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NONLINEAR BOUNDARY DISSIPATION FOR A COUPLED SYSTEM OF KLEIN-GORDON EQUATIONS

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Abstract. This article concerns the existence of solutions and the decay of the energy of the mixed problem for the coupled system of Klein-Gordon equations

$$u'' - \Delta u + \alpha v^2 u = 0 \quad \text{in } \Omega \times (0,\infty),$$
$$v'' - \Delta v + \alpha u^2 v = 0 \quad \text{in } \Omega \times (0,\infty),$$

with the nonlinear boundary conditions,

$$\frac{\partial u}{\partial \nu} + h_1(.,u') = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$\frac{\partial v}{\partial \nu} + h_2(.,v') = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$

and boundary conditions $u = v = 0$ on $(\Gamma \setminus \Gamma_1) \times (0,\infty)$, where Ω is a bounded open set of $\mathbb{R}^n$ $(n \le 3)$, $\alpha > 0$ a real number, Γ_1 a subset of the boundary Γ of Ω and h_i a real function defined on $\Gamma_1 \times (0,\infty)$.

Assuming that each h_i is strongly monotone in the second variable, the existence of global solutions of the mixed problem is obtained. For that it is used the Galerkin method, the Strauss' approximations of real functions and trace theorems for non-smooth functions. The exponential decay of the energy for a particular stabilizer is derived by application of a Lyapunov functional.

1. Introduction

A mathematical model that describes the interaction of two electromagnetic fields u and v with masses a and b, respectively, and with interaction constant $\alpha > 0$ is given by the following Klein-Gordon system

$$\begin{aligned} &u_{tt}(x,t) - \Delta u(x,t) + a^2 u(x,t) + \alpha v^2(x,t) u(x,t) = 0, \quad x \in \Omega,\ t > 0,\\ &v_{tt}(x,t) - \Delta v(x,t) + b^2 v(x,t) + \alpha u^2(x,t) v(x,t) = 0, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^3$. This model was proposed by Segal [18].

As the interest of this paper is to make the mathematical analysis of the model (1.1), we can assume, without loss of generality, that $a = b = 0$.

Let Ω be a bounded open set of the $\mathbb{R}^n$ with boundary Γ. The existence and uniqueness of solutions of the mixed problem with null Dirichlet boundary conditions on Γ for system (1.1) with coupled nonlinear terms $\alpha|v|^{\sigma+2}|u|^{\sigma}u$ and $\alpha|u|^{\sigma+2}|v|^{\sigma}v$ was studied by Medeiros and second author, in the cases $\alpha > 0$ and

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$\alpha < 0$, in [14] and [15], respectively. Here $\sigma \geq 0$ is related with the dimension n of the $\mathbb{R}^n$ and the embedding of Sobolev spaces.

Let $\{u, v\}$ be a solution of system (1.1) with null Dirichlet boundary conditions on Γ and

$$\begin{aligned} E(t) = \|u'(t)\|^2_{L^2(\Omega)} + \|v'(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{(L^2(\Omega))^n} \\ + \|\nabla v(t)\|^2_{(L^2(\Omega))^n} + \alpha \|u(t)v(t)\|^2_{L^2(\Omega)} \end{aligned}$$

the energy associated to the problem. Then

$$E(t) = E(0), \quad \forall t \geq 0.$$

Thus, to obtain a decay of the energy, we need to introduce a dissipation in the problem, on the boundary Γ, for instance. In what follows we describe this problem.

Let Ω be a bounded open domain of $\mathbb{R}^n$ where $n \leq 3$ with boundary Γ of class C^2. Assume that Γ is constituted by two disjoint closed parts Γ_0 and Γ_1 both with positive Lebesgue measures (Thus Γ is not connected). By $\nu(x)$ is represented the unit outward normal at $x \in \Gamma_1$. Consider two real valued functions $h_1(x, s)$ and $h_2(x, s)$ defined in $x \in \Gamma_1$ and $s \in \mathbb{R}$. With these notations we have the problem

$$\begin{gathered} u'' - \Delta u + \alpha v^2 u = 0 \quad \text{in } \Omega \times (0, \infty), \\ v'' - \Delta v + \alpha u^2 v = 0 \quad \text{in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \\ v = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + h_1(., u') = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial v}{\partial \nu} + h_2(., v') = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\ u(0) = u_0, \quad v(0) = v_0 \quad \text{in } \Omega, \\ u'(0) = u_1 \quad v'(0) = v_1 \quad \text{in } \Omega. \end{gathered} \tag{1.2}$$

In the case of one equation (that is, when $\alpha = 0$), Ω a bounded open set of $\mathbb{R}^n$, $h(x, s) = \delta(x)s$, Komornik and Zuazua [8], using the semigroup theory, showed the existence of solutions. Under the same hypotheses, but applying the Galerkin method with a special basis, the second author and Medeiros [16], obtained a similar result. The second method, furthermore to be constructive, has the advantage of showing the Sobolev space where lies $\frac{\partial u}{\partial \nu}$. Applying this second method to a wave equation with a nonlinear term, Araruna and Maciel [1], derived an analogous result.

The existence of solutions of the wave equation with a nonlinear dissipation on Γ_1 has been obtained, using the theory of monotone operators, among others, by Zuazua [21], Lasiecka and Tataru [9], Komornik [6], and applying the method of Galerkin, by Vitillaro [20] and Cavalcanti *et al.* [4].

In Alabau-Boussouira [2], as in all above works, the exponential decay of the energy associated to the wave equation is obtained by applying functionals of Lyapunov and the technique of multipliers.

It is worth emphasizing that the known results on the exponential decay of the energy associated to the wave equation with a nonlinear boundary dissipation were obtained by supposing that $h(s)$ has a linear behavior in the infinite; that is,

$$d_0|s| \leq |h(s)| \leq d_1|s|, \quad \forall |s| \geq R, \tag{1.3}$$

where R sufficiently large (d_0 and d_1 positiveconstants). See Komornik [6] and the references therein.

Returning to system (1.2) we can mention the work of Cousin *et al.* [5] where the conditions on the boundary are linear. We will also mention the work of Komornik and Rao [7] where the coupled terms are the form $\alpha(u-v)$ and $\alpha(v-u)$ and the boundary conditions are similar to (1.2). More precisely, in this work under the hypotheses

$\alpha \in L^\infty(\Omega)$, $\alpha \geq 0$
h is continuous, nondecreasing, $h(s) = 0$ if $s = 0$;
$|h(s)| \leq 1 + c|s|$, for all $s \in \mathbb{R}$ where c is a positive constant;

and using results of maximum monotone operators, they showed the existence of solutions. With h satisfying (1.3) for all $s \in \mathbb{R}$ and applying the technique of the multipliers, they obtained the exponential decay of the energy associated to the problem.

