2.2 we have

$$\begin{aligned}
N_2' &= \mathbf{E}\|\int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} \frac{\partial}{\partial\tau} U(\tau,s)\, d\tau\Big] f_2(s)\, d\mathbb{W}(s)\|_\alpha^2 \\
&= \mathbf{E}\|\int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} A(\tau)U(\tau,s)\, d\tau\Big] f_2(s)\, d\mathbb{W}(s)\|_\alpha^2 \\
&= \mathbf{E}\|\int_{t_1}^{t_2} A(\tau)U(\tau,t_1)\Big\{\int_{-\infty}^{t_1} U(t_1,s) f_2(s)\, d\mathbb{W}(s)\Big\}\, d\tau\|_\alpha^2 \\
&\le \mathbf{E}\Big[\int_{t_1}^{t_2} \|\int_{-\infty}^{t_1} A(\tau)U(\tau,s) f_2(s)\, d\mathbb{W}(s)\|_\alpha^2\, d\tau\Big]^2 \\
&\le r(\mu,\alpha)^2 (t_2 - t_1) \int_{t_1}^{t_2} \Big\{\int_{-\infty}^{t_1} (\tau - s)^{-2\alpha} e^{-\frac{\delta}{2}(\tau-s)} \mathbf{E}\|f_2(s)\|_{\mathbb{L}_2^0}^2\, ds\Big\}\, d\tau \\
&\le r(\mu,\alpha)^2 (t_2 - t_1) \Big(\int_{t_1}^{t_2} (\tau - t_1)^{-2\alpha}\, d\tau\Big)\Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{2}(t_1-s)} \mathbf{E}\|f_2(s)\|_{\mathbb{L}_2^0}^2\, ds\Big) \\
&\le r(\mu,\alpha)^2 (1-2\alpha)^{-1} \mathcal{M}_2\big(\|X\|_\infty\big)(t_2 - t_1)^{2(1-\alpha)}\,.
\end{aligned}$$

For $\gamma = \frac{1}{2} - \alpha$, one has

$$\Big[\mathbf{E}\|(\Gamma_2 X)(t_2) - (\Gamma_2 X)(t_1)\|_\alpha^2\Big]^{1/2} \le r(\xi,\beta,\delta)(1-2\beta)^{-1/2}\Big[\mathcal{M}_2\big(\|X\|_\infty\big)\Big]^{1/2}(t_2 - t_1)^\gamma\,.$$

Therefore, for each $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ such that $\mathbf{E}\|X(t)\|^p \le R$ for all $t \in \mathbb{R}$, then $\Gamma_i X(t)$ belongs to $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ with $\mathbf{E}\|\Gamma_i X(t)\|^p \le R'$ where R' depends on R. $\square$

Lemma 3.6. *The integral operators Γ_i map bounded sets of $AP(\Omega, L^p(\Omega,\mathbb{H}))$ into bounded sets of $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \cap AP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ for $0<\gamma<\alpha$, $i=1,2$.*

The proof of the above lemma follows the same lines as that of Lemma 3.4, and hence it is omitted. Similarly, the next lemma is a consequence of [30, Proposition 3.3]. Note in this context that $\mathbb{X} = L^p(\Omega,\mathbb{H})$ and $\mathbb{Y} = L^p(\Omega,\mathbb{H}_\alpha)$.

Lemma 3.7. *For $0<\gamma<\alpha$, $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ is compactly contained in $BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$; that is, the canonical injection*

$$\mathrm{id} : BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \hookrightarrow BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$$

is compact, which yields

$$\mathrm{id} : BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \cap AP(\mathbb{R}, L^p(\Omega,\mathbb{H})) \to AP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$$

is also compact.

The next theorem is the main result of Section 3 and is a nondeterministic counterpart of the main result in Diagana [21].

Theorem 3.8. *Suppose assumptions* (H1)–(H6) *hold, then the nonautonomous differential equation Equation* (1.2) *has at least one p-th mean almost periodic solution.*

Proof. Let us recall that in view of Lemmas 3.7 and 3.3, we have

$$\|\big(\Gamma_1+\Gamma_2\big)X\|_{\alpha,\infty} \le d(\beta,\delta)\Big(\mathcal{M}_1\big(\|X\|_\infty\big) + \mathcal{M}_2\big(\|X\|_\infty\big)\Big)$$

and

$$\begin{aligned}&\mathbf{E}\|\big(\Gamma_1+\Gamma_2\big)X(t_2) - \big(\Gamma_1+\Gamma_2\big)X(t_1)\|_\alpha^p \\ &\le s(\alpha,\beta,\delta)\Big(\mathcal{M}_1\big(\|X\|_\infty\big)\Big) + \mathcal{M}_2\big(\|X\|_\infty\big)\Big)\big|t_2-t_1\big|^\gamma\end{aligned}$$

for all $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$, $t_1,t_2\in\mathbb{R}$ with $t_1\neq t_2$, where $d(\beta,\delta)$ and $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $\|X\|_\infty < R$ yield $(\Gamma_1+\Gamma_2)X \in BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ and $\|\big(\Gamma_1+\Gamma_2\big)X\|_{\alpha,\infty}^p < R_1$ where $R_1 = c(\alpha,\beta,\delta)\big(\mathcal{M}_1(R)+\mathcal{M}_2(R)\big)$. since $\mathcal{M}(R)/R \to 0$ as $R\to\infty$, and since $\mathbf{E}\|X\|^p \le c\mathbf{E}\|X\|_\alpha^p$ for all $X \in L^p(\Omega,\mathbb{H}_\alpha)$, it follows that exists an $r>0$ such that for all $R \ge r$, the following hold

$$\big(\Gamma_1+\Gamma_2\big)\Big(B_{AP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\Big) \subset B_{BC^\gamma(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))} \cap B_{AP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\,.$$

In view of the above, it follows that $\big(\Gamma_1+\Gamma_2\big) : D \to D$ is continuous and compact, where D is the ball in $AP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ of radius R with $R \ge r$. Using the Schauder fixed point it follows that $\big(\Gamma_1+\Gamma_2\big)$ has a fixed point, which is obviously a p-th mean almost periodic mild solution to (1.2). □

4. Square-mean almost periodic solutions to some second order stochastic differential equations

In this section we study and obtain under some reasonable assumptions, the existence of square-mean almost periodic solutions to some classes of nonautonomous second-order stochastic differential equations of type (1.3) on a Hilbert space $\mathbb{H}$ using Schauder's fixed-point theorem.

For that, the main idea consists of rewriting (1.3) as a nonautonomous first-order differential equation on $\mathbb{H}\times\mathbb{H}$ involving the family of 2×2-operator matrices $\mathfrak{L}(t)$.

Indeed, setting $Z := \begin{pmatrix} X \\ dX(t) \end{pmatrix}$, Equation (1.3) can be rewritten in the Hilbert space $\mathbb{H} \times \mathbb{H}$ in the form

$$dZ(\omega,t) = [\mathfrak{L}(t)Z(\omega,t) + F_1(t, Z(\omega,t))]\,dt + F_2(t, Z(\omega,t))d\mathbb{W}(\omega,t), \tag{4.1}$$

where $t \in \mathbb{R}$, $\mathfrak{L}(t)$ is the family of 2×2-operator matrices defined on $\mathcal{H} = \mathbb{H}\times\mathbb{H}$ by

$$\mathfrak{L}(t) = \begin{pmatrix} 0 & I_{\mathbb{H}} \\ -b(t)\mathcal{A} & -a(t)I_{\mathbb{H}} \end{pmatrix} \tag{4.2}$$

whose domain $D = D(\mathfrak{L}(t))$ is constant in $t\in\mathbb{R}$ and is given by $D(\mathfrak{L}(t)) = D(\mathcal{A})\times\mathbb{H}$. Moreover, the semilinear term $F_i (i = 1, 2)$ appearing in (4.1) is defined on $\mathbb{R}\times\mathcal{H}_\alpha$ for some $\alpha\in(0,1)$ by

$$F_i(t,Z) = \begin{pmatrix} 0 \\ f_i(t,X) \end{pmatrix},$$

where $\mathcal{H}_\alpha = \tilde{\mathcal{H}}_\alpha \times \mathbb{H}$ with $\tilde{\mathcal{H}}_\alpha$ is the real interpolation space between $\mathcal{B}$ and $D(\mathcal{A})$ given by $\tilde{\mathcal{H}}_\alpha := \Big(\mathbb{H}, D(\mathcal{A})\Big)_{\alpha,\infty}$.