In this work we are interested in studying the existence of solutions of Problem (1.2) under very general conditions on h_i, $i = 1, 2$. In fact, assuming that

$$h_i \in C^0(\mathbb{R}; L^\infty(\Gamma_1)), \quad h_i(x, 0) = 0, \quad \text{a.e. } x \in \Gamma_1$$

and h_i is strongly monotone in the second variable; that is,

$$[h_i(x, s) - h_i(x, r)](s - r) \geq d_i(s - r)^2, \quad \forall s, r \in \mathbb{R},$$

where d_i are positive constant for $i = 1, 2$. We obtain the existence of global solutions for (1.2). In our approach, we apply the Galerkin method with a special basis, an appropriate Strauss' Lipschitz approximation of h_i and results on the trace of non-smooth functions. In the passage to the limit in the nonlinear boundary term $h_{il}(., u_l')$ ((h_{il}) are the Strauss' approximations of h_i and (u_l), approximate solutions of (1.2)), we use the compactness method (In what follows $i = 1, 2$). For that we need to obtain estimates for (u_l') and (u_l''). It is possible thanks to the strong monotonicity of h_i. These estimates allow us to obtain the strong convergence

$$u_l' \to u' \quad \text{in } \ L^2(0, T; L^2(\Gamma_1)), \forall T > 0.$$

This, Strauss' Theorem [19] and results on trace of non-smooth functions (Lemma 3.2) give

$$h_{il}(., u_l') \to h_i(., u') \quad \text{in } L^1(0, T; L^1(\Gamma_1)), \quad \forall T > 0.$$

As consequence of the mentioned estimates, we are driven to obtain global strong solutions of (1.2). The existence of global weak solution for (1.2) with the general hypotheses on h_i is an open problem.

The exponential decay of the energy of (1.2) is derived for the particular case

$$h_i(x, s) = m(x).\nu(x) g_i(s),$$

$g_i \in C^0(\mathbb{R})$, g_i satisfying (1.3) and $m(x) = x - x^0$, $x^0 \in \mathbb{R}^n$. In this part we use a functional of Lyapunov (see Komornik and Zuazua [8]) and the technique of multipliers (see [17]). The exponential decay for more general stabilizers is an open problem.

In Section 2 we state our main results and in Section 3, we prove these results.

2. Notation and main results

Let Ω be a bounded open set of $\mathbb{R}^n$ with boundary Γ of class C^2 and Γ_0, Γ_1, $\nu(x)$ as in the Introduction. The scalar product and norm of $L^2(\Omega)$ are represented, respectively, by (u, v) and $|u|$. By V is denoted the Hilbert space

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_0\}$$

equipped with the scalar product

$$((u, v)) = \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\Big) \quad \text{and norm} \quad \|u\|^2 = \sum_{i=1}^{n} \Big|\frac{\partial u}{\partial x_i}\Big|^2.$$

To state our results, we introduce some hypotheses. Consider real functions $h_1(x, s)$ and $h_2(x, s)$ defined on $\Gamma_1 \times \mathbb{R}$ satisfying the following hypotheses:

- (H1) $h_i \in C^0(\mathbb{R}; L^\infty(\Gamma_1))$;
  $h_i(x, s)$ is nondecreasing in s for a.e. x in Γ_1;
  $h_i(x, 0) = 0$ a.e. $x \in \Gamma_1$;
  $[h_i(x, s) - h_i(x, r)](s - r) \geq d_i(s - r)^2$, for all $s, r \in \mathbb{R}$ and a.e. x in Γ_1, where $i = 1, 2$. Here d_1 and d_2 are positive constants and we use the notation $(h_i(s))(x) = h_i(x, s)$.
- (H2) $n \leq 3$ and $\alpha \geq 0$;
- (H3) $\{u^0, v^0\} \in [D(-\Delta)]^2$ and $\{u^1, v^1\} \in [H_0^1(\Omega)]^2$ where

$$D(-\Delta) = \{u \in V \cap H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1\}$$

Theorem 2.1. *Assume* (H1)–(H3). *Then there exist a pair of functions* $\{u, v\}$ *in the class*

- (C) $\{u, v\} \in [L^\infty(0, \infty; V)]^2$, $\{u', v'\} \in [L^\infty_{\rm loc}(0, \infty; V)]^2$,
  $\{u'', v''\} \in [L^\infty_{\rm loc}(0, \infty; L^2(\Omega))]^2$,

satisfying the equations

$$\begin{gathered} u'' - \Delta u + \alpha u v^2 = 0 \quad \text{in } L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \\ v'' - \Delta v + \alpha v u^2 = 0 \quad \text{in } L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \end{gathered} \tag{2.1}$$

the boundary conditions

$$\begin{gathered} \frac{\partial u}{\partial \nu} + h_1(., u') = 0 \quad \text{in } L^1_{\rm loc}(0, \infty; L^1(\Gamma_1)), \\ \frac{\partial v}{\partial \nu} + h_2(., v') = 0 \quad \text{in } L^1_{\rm loc}(0, \infty; L^1(\Gamma_1)), \end{gathered} \tag{2.2}$$

and the initial conditions

$$\begin{gathered} u(0) = u^0, \quad v(0) = v^0 \quad \text{in } \Omega, \\ u'(0) = u^1, \quad v'(0) = v^1 \quad \text{in } \Omega. \end{gathered} \tag{2.3}$$

Theorem 2.2. *If in addition to the hypotheses of Theorem 2.1 we have*

- (H4) *there are positive constant* k_1, k_2 *such that*

$$|h_1(x, s)| \leq k_1|s|, \quad |h_2(x, s)| \leq k_2|s|$$

for all $s \in \mathbb{R}$ *and a.e.* x *in* Γ_1.

Then the solution $\{u, v\}$ *given by Theorem 2.1 belongs to the class*

- (C*) $\{u, v\} \in [L^\infty(0, \infty; V) \cap L^2_{\rm loc}(0, \infty; H^{\frac{3}{2}}(\Omega))]^2$;

this solution is unique in the classes (C), (C), and satisfies the boundary conditions*

$$\begin{aligned}
&\frac{\partial u}{\partial \nu} + h_1(.,u') = 0 \quad \text{in } L^2(0,\infty;L^2(\Gamma_1)),\\
&\frac{\partial v}{\partial \nu} + h_2(.,v') = 0 \quad \text{in } L^2(0,\infty;L^2(\Gamma_1)).
\end{aligned}$$

Remark 2.3. By (H3), we have $\frac{\partial u^0}{\partial \nu} = 0$, $\frac{\partial v^0}{\partial \nu} = 0$ on Γ_1, and $u^1 = 0$, $v^1 = 0$ on Γ_1. Therefore, since $h_i(.,0) = 0$,

$$\begin{aligned}
&\frac{\partial u^0}{\partial \nu} + h_1(.,u^1) = 0 \quad \text{on } \Gamma_1,\\
&\frac{\partial v^0}{\partial \nu} + h_2(.,v^1) = 0 \quad \text{on } \Gamma_1.
\end{aligned}$$

In the general case, that is, when $\{u^0, v^0\} \in [V \cap H^2(\Omega)]^2$ and $\{u^1, v^1\} \in V^2$ satisfying the compatibility conditions

$$\begin{aligned}
&\frac{\partial u^0}{\partial \nu} + h_1(.,u^1) = 0 \quad \text{on } \Gamma_1,\\
&\frac{\partial v^0}{\partial \nu} + h_2(.,v^1) = 0 \quad \text{on } \Gamma_1,
\end{aligned}$$

the existence of global solutions of (1.2) with initial data $\{u^0, v^0\}$ and $\{u^1, v^1\}$ is an open problem. In our approach, when $u^0, u^1 \in V \cap H^2(\Omega)$, the condition

$$\frac{\partial u^0}{\partial \nu} + h_1(.,u^1) = 0 \quad \text{on } \Gamma_1,$$

does not imply necessarily

$$\frac{\partial u^0}{\partial \nu} + h_{1l}(.,u^1) = 0 \quad \text{on } \Gamma_1, \quad \forall l.$$

Thus in this case, we cannot to construct a special basis of $V \cap H^2(\Omega)$ in order to apply the Galerkin method.

Next we state the result on the decay of solutions of Problem (1.2). We assume that there exists a point $x^0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu(x) \le 0\}, \quad \Gamma_1 = \{x \in \Gamma : m(x).\nu(x) > 0\},$$

where $m(x) = x - x^0$, $x \in \mathbb{R}^n$, and $\eta.\xi$ denotes the scalar product of $\mathbb{R}^n$ of the vectors $\eta, \xi \in \mathbb{R}^n$. Consider the particular functions

$$h_1(x,s) = m(x).\nu(x) g_1(s), \quad h_2(x,s) = m(x).\nu(x) g_2(s), \quad x \in \Gamma_1,\ s \in \mathbb{R}, \tag{2.4}$$

where $g_1(s)$ and $g_2(s)$ are continuous real functions with $g_i(0) = 0$, $i = 1,2$ and satisfy

- (H5) $[g_i(s) - g_i(r)](s - r) \ge d_i^*(s - r)^2$, for all $s, r \in \mathbb{R}$, $i = 1,2$;
- (H6) $|g_i(s)| \le k_i^*|s|$, for all $s \in \mathbb{R}$, where d_i^* and k_i^* are positive constants, $i = 1,2$.

Introduce the following constants (K, K^* positive) such that

$$\|w\|_{L^6(\Omega)} \le K\|w\|, \quad \|w\|^2_{L^2(\Gamma_1)} \le K^*\|w\|^2, \quad \forall w \in V; \tag{2.5}$$

$$R = \max_{x\in\overline{\Omega}} \|m(x)\|; \tag{2.6}$$

$$N(\alpha, u^0, v^0, u^1, v^1) = N = |u^1|^2 + |v^1|^2 + \|u^0\|^2 + \|v^0\|^2 + \alpha\|u^0\|^2\|v^0\|^2 + 1, \quad \text{with } \alpha \ge 0; \tag{2.7}$$

$$L_i = \frac{3}{4}(n-1)^2 k_i^* R(K^*)^2, \quad i = 1, 2; \tag{2.8}$$

$$L = \max\big\{R^2\big(\frac{3}{2}k_1^*\big)^2 + L_1 + 1,\ R^2\big(\frac{3}{2}k_2^*\big)^2 + L_2 + 1\big\}; \tag{2.9}$$

$$M = 2\Big(R + \frac{n-1}{2} + \frac{n-1}{2\lambda_1}\Big) \tag{2.10}$$

where λ_1 is the first eigenvalue of the Laplacian operator associated to the triplet $\{V, L^2(\Omega), ((u, v))\}$ (see [10]). Define the energy

$$E(t) = \frac{1}{2}\left[|u'(t)|^2 + |v'(t)|^2 + \|u(t)\|^2 + \|v(t)\|^2 + \alpha|u(t)v(t)|^2\right], \quad t \ge 0,$$

where $|\cdot|$ is the L^2 norm.

Theorem 2.4. *Consider*

$$\{u^0, v^0\} \in [D(-\Delta)]^2 \quad \textit{and} \quad \{u^1, v^1\} \in [H^1_0(\Omega)]^2$$

and a positive real number α_0 such that

(H7) $\alpha_0 N \le 1/(8RK^3)$.

Let $\{u, v\}$ be the solution obtained in Theorem 2.1 with hypotheses (H4)–(H6) *and* $0 \le \alpha \le \alpha_0$. *Then*

$$E(t) \le 3E(0)e^{-2\omega t/3}, \quad \forall t \ge 0. \tag{2.11}$$

where

$$\omega = \min\big\{\frac{d_1^*}{L}, \frac{d_2^*}{L}, \frac{1}{2M}\big\}.$$

We make some comments. The open sets Ω of $\mathbb{R}^n$ satisfying the geometrical condition given above (existence of $x^0 \in \mathbb{R}^n$ which permits to determine Γ_0 and Γ_1 satisfying conditions of Theorem 2.1) were introduced by Lions [12]. The decay of solutions of Problem (1.2) for more general Ω, for example, when Ω satisfy the geometrical control condition of Bardos, Lebeau and Rauch (see [12]), is an open problem.

Hypothesis (H6) says that our feedback is between two linear feedbacks. This, hypothesis ($H5$) and α_0 small state that Problem (1.2) of Theorem 2.4 is a small perturbation of the linear problems associated to (1.2), that is, $\alpha = 0$ and $h_i(x, s)$ linear in s.

When $\alpha = 0$, all our results can be applied to the equation given by (1.2). In this case Ω is an open bounded domain of $\mathbb{R}^n$.

Consider the equation

$$u''(x, t) - \Delta u(x, t) + f(u(x, t)) = 0, \quad x \in \Omega,\ t > 0$$

with

$$f \in W^{1,\infty}_{\rm loc}(\mathbb{R}), \quad f(s)s \geq 0, \quad \forall s \in \mathbb{R},$$
$$(f(s) - f(r)) \leq a(1 + |s|^{p-1} + |r|^{p-1})(s - r), \quad \forall s, r \in \mathbb{R},\ a > 0,$$

where $1 < p \leq \frac{n}{n-2}$ if $n \geq 3$, and $p > 1$ if $n = 1, 2$; and the nonlinear dissipation of (1.2). Then our results can be applied to obtain the existence of solutions of this problem. This result is a nonlinear boundary version of the work of Araruna and Maciel [1].

3. Proof of Results

To prove Theorem 2.1 we need the following two lemmas.