First of all, note that for $0<\alpha<\beta<1$, then

$$L^2(\Omega,\mathcal{H}_\beta) \hookrightarrow L^2(\Omega,\mathcal{H}_\alpha) \hookrightarrow L^2(\Omega;\mathcal{H})$$

are continuously embedded and hence therefore exist constants $k_1>0$, $k(\alpha)>0$ such that

$$\mathbf{E}\|Z\|^2 \le k_1\mathbf{E}\|Z\|_\alpha^2 \quad \text{for each } Z\in L^2(\Omega,\mathcal{H}_\alpha),$$
$$\mathbf{E}\|Z\|_\alpha^2 \le k(\alpha)\mathbf{E}\|Z\|_\beta^2 \quad \text{for each } Z\in L^2(\Omega,\mathcal{H}_\beta).$$

To study the existence of square-mean solutions of (4.1), in addition to (H1) we adopt the following assumptions.

(H7) Let $f_i (i=1,2) : \mathbb{R}\times L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ be square-mean almost periodic. Furthermore, $X\mapsto f_i(t,X)$ is uniformly continuous on any bounded subset K of $L^2(\Omega;\mathbb{H})$ for each $t\in\mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|f_i(t,X)\|^2 \le \mathcal{M}_i\big(\|X\|_\infty\big)$$

where $\mathcal{M}_i : \mathbb{R}^+\to\mathbb{R}^+$ is continuous function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}_i(r)}{r} = 0\,.$$

Under the above assumptions, it will be shown that the linear operator matrices $\mathfrak{L}(t)$ satisfy the well-known Acquistapace-Terreni conditions, which does guarantee the existence of an evolution family $\mathfrak{U}(t,s)$ associated with it. Moreover, it will be shown that $\mathfrak{U}(t,s)$ is exponentially stable under those assumptions.

4.1. **Square-Mean Almost Periodic Solutions.** To analyze (4.1), our strategy consists in studying the existence of square-mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$dZ(t) = [L(t)Z(t) + F_1(t, Z(t))]dt + F_2(t, Z(t))d\mathbb{W}(t) \tag{4.3}$$

for all $t\in\mathbb{R}$, where the operators $L(t) : D(L(t)) \subset L^2(\Omega,\mathcal{H}) \to L^2(\Omega,\mathcal{H})$ satisfy Acquistapace-Terreni conditions, $F_i (i = 1, 2)$ as before, and $\mathbb{W}$ is a one-dimensional Brownian motion.

Note that each $Z \in L^2(\Omega, \mathcal{H})$ can be written in terms of the sequence of orthogonal projections E_n as

$$X = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle X, e_n^k \rangle e_n^k = \sum_{n=1}^{\infty} E_n X.$$

Moreover, for each $X \in D(A)$,

$$AX = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle X, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j X.$$

Therefore, for all $Z := \begin{pmatrix} X \\ Y \end{pmatrix} \in D(L) = D(A) \times L^2(\Omega, \mathcal{H})$, we obtain

$$\begin{aligned}
L(t)Z &= \begin{pmatrix} 0 & I_{L^2(\Omega,\mathbb{H})} \\ -b(t)A & -a(t)I_{L^2(\Omega,\mathbb{H})} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \begin{pmatrix} Y \\ -b(t)AX - a(t)Y \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n Y \\ -b(t) \sum_{n=1}^{\infty} \lambda_n E_n X - a(t) \sum_{n=1}^{\infty} E_n Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n(t) P_n Z,
\end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \quad n \geq 1,$$

and

$$A_n(t) := \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix}, \quad n \geq 1.$$

Now, the characteristic equation for $A_n(t)$ is

$$\lambda^2 + a(t)\lambda + \lambda_n b(t) = 0 \tag{4.4}$$

with discriminant $\Delta_n(t) = a^2(t) - 4\lambda_n b(t)$ for all $t \in \mathbb{R}$. We assume that there exists $\delta_0, \gamma_0 > 0$ such that

$$\inf_{t \in \mathbb{R}} a(t) > 2\delta_0 > 0, \quad \inf_{t \in \mathbb{R}} b(t) > \gamma_0 > 0. \tag{4.5}$$