Lemma 3.1. *Let $h(x, s)$ be a real function defined on $\Gamma_1 \times \mathbb{R}$ satisfying (H1) with strongly monotone constant d_0. Then there exists a sequence (h_l) in $C^0(\mathbb{R}; L^\infty(\Gamma_1))$ satisfying*

(i) $h_l(x, 0) = 0$ *for a.e. x in Γ_1;*
(ii) $[h_l(x, s) - h_l(x, r)](s - r) \geq d_0(s - r)^2$, *for all $s, r \in \mathbb{R}$, for a.e. x in Γ_1;*
(iii) *there exists a function $c_l \in L^\infty(\Gamma_1)$ such that*
$$|h_l(x, s) - h_l(x, r)| \leq c_l(x)|s - r|, \quad \forall s, r \in \mathbb{R}, \quad \text{for a.e. } x \text{ in } \Gamma_1;$$
(iv) *(h_l) converges to h uniformly on bounded sets of $\mathbb{R}$, for a.e.x in Γ_1.*

Proof. For each $l \in \mathbb{N}$ we define

$$h_l(x, s) = \begin{cases} C_{1l}(x)s, & \text{if } 0 \leq s \leq \frac{1}{l}, \\ l \int_s^{s+\frac{1}{l}} h(x, \tau)d\tau, & \text{if } \frac{1}{l} \leq s \leq l, \\ C_{2l}(x)s, & \text{if } s > l, \\ C_{3l}(x)s, & \text{if } -\frac{1}{l} \leq s \leq 0, \\ -l \int_{s-\frac{1}{l}}^{s} h(x, \tau)d\tau, & \text{if } -l \leq s \leq -\frac{1}{l}, \\ C_{4l}(x)s, & \text{if } s < -l, \end{cases}$$

where

$$C_{1l}(x) = l^2 \int_{\frac{1}{l}}^{\frac{2}{l}} h(x, \tau)d\tau, \quad C_{2l} = \int_l^{l+\frac{1}{l}} h(x, \tau)d\tau,$$
$$C_{3l}(x) = -l^2 \int_{-\frac{2}{l}}^{-\frac{1}{l}} h(x, \tau)d\tau, \quad C_{4l}(x) = -\int_{-l-\frac{1}{l}}^{-l} h(x, \tau)d\tau.$$

The sequence (h_l) satisfies the conditions of the lemma. □

Lemma 3.2. *Let $T > 0$ be a real number. Consider a sequence (w_l) of vectors of $L^2(0, T; H^{-1/2}(\Gamma_1)) \cap L^1(0, T; L^1(\Gamma_1))$ and vectors $w \in L^2(0, T; H^{-1/2}(\Gamma_1))$, $\chi \in L^1(0, T; L^1(\Gamma_1))$ such that*

(i) $w_l \to w$ *weak in* $L^2(0, T; H^{-1/2}(\Gamma_1))$,
(ii) $w_l \to \chi$ *in* $L^1(0, T; L^1(\Gamma_1))$.

Then $w = \chi$.

Proof. The preceding lemma follows by noting that convergence (i) and (ii) imply

$$\begin{gathered} w_l \to w \quad \text{in } \mathcal{D}'(0,T;\mathcal{D}'(\Gamma_1)), \\ w_l \to \chi \quad \text{in } \mathcal{D}'(0,T;\mathcal{D}'(\Gamma_1)). \end{gathered}$$

Therefore, $w = \chi$. □

Proof of Theorem 2.1. Let (h_{1l}) and (h_{2l}) be two sequences of real functions in the conditions of Lemma 3.1 that approximate h_1 and h_2, respectively. Also let (u_l^1) and (v_l^1) be two sequences of vectors of $\mathcal{D}(\Omega)$ such that

$$u_l^1 \to u^1 \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad v_l^1 \to v^1 \quad \text{in } H_0^1(\Omega). \tag{3.1}$$

Note that

$$\frac{\partial u^0}{\partial \nu} + h_{1l}(.,u_l^1) = 0 \quad \text{on } \Gamma_1,\ \forall l,$$

since $u_l^1 = 0$ and $\frac{\partial u^0}{\partial \nu} = 0$ on Γ_1. Analogously

$$\frac{\partial v^0}{\partial \nu} + h_{2l}(.,v_l^1) = 0 \quad \text{on } \Gamma_1,\ \forall l.$$

Fix $l \in \mathbb{N}$. We apply the Faedo-Galerkin's method with a special basis. In fact, consider the basis

$$\{w_1^l, w_2^l, w_3^l, w_4^l, \dots\},$$

of $V \cap H^2(\Omega)$ where u^0, v^0, u_l^1 and v_l^1 belong to the subspace generated by w_1^l, w_2^l, w_3^l and w_4^l. Note that u_l^1 and v_l^1 belong to $V \cap H^2(\Omega)$. With this basis we determine approximate solutions $u_{lm}(t)$ and $v_{lm}(t)$ of Problem (1.2); that is,

$$u_{lm}(t) = \sum_{j=1}^m g_{jlm}(t) w_j^l \quad \text{and} \quad v_{lm}(t) = \sum_{j=1}^m h_{jlm}(t) w_j^l,$$

when $g_{jlm}(t)$ and $h_{jlm}(t)$ are defined by the system:

$$\begin{gathered} (u''_{lm}(t), w_k) + ((u_{lm}(t), w_k)) + \alpha(u_{lm}(t)v_{lm}^2(t), w_k) + \int_{\Gamma_1} h_{1l}(., u'_{lm}(t)) w_k d\Gamma = 0, \\ (v''_{lm}(t), w_p) + ((v_{lm}(t), w_p)) + \alpha(v_{lm}(t)u_{lm}^2(t), w_p) + \int_{\Gamma_1} h_2(., v'_{lm}(t)) w_p d\Gamma = 0, \\ u_{lm}(0) = u^0, \quad v_{lm}(0) = v^0 \quad \text{in } \Omega; \\ u'_{lm}(0) = u_l^1, \quad v'_{lm}(0) = v_l^1 \quad \text{in } \Omega. \end{gathered} \tag{3.2}$$

for all $k = 1, 2, \dots, m$ and all $p = 1, 2, \dots, m$.

The above finite-dimensional system has a solution $\{u_{lm}(t), v_{lm}(t)\}$ defined on $[0, t_{lm}[$. The following estimate allows us to extend this solution to the interval $[0, \infty[$.

First Estimate. Considering $2u'_{lm}(t)$ instead of w_k in $(3.2)_1$ and $2v'_{lm}(t)$ instead of w_p in $(3.2)_2$ and adding these results, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[|u'_{lm}(t)|^2 + \|u_{lm}(t)\|^2 + |v'_{lm}(t)|^2 + \|v_{lm}(t)\|^2\right] \\ &+ \alpha \int_\Omega v_{lm}^2(t) \frac{d}{dt}(u_{lm}^2(t)) dx + \alpha \int_\Omega u_{lm}^2(t) \frac{d}{dt}(v_{lm}^2(t)) dx \\ &+ 2\int_{\Gamma_1} h_{1l}(., u'_{lm}(t)) u'_{lm}(t) d\Gamma + 2\int_{\Gamma_1} h_{2l}(., v'_{lm}(t)) v'_{lm}(t) d\Gamma = 0. \end{aligned}$$

Noting that

$$\int_\Omega v_{lm}^2(t)\frac{d}{dt}u_{lm}^2(t)dx+\int_\Omega u_{lm}^2(t)\frac{d}{dt}v_{lm}^2(t)dx=\int_\Omega \frac{d}{dt}\left[u_{lm}(t)v_{lm}(t)\right]^2dx,$$

By the two preceding expressions, after integrate on $[0,t[$, $0<t\le t_{lm}$, we obtain

$$\begin{aligned}
&|u'_{lm}(t)|^2+\|u_{lm}(t)\|^2+|v'_{lm}(t)|^2+\|v_{lm}(t)\|^2+\alpha|u_{lm}(t)v_{lm}(t)|^2\\
&+2\int_0^t\int_{\Gamma_1}h_{1l}(.,u'_{lm}(s))u'_{lm}(s)d\Gamma ds+2\int_0^t\int_{\Gamma_1}h_{2l}(.,v'_{lm}(s))v'_{lm}(s)d\Gamma ds\\
&=|u_l^1|^2+\|u^0\|^2+|v_l^1|^2+\|v^0\|^2+\alpha|u^0v^0|^2.
\end{aligned}\tag{3.3}$$

By Part (ii) of Lemma 3.1, we have

$$h_{il}(x,s)s\ge d_is^2,\quad \forall s\in\mathbb{R}\text{ and a.e. } x \text{ in } \Gamma_1,\ \forall l,\ i=1,2.$$

Note that $|u^0v^0|<\infty$ because $n\le 3$ and $u^0,v^0\in H_0^1(\Omega)$. Taking into account these two considerations and convergence (3.1), in (3.3), we obtain

$$\begin{aligned}
&|u'_{lm}(t)|^2+\|u_{lm}(t)\|^2+|v'_{lm}(t)|^2+\|v_{lm}(t)\|^2+\alpha|u_{lm}(t)v_{lm}(t)|^2\\
&+2d_1\int_0^t\int_{\Gamma_1}[u'_{lm}(s)]^2d\Gamma ds+2d_2\int_0^t\int_{\Gamma_1}[v'_{lm}(s)]^2d\Gamma ds\\
&\le\left[|u^1|^2+\|u^0\|^2+|v^1|^2+\|v^0\|^2+\alpha|u^0v^0|^2+1\right]=N_1,\quad \forall l\ge l_0,
\end{aligned}$$

where the constant N_1 is independent of t, m and $l\ge l_0$. Thus

$$\begin{gathered}
(u_{lm})\text{is bounded in } L^\infty(0,\infty;V),\quad \forall l\ge l_0,\ \forall m\\
(u'_{lm})\text{is bounded in } L^\infty(0,\infty;L^2(\Omega)),\quad \forall l\ge l_0,\ \forall m\\
(u'_{lm})\text{is bounded in } L^2(0,\infty;L^2(\Gamma_1)),\quad \forall l\ge l_0,\ \forall m
\end{gathered}\tag{3.4}$$

Analogous boundedness holds for (v_{lm}) and (v'_{lm}). Also

$$(u_{lm}v_{lm})\text{is bounded in } L^\infty(0,\infty;L^2(\Omega)),\quad \forall l\ge l_0,\ \forall m$$

As we are in a finite dimensional setting, the above estimates allows us to prolong the approximate solutions $\{u_{lm}(t),v_{lm}(t)\}$ to the interval $[0,\infty[$.

Second Estimate. Derive with respect to t equations $(3.2)_1$ and $(3.2)_2$ and consider $2u''_{lm}(t)$ and $2v''_{lm}(t)$ instead w_k and w_p in $(3.2)_1$ and $(3.2)_2$, respectively. We obtain

$$\begin{aligned}
&\frac{d}{dt}|u''_{lm}(t)|^2+\frac{d}{dt}\|u'_{lm}(t)\|^2+2\alpha(u'_{lm}(t)v_{lm}^2(t),u''_{lm}(t))\\
&+4\alpha(u_{lm}(t)v_{lm}(t)v'_{lm}(t),u''_{lm}(t))+2\int_{\Gamma_1}(u''_{lm}(t))^2h'_{1l}(.,u'_{lm}(t))d\Gamma=0,
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
&\frac{d}{dt}|v''_{lm}(t)|^2+\frac{d}{dt}\|v'_{lm}(t)\|^2+2\alpha(v'_{lm}(t)u_{lm}^2(t),v''_{lm}(t))\\
&+4\alpha(v_{lm}(t)u_{lm}(t)u'_{lm}(t),v''_{lm}(t))+2\int_{\Gamma_1}(v''_{lm}(t))^2h'_{2l}(.,v'_{lm}(t))d\Gamma=0.
\end{aligned}\tag{3.6}$$

• Analysis of the term: $(u'_{lm}(t)v^2_{lm}(t), u''_{lm}(t))$. Using the Holder inequality, the Sobolev embedding $V \hookrightarrow L^6(\Omega)$ (note that $n \leq 3$) and estimates (3.4), we obtain

$$
\begin{aligned}
|(u'_{lm}(t)v^2_{lm}(t), u''_{lm}(t))| &\leq \int_\Omega |u'_{lm}(t)|_{\mathbb{R}} |v^2_{lm}(t)|_{\mathbb{R}} |u''_{lm}(t)|_{\mathbb{R}} dx \\
&\leq \|u'_{lm}(t)\|_{L^6(\Omega)} \|v_{lm}(t)\|^2_{L^6(\Omega)} |u''_{lm}(t)| \\
&\leq C\|u'_{lm}(t)\| |u''_{lm}(t)| \\
&\leq C(\|u'_{lm}(t)\|^2 + |u''_{lm}(t)|^2),
\end{aligned}
\tag{3.7}
$$

where C denotes the several constants independent of l and m.

• Analysis of the term $(u_{lm}(t)v_{lm}(t)v'_{lm}(t), u''_{lm}(t))$. Applying the same arguments used fo (3.7), we obtain

$$
|(u_{lm}(t)v_{lm}(t)v'_{lm}(t), u''_{lm}(t))| \leq C(\|v'_{lm}(t)\| |u''_{lm}(t)|) \leq C(\|v'_{lm}(t)\|^2 + |u''_{lm}(t)|^2).
\tag{3.8}
$$

In a similar way, we obtain estimates for

$$
(v'_{lm}(t)u^2_{lm}(t), v''_{lm}(t)) \quad \text{and} \quad (v_{lm}(t)u_{lm}(t)u'_{lm}(t), v''_{lm}(t)).
\tag{3.9}
$$

Integrating (3.5) and (3.6) on $[0, t]$, adding these results, using estimates (3.7)-(3.9) and noting that

$$
\frac{\partial}{\partial s} h_{1l}(x, s) \geq d_1 > 0, \quad \frac{\partial}{\partial s} h_{2l}(x, s) \geq d_2 > 0
$$

for a.e. x in Γ_1 and a.e s in $\mathbb{R}$, we derive

$$
\begin{aligned}
&|u''_{lm}(t)|^2 + |v''_{lm}(t)|^2 + \|u'_{lm}(t)\|^2 + \|v'_{lm}(t)\|^2 \\
&+ 2d_1 \int_0^t \int_{\Gamma_1} (u''_{lm}(s))^2 d\Gamma ds + 2d_2 \int_0^t \int_{\Gamma_1} (v''_{lm}(s))^2 d\Gamma ds \\
&\leq |u''_{lm}(0)|^2 + |v''_{lm}(0)|^2 + \|u'_{lm}(0)\|^2 + \|v'_{lm}(0)\|^2 \\
&\quad + \int_0^t C[|u''_{lm}(s)|^2 + |v''_{lm}(s)|^2 + \|u'_{lm}(s)\|^2 + \|v'_{lm}(s)\|^2] ds.
\end{aligned}
\tag{3.10}
$$

The Gronwall's Lemma implies that there exists $C(t)$, $t > 0$, such that

$$
\begin{aligned}
&|u''_{lm}(t)|^2 + |v''_{lm}(t)|^2 + \|u'_{lm}(t)\|^2 + \|v'_{lm}(t)\|^2 \\
&+ 2d_1 \int_0^t \int_{\Gamma_1} (u''_{lm}(s))^2 d\Gamma ds + 2d_2 \int_0^t \int_{\Gamma_1} (v''_{lm}(s))^2 d\Gamma ds \\
&\leq C(t)(|u''_{lm}(0)|^2 + |v''_{lm}(0)|^2 + \|u'_{lm}(0)\|^2 + \|v'_{lm}(0)\|^2).
\end{aligned}
$$

We need to bound $|u''_{lm}(0)|^2$ and $|v''_{lm}(0)|^2$ by a constant independent of l and m. This is one of the key points of the proof. These bounds are obtained thanks to the choice of the special basis of $V \cap H^2(\Omega)$. It is showed in the next estimate.