From (4.5) it easily follows that all the roots of (4.4) are nonzero (with nonzero real parts) given by

$$\lambda_1^n(t) = \frac{-a(t) + \sqrt{\Delta_n(t)}}{2}, \quad \lambda_2^n(t) = \frac{-a(t) - \sqrt{\Delta_n(t)}}{2};$$

that is,

$$\sigma(A_n(t)) = \Big\{ \lambda_1^n(t), \lambda_2^n(t) \Big\}.$$

In view of the above, it is easy to see that there exist $\gamma_0 \geq 0$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that

$$S_\theta \cup \{0\} \subset \rho\left(L(t) - \gamma_0 I\right)$$

for each $t \in \mathbb{R}$ where

$$S_\theta = \Big\{ z \in \mathbb{C} \setminus \{0\} : \big|\arg z\big| \leq \theta \Big\}.$$

On the other hand, one can show without difficulty that $A_n(t) = K_n^{-1}(t)J_n(t)K_n(t)$, where
$$J_n(t) = \begin{pmatrix} \lambda_1^n(t) & 0 \\ 0 & \lambda_2^n(t) \end{pmatrix}, \quad K_n(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1^n(t) & \lambda_2^n(t) \end{pmatrix}$$
and
$$K_n^{-1}(t) = \frac{1}{\lambda_1^n(t) - \lambda_2^n(t)} \begin{pmatrix} -\lambda_2^n(t) & 1 \\ \lambda_1^n(t) & -1 \end{pmatrix}.$$
For $\lambda \in S_\theta$ and $Z \in L^2(\Omega, \mathcal{H})$, one has
$$\begin{aligned} R(\lambda, L)Z &= \sum_{n=1}^{\infty} (\lambda - A_n(t))^{-1} P_n Z \\ &= \sum_{n=1}^{\infty} K_n(t)(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t) P_n Z. \end{aligned}$$
Hence,
$$\begin{aligned} \mathbf{E}\|R(\lambda, L)Z\|^2 &\le \sum_{n=1}^{\infty} \|K_n(t)P_n(\lambda - J_n(t)P_n)^{-1}K_n^{-1}(t)P_n\|^2_{B(\mathcal{H})} \mathbf{E}\|P_n Z\|^2 \\ &\le \sum_{n=1}^{\infty} \|K_n(t)P_n\|^2_{B(\mathcal{H})} \|(\lambda - J_n(t)P_n)^{-1}\|^2_{B(\mathcal{H})} \|K_n^{-1}(t)P_n\|^2_{B(\mathcal{H})} \mathbf{E}\|P_n Z\|^2. \end{aligned}$$
Moreover, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega, \mathcal{H})$, we obtain
$$\begin{aligned} \mathbf{E}\|K_n(t)P_n Z\|^2 &= \mathbf{E}\|E_n Z_1 + E_n Z_2\|^2 + \mathbf{E}\|\lambda_1^n E_n Z_1 + \lambda_2^n E_n Z_2\|^2 \\ &\le 3\Big(1 + \big|\lambda_n^1(t)\big|^2\Big) \mathbf{E}\|Z\|^2. \end{aligned}$$
Thus, there exists $C_1 > 0$ such that
$$\mathbf{E}\|K_n(t)P_n Z\|^2 \le C_1 \big|\lambda_n^1(t)\big| \mathbf{E}\|Z\|^2 \quad \text{for all } n \ge 1.$$
Similarly, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega, \mathcal{H})$, one can show that there is $C_2 > 0$ such that
$$\mathbf{E}\|K_n^{-1}(t)P_n Z\|^2 \le \frac{C_2}{\big|\lambda_n^1(t)\big|} \mathbf{E}\|Z\|^2 \quad \text{for all } n \ge 1.$$
Now, for $Z \in L^2(\Omega, \mathcal{H})$, we have
$$\begin{aligned} \mathbf{E}\|(\lambda - J_n(t)P_n)^{-1} Z\|^2 &= \mathbf{E}\Big\| \begin{pmatrix} \frac{1}{\lambda - \lambda_n^1(t)} & 0 \\ 0 & \frac{1}{\lambda - \lambda_n^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \Big\|^2 \\ &\le \frac{1}{|\lambda - \lambda_n^1(t)|^2} \mathbf{E}\|Z_1\|^2 + \frac{1}{|\lambda - \lambda_n^2(t)|^2} \mathbf{E}\|Z_2\|^2. \end{aligned}$$
Let $\lambda_0 > 0$. Define the function
$$\eta_t(\lambda) := \frac{1 + |\lambda|}{|\lambda - \lambda_n^2(t)|}.$$
It is clear that η_t is continuous and bounded on the closed set
$$\Sigma := \{\lambda \in \mathbb{C} : |\lambda| \le \lambda_0, \ |\arg \lambda| \le \theta\}.$$