Third Estimate. Note that $u_{lm}(0) = u^0$ e $v_{lm}(0) = v^0$, respectively, for all l, m, and $\frac{\partial u^0}{\partial \nu} + h_{1l}(., u^1_l) = 0$ on Γ_1. Take $t = 0$ in $(3.2)_1$. Then these two results and Green formulae, give

$$
(u''_{lm}(0), \varphi) + (-\Delta u^0, \varphi) + \alpha(u^0(v^0)^2, \varphi) = 0
$$

Taking $\varphi = u''_{lm}(0)$ in this equality, we derive

$$
|u''_{lm}(0)| \leq |\Delta u^0| + \alpha |u^0(v^0)^2| \leq C, \quad \forall l, m
$$

Thus $(u''_{lm}(0))$ is bounded in $L^2(\Omega)$, for all l, m. Analogously $(v''_{lm}(0))$ is bounded in $L^2(\Omega)$, for all l, m. Taking into account these last two boundness in (3.10), we obtain

$$\begin{gathered}(u'_{lm}) \quad \text{is bounded in } L^\infty_{\rm loc}(0,\infty;V), \quad \forall l \geq l_0,\ m;\\ (u''_{lm}) \quad \text{is bounded in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega)), \quad \forall l \geq l_0,\ \forall m;\\ (u''_{lm}) \quad \text{is bounded in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)), \quad \forall l \geq l_0,\ \forall m.\end{gathered} \tag{3.11}$$

Analogous boundedness hold for (v'_{lm}) and (v''_{lm}).

Fourth Estimate. By the Holder inequality, the embedding $V \hookrightarrow L^6(\Omega)$ and estimate (3.4), we obtain

$$|u_{lm}(t)v^2_{lm}(t)|^2 \leq \|u_{lm}(t)\|^2_{L^6(\Omega)}\|v_{lm}(t)\|^4_{L^6(\Omega)} \leq C.$$

Thus

$$(u_{lm}v^2_{lm}) \quad \text{is bounded in } L^\infty(0,\infty;L^2(\Omega)), \quad \forall l \geq l_0,\ \forall m. \tag{3.12}$$

Analogously,

$$(v_{lm}u^2_{lm}) \quad \text{is bounded in } L^\infty(0,\infty;L^2(\Omega)), \quad \forall l \geq l_0,\ \forall m. \tag{3.13}$$

As the estimates obtained are independent of l and m, it is natural to take the limit in l and m in (3.2), but there are a difficulty in the passage to the limit in the nonlinear term on the boundary Γ_1. For that, first we take the limit in m in (3.2) and then in l.

Passage to the Limit in m**.** The index l is fixed. Estimates (3.4) and (3.11) allow us, by induction and diagonal process (in order to have sequences converging on all $[0,\infty)$), to obtain a subsequences of (u_{lm}) and (v_{lm}), still denoted by (u_{lm}) and (v_{lm}), and functions $u_l, v_l : \Omega\times]0,\infty[\to \mathbb{R}$ satisfying:

$$\begin{gathered}u_{lm} \to u_l, m \to \infty, \quad \text{weak star in } L^\infty(0,\infty;V),\\ u'_{lm} \to u'_l, m \to \infty, \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;V),\\ u''_{lm} \to u''_l, m \to \infty, \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega)),\\ u'_{lm} \to u'_l, m \to \infty, \quad \text{weak in } L^2(0,\infty;L^2(\Gamma_1)),\\ u''_{lm} \to u''_l, m \to \infty, \quad \text{weak in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)).\end{gathered} \tag{3.14}$$

Analogous convergence holds for (v_{lm}), (v'_{lm}) and (v''_{lm}) to v_l, v'_l and v''_l, respectively.

In what follows we work with subsequence of (u_{lm}), always denoted by (u_{lm}), obtained by induction and diagonal process. We analyze the nonlinear terms. By $(3.14)_2$ we have

$$u'_{lm} \to u'_l \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \text{ as } m \to \infty.$$

This convergence, $(3.14)_5$ and Compactness Aubin-Lions' Theorem give

$$u'_{lm} \to u'_l \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)) \text{ as } m \to \infty. \tag{3.15}$$

By part (iii) of Lemma 3.1, we have

$$\begin{aligned}&\int_0^T \int_{\Gamma_1} \left[h_{1l}(x,u'_{lm}(x,t)) - h_{1l}(x,u'_l(x,t)\right]^2 d\Gamma dt\\ &\leq \|c_{1l}\|^2_{L^\infty(\Gamma_1)}\|u'_{lm} - u'_l\|^2_{L^2(0,T;L^2(\Gamma_1))}.\end{aligned}$$

Applying the above convergence in this inequality, we obtain

$$h_{1l}(.,u'_{lm}) \to h_{1l}(.,u'_l) \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)) \text{ as } m \to \infty. \tag{3.16}$$

Analogously,

$$h_{2l}(.,v'_{lm}) \to h_{2l}(.,v'_l) \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1))\text{as } ,m\to\infty. \tag{3.17}$$

Convergence $(3.14)_1$, $(3.14)_2$, and Compactness Aubin-Lions' Theorem imply

$$\begin{gathered}u_{lm}\to u_l, \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega)) \text{ as } m\to\infty,\\ v_{lm}\to v_l \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega)) \text{ as } m\to\infty,\end{gathered}$$

which implies

$$\begin{gathered}u_{lm}v^2_{lm}\to u_lv^2_l \quad \text{a.e. in } Q=\Omega\times]0,T[\text{ as } m\to\infty,\\ v_{lm}u^2_{lm}\to v_lu^2_l \quad \text{a. e. in } Q=\Omega\times]0,T[\text{ as } m\to\infty.\end{gathered}$$

This convergence, the fourth estimate and Lions' Lemma [11], give

$$\begin{gathered}u_{lm}v^2_{lm}\to u_lv^2_l \quad \text{weak in } L^2_{\rm loc}(0,\infty;L^2(\Omega)) \text{ as } m\to\infty,\\ v_{lm}u^2_{lm}\to v_lu^2_l \quad \text{weak in } L^2_{\rm loc}(0,\infty;L^2(\Omega)) \text{ as } m\to\infty.\end{gathered} \tag{3.18}$$

Convergence (3.14), (3.16)-(3.18) allow us to take the limit in m in $(3.2)_1$ and $(3.2)_2$. Thus by these convergence and the density of $V\cap H^2(\Omega)$ in V, we obtain

$$\begin{aligned}&\int_0^\infty (u''_l(s),\varphi)\theta(s)ds+\int_0^\infty((u_l(s),\varphi))\theta(s)ds+\alpha\int_0^\infty(u_l(s)v^2_l(s),\varphi)\theta(s)ds\\ &+\int_0^\infty\int_{\Gamma_1}h_{1l}(.,u'_l(s))\varphi\theta(s)d\Gamma ds=0,\quad \forall\varphi\in V,\ \forall\theta\in\mathcal{D}(0,\infty)\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\int_0^\infty (v''_l(s),\psi)\theta(s)ds+\int_0^\infty((v_l(s),\psi))\theta(s)ds+\alpha\int_0^\infty(v_l(s)u^2_l(s),\psi)\theta(s)ds\\ &+\int_0^\infty\int_{\Gamma_1}h_{2l}(.,v'_l(s))\psi\theta(s)d\Gamma ds=0,\quad \forall\psi\in V,\ \forall\theta\in\mathcal{D}(0,\infty).\end{aligned} \tag{3.20}$$

Now considering $\varphi,\psi\in\mathcal{D}(\Omega)$ and $\theta\in\mathcal{D}(0,\infty)$ in the last two equalities and taking into account that $u''_l,v''_l,u_lv^2_l$ and $v_lu^2_l$ belong to $L^2_{\rm loc}(0,\infty;L^2(\Omega))$, we get

$$\begin{gathered}u''_l-\Delta u_l+\alpha u_lv^2_l=0 \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega)),\\ v''_l-\Delta v_l+\alpha v_lu^2_l=0 \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega)).\end{gathered} \tag{3.21}$$

The above equalities give $\Delta u_l,\Delta v_l\in L^2_{\rm loc}(0,\infty;L^2(\Omega))$. As $u_l,v_l\in L^2_{\rm loc}(0,\infty;V)$, we obtain

$$\frac{\partial u_l}{\partial\nu},\frac{\partial v_l}{\partial\nu}\in L^2_{\rm loc}(0,\infty;H^{-1/2}(\Gamma_1)). \tag{3.22}$$

(see [13] and [10]).

Multiplying both sides of equation $(3.21)_1$ by $\varphi\theta$ with $\varphi\in V$ and $\theta\in\mathcal{D}(0,\infty)$, integrating on $[0,\infty[$, using Green formulae and regularity (3.22), we obtain

$$\begin{aligned}&\int_0^\infty (u''_l(s),\varphi)\theta(s)ds+\int_0^\infty((u_l(s),\varphi))\theta(s)ds+\alpha\int_0^\infty(u_l(s)v^2_l(s),\varphi)\theta(s)ds\\ &-\int_0^\infty\langle\frac{\partial u_l(s)}{\partial\nu},\varphi\rangle\theta(s)ds=0,\end{aligned}$$

where $\langle ; \rangle$ represents the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. Comparing this result with equation (3.19), we deduce

$$\frac{\partial u_l}{\partial \nu} + h_{1l}(., u_l') = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; L^2(\Gamma_1)). \tag{3.23}$$

By similar arguments, we obtain

$$\frac{\partial v_l}{\partial \nu} + h_{2l}(., v_l') = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; L^2(\Gamma_1)). \tag{3.24}$$

Passage to the Limit in l. As estimates (3.4), (3.11), (3.12) and (3.13) are independent of l and m, we obtain with (u_l) and (v_l) similar convergence to (3.14) and (3.18), that is, we have functions $u, v : \Omega \times]0, \infty[\to \mathbb{R}$ such that

$$\begin{gathered} u_l \to u \quad \text{weak star in } L^\infty(0, \infty; V), \\ u_l' \to u' \quad \text{weak star in } L^\infty_{\rm loc}(0, \infty; V), \\ u_l'' \to u'' \quad \text{weak star in } L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \\ u_l' \to u' \quad \text{weak in } L^2(0, \infty; L^2(\Gamma_1)), \\ u_l'' \to u'' \quad \text{weak in } L^2_{\rm loc}(0, \infty; L^2(\Gamma_1)), \end{gathered} \tag{3.25}$$

analogous convergence holds for $(v_l), (v_l')$ and (v_l'') to v, v' and v'' respectively. Also

$$\begin{gathered} u_l v_l^2 \to u v^2 \quad \text{weak in } L^2_{\rm loc}(0, \infty; L^2(\Omega)), \\ v_l u_l^2 \to v u^2 \quad \text{weak in } L^2_{\rm loc}(0, \infty; L^2(\Omega)). \end{gathered} \tag{3.26}$$

Considering $\varphi \in \mathcal{D}(\Omega)$ and $\theta \in \mathcal{D}(0, \infty)$ in (3.19), using convergence (3.25), (3.26) and applying similar arguments as in (3.21), we obtain

$$u'' - \Delta u + \alpha u v^2 = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; L^2(\Omega)) \tag{3.27}$$

Similarly

$$v'' - \Delta v + \alpha v u^2 = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; L^2(\Omega)) \tag{3.28}$$

We analyze the convergence in (3.23). As in (3.15), we get the convergence

$$u_l' \to u' \quad \text{in } L^2_{\rm loc}(0, \infty; L^2(\Gamma_1))$$

Fix $T > 0$. The preceding convergence implies

$$u_l'(x, t) \to u'(x, t) \quad \text{a.e. in } \Sigma_1 = \Gamma_1 \times]0, T[\tag{3.29}$$

Fix $(x, t) \in \Sigma_1$. Then by (3.29) the set $\{u_l'(x, t); l \in \mathbb{N}\}$ is bounded. Part (iv) of Lemma 3.1 says that (h_{1l}) converges to h_1 uniformly on bounded sets of $\mathbb{R}$, a.e. x in Γ_1. These two results and (3.29) imply

$$h_{1l}(x, u_l'(x, t)) \to h_1(x, u'(x, t)) \quad \text{a.e.in } \Sigma_1. \tag{3.30}$$

Analogously,

$$h_{2l}(x, v_l'(x, t)) \to h_2(x, v'(x, t)) \quad \text{a.e. in } \Sigma_1. \tag{3.31}$$

On the other hand, by $(3.21)_1$ we obtain

$$(u_l''(t), u_l'(t)) + ((u_l(t), u_l'(t))) + \alpha(u_l(t) v_l^2(t)), u_l'(t)) + \int_{\Gamma_1} h_{1l}(., u_l'(t)) u_l'(t) d\Gamma = 0,$$

or

$$\int_{\Gamma_1} h_{1l}(., u_l'(t)) u_l'(t) d\Gamma = -\frac{1}{2}\frac{d}{dt}|u_l'(t)|^2 - \frac{1}{2}\frac{d}{dt}\|u_l(t)\|^2 - \alpha(u_l(t) v_l^2(t), u_l'(t)).$$