On the other hand, it is clear that η is bounded for $|\lambda| > \lambda_0$. Thus η is bounded on S_θ. If we take

$$N = \sup\Big\{\frac{1+|\lambda|}{|\lambda - \lambda_n^j(t)|} : \lambda \in S_\theta,\ n \geq 1,\ j = 1, 2, \Big\}.$$

Therefore,

$$\mathbf{E}\|(\lambda - J_n(t)P_n)^{-1}Z\|^2 \leq \frac{N}{1+|\lambda|}\mathbf{E}\|Z\|^2, \quad \lambda \in S_\theta.$$

Consequently,

$$\|R(\lambda, L(t))\| \leq \frac{K}{1+|\lambda|}$$

for all $\lambda \in S_\theta$.

First of all, note that the domain $D = D(L(t))$ is independent of t. Now note that the operator $L(t)$ is invertible with

$$L(t)^{-1} = \begin{pmatrix} -a(t)b^{-1}(t)A^{-1} & -b^{-1}(t)A^{-1} \\ I_{\mathbb{H}} & 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Hence, for $t, s, r \in \mathbb{R}$, computing $\big(L(t) - L(s)\big)L(r)^{-1}$ and assuming that there exist $L_a, L_b \geq 0$ and $\mu \in (0, 1]$ such that

$$\big|a(t) - a(s)\big| \leq L_a\big|t - s\big|^\mu, \quad \big|b(t) - b(s)\big| \leq L_b\big|t - s\big|^\mu, \tag{4.6}$$

it easily follows that there exists $C > 0$ such that

$$\mathbf{E}\|(L(t) - L(s))L(r)^{-1}Z\|^2 \leq C\big|t - s\big|^{2\mu}\mathbf{E}\|Z\|^2.$$

In summary, the family of operators $\big\{L(t)\big\}_{t\in\mathbb{R}}$ satisfy Acquistpace-Terreni conditions. Consequently, there exists an evolution family $U(t, s)$ associated with it. Let us now check that $U(t, s)$ has exponential dichotomy. First of all note that For every $t \in \mathbb{R}$, the family of linear operators $L(t)$ generate an analytic semigroup $(e^{\tau L(t)})_{\tau\geq 0}$ on $L^2(\Omega, \mathcal{H})$ given by

$$e^{\tau L(t)}Z = \sum_{l=1}^{\infty} K_l(t)^{-1}P_l e^{\tau J_l}P_l K_l(t)P_l Z,\ Z \in L^2(\Omega, \mathcal{H}).$$

On the other hand,

$$\mathbf{E}\|e^{\tau L(t)}Z\|^2 = \sum_{l=1}^{\infty} \|K_l(t)^{-1}P_l\|^2_{B(\mathcal{H})}\|e^{\tau J_l}P_l\|^2_{B(\mathcal{H})}\|K_l(t)P_l\|^2_{B(\mathcal{H})}\mathbf{E}\|P_l Z\|^2,$$

with for each $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$,

$$\begin{aligned}
\mathbf{E}\|e^{\tau J_l}P_l Z\|^2 &= \Big\| \begin{pmatrix} e^{\rho_1^l \tau}E_l & 0 \\ 0 & e^{\rho_2^l \tau}E_l \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \Big\|^2 \\
&\leq \mathbf{E}\|e^{\rho_1^l \tau}E_l Z_1\|^2 + \mathbf{E}\|e^{\rho_2^l \tau}E_l Z_2\|^2 \\
&\leq e^{-2\delta_0 \tau}\mathbf{E}\|Z\|^2.
\end{aligned}$$

Therefore,

$$\|e^{\tau L(t)}\| \leq Ce^{-\delta_0 \tau}, \quad \tau \geq 0. \tag{4.7}$$

Using the continuity of a, b and the equality

$$R(\lambda, L(t)) - R(\lambda, L(s)) = R(\lambda, L(t))(L(t) - L(s))R(\lambda, L(s)),$$

it follows that the mapping $J \ni t \mapsto R(\lambda, L(t))$ is strongly continuous for $\lambda \in S_\omega$ where $J \subset \mathbb{R}$ is an arbitrary compact interval. Therefore, $L(t)$ satisfies the assumptions of [42, Corollary 2.3], and thus the evolution family $(U(t,s))_{t\geq s}$ is exponentially stable.

It remains to verify that $R(\gamma_0, L(\cdot)) \in AP(\mathbb{R}, B(L^2(\Omega;\mathcal{H})))$. For that we need to show that $L^{-1}(\cdot) \in AP(\mathbb{R}, B(L^2(\Omega,\mathcal{H})))$. Since $t \to a(t)$, $t \to b(t)$, and $t \to b(t)^{-1}$ are almost periodic it follows that $t \to d(t) = -\frac{a(t)}{b(t)}$ is almost periodic, too. So for all $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a τ such that

$$\Big|\frac{1}{b(t+\tau)} - \frac{1}{b(t)}\Big| < \frac{\varepsilon}{\|A^{-1}\|\sqrt{2}}, \quad \big|d(t+\tau) - d(t)\big| < \frac{\varepsilon}{\|A^{-1}\|\sqrt{2}}$$

for all $t \in \mathbb{R}$. Clearly,

$$\begin{aligned}\|L^{-1}(t+\tau) - L^{-1}(t)\| &\leq \Big(\Big|\frac{1}{b(t+\tau)} - \frac{1}{b(t)}\Big|^2 + \big|d(t+\tau) - d(t)\big|^2\Big)^{1/2}\|A^{-1}\|_{B(\mathbb{H})}\\ &< \varepsilon\end{aligned}$$

and hence $t \to L^{-1}(t)$ is almost periodic with respect to $L^2(\Omega,\mathcal{H})$-operator topology. Therefore, $R(\gamma_0, L(\cdot)) \in AP(\mathbb{R}, B(L^2(\Omega;\mathcal{H})))$.

To study the existence of square-mean almost periodic solutions of (4.3), we use the general results obtained in Section 3.

Definition 4.1. A continuous random function, $Z : \mathbb{R} \to L^2(\Omega;\mathcal{H})$ is said to be a bounded solution of (4.3) on $\mathbb{R}$ provided that

$$Z(t) = \int_s^t U(t,s)F_1(s, Z(s))\,ds + \int_s^t U(t,s)P(s)\,F_2(s, Z(s))\,d\mathbb{W}(s)$$

for each $t \geq s$ and for all $t, s \in \mathbb{R}$.

Remark 4.2. Note that it follows from (H7) that $F_i (i = 1, 2) : \mathbb{R} \times L^2(\Omega;\mathcal{H}) \to L^2(\Omega;\mathcal{H})$ is square-mean almost periodic. Furthermore, $Z \mapsto F_i(t, Z)$ is uniformly continuous on any bounded subset K of $L^2(\Omega;\mathcal{H})$ for each $t \in \mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|F_i(t, Z)\|^2 \leq \mathcal{M}_i\big(\|Z\|_\infty\big)$$

where $\mathcal{M}_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}_i(r)}{r} = 0\,.$$

Theorem 4.3. *Suppose assumptions* (H1), (H3), (H7) *hold, then the nonautonomous differential equation* (4.3) *has at least one square-mean almost periodic solution.*

In view of Remark 4.2, the proof of the above theorem follows along the same lines as that of Theorem 3.8 and hence it is omitted.

Acknowledgments. The authors would like thanks the anonymous referee for the careful reading of the manuscript and insightful comments.

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