By analogous arguments used to obtain (3.7), we deduce

$$|(u_l(t)v_l^2(t), u_l'(t))| \le C[\|u_l(t)\|^2 + |u_l'(t)|^2].$$

Note that $u_l \in C^0([0,T];V)$, $u_l' \in C^0([0,T];L^2(\Omega))$ and that $(u_l(T))$ and $(u_l'(T))$ are bounded in V and $L^2(\Omega)$, respectively (see similar estimates (3.4) and (3.11) for (u_l)). By the last two expressions and preceding considerations, we have

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} h_{1l}(., u_l'(t))u_l'(t) d\Gamma dt \\ &\le -\frac{1}{2}|u_l'(T)|^2 + \frac{1}{2}|u_l^1| - \frac{1}{2}\|u_l(T)\|^2 + \frac{1}{2}\|u^0\|^2 + \alpha C \int_0^T [\|u_l(t)\|^2 + |u_l'(t)|^2] dt \le C \end{aligned}$$

for all $t \in [0,T]$ for all $l \ge l_0$. As $h_{1l}(x,s)s \ge 0$, we obtain

$$\int_0^T \int_{\Gamma_1} h_{1l}(., u_l'(t))u_l'(t) d\Gamma dt \le C, \ \forall t \in [0,T], \quad \forall l \ge l_0. \tag{3.32}$$

where $C > 0$ is a constant independent of $l \ge l_0$ and $t \in [0,T]$. By (3.30), (3.32) and Strauss' Theorem [19], we have

$$h_{1l}(., u_l') \to h_1(., u') \quad \text{in } L^1(\Gamma_1 \times]0,T[). \tag{3.33}$$

By similar considerations,

$$h_{2l}(., v_l') \to h_2(., v') \quad \text{in } L^1(\Gamma_1 \times]0,T[). \tag{3.34}$$

On the other hand, by convergence (3.25), we find $u_l \to u$ weak in $L^2(0,T;V)$ and by (3.21) and convergence (3.25),

$$\Delta u_l \to \Delta u \quad \text{weak in } L^2(0,T:L^2(\Omega)).$$

These two convergences imply

$$\frac{\partial u_l}{\partial \nu} \to \frac{\partial u}{\partial \nu} \quad \text{weak in } L^2(0,T;H^{-1/2}(\Gamma_1))$$

(see [13]). As $\frac{\partial u_l}{\partial \nu} = -h_{1l}(., u_l')$ in $L^2(0,T:L^2(\Gamma_1))$ (see 3.23) we have that $\frac{\partial u_l}{\partial \nu} \in L^1(0,T:L^1(\Gamma_1))$. Then convergence (3.33) gives

$$\frac{\partial u_l}{\partial \nu} \to h_1(., u^1) \quad \text{in } L^1(0,T;L^1(\Gamma_1)).$$

These two last convergences and Lemma 3.2 provide

$$\frac{\partial u}{\partial \nu} + h_1(., u') = 0 \quad \text{in } L^1(0,T;L^1(\Gamma_1)).$$

By induction and diagonal process we obtain

$$\frac{\partial u}{\partial \nu} + h_1(., u') = 0 \quad \text{in } L^1_{\rm loc}(0,\infty;L^1(\Gamma_1)). \tag{3.35}$$

Similarly,

$$\frac{\partial v}{\partial \nu} + h_2(., v') = 0 \quad \text{in } L^1_{\rm loc}(0,\infty;L^1(\Gamma_1)). \tag{3.36}$$

Convergence (3.25) shows that $\{u, v\}$ belongs to class (C), expressions (3.27) and (3.28) are equations (2.1) and (3.35), (3.36) are the boundary conditions (2.2) of the theorem. The verification of the initial conditions (2.3) follows by convergence $(3.19)_l$. □

Proof of Theorem 2.2. Hypothesis $(H4)_1$ and estimate $(3.4)_3$ give
$$(h_{1l}(.,u_l')) \quad \text{is bounded in } L^2(0,\infty;L^2(\Gamma_1)).$$
Hence there exists χ in $L^2(0,\infty;L^2(\Omega))$ such that
$$h_{1l}(.,u_l') \to \chi \quad \text{weak in } L^2(0,\infty;L^2(\Gamma_1)).$$
By (3.33), we have
$$h_{1l}(.,u_l') \to h_1(.,u') \quad \text{in } L^1_{\rm loc}(0,\infty;L^1(\Gamma_1)).$$
Writing these two convergences in $\mathcal{D}'(0,\infty;L^1(\Gamma_1))$, we obtain by the uniqueness of limits,
$$h_{1l}(.,u_l') \to h_1(.,u') \quad \text{weak in } L^2(0,\infty;L^2(\Gamma_1)).$$
This and (3.35) provides
$$\frac{\partial u}{\partial \nu} + h_1(.,u') = 0 \quad \text{in } L^2(0,\infty;L^2(\Gamma_1)).$$
In a similar way,
$$\frac{\partial v}{\partial \nu} + h_2(.,v') = 0 \quad \text{in } L^2(0,\infty;L^2(\Gamma_1)).$$
The facts
$$u \in L^\infty(0,\infty;V), \quad \Delta u \in L^\infty_{\rm loc}(0,\infty;L^2(\Omega)), \quad \frac{\partial u}{\partial \nu} \in L^2(0,\infty;L^2(\Gamma_1))$$
give $u \in L^2_{\rm loc}(0,\infty;H^{3/2}(\Omega))$ (see [13] and [10]).

Regularity of solutions $\{u,v\}$ given by class (C) allows us to apply the energy method in equations (2.1) and to obtain the uniqueness of solutions (see [11]) □

Proof of Theorem 2.4. Let (g_{1l}) and (g_{2l}) be the sequences obtained in Lemma 3.1 for the functions g_1 and g_2, respectively. By direct computations, we show
$$|g_{1l}(s)| \le \frac{3}{2}k_1^*|s|, \quad |g_{2l}(s)| \le \frac{3}{2}k_2^*|s|, \quad \forall s \in \mathbb{R}. \tag{3.37}$$
Consider the approximate solutions $\{u_l,v_l\}$ of $\{u,v\}$ satisfying (3.21) and boundary conditions (3.23) and (3.24) constructed with
$$h_{1l}(.,u_l') = (m.\nu)g_{1l}(u_l'), \quad h_{2l}(.,u_l') = (m.\nu)g_{2l}(u_l').$$
Introduce the energy
$$E_l(t) = \frac{1}{2}\left[|u_l'(t)|^2 + |v_l'(t)|^2 + \|u_l(t)\|^2 + \|v_l(t)\|^2 + \alpha|u_l(t)v_l(t)|^2\right], \quad t \ge 0. \tag{3.38}$$
We prove inequality (2.11) for $E_l(t)$. The theorem will follow by taking the $\lim inf$ of both sides of this inequality. First of all, we note that
$$u_l, v_l \in L^\infty_{\rm loc}(0,\infty;V\cap H^2(\Omega)), \quad \forall l \ge l_0. \tag{3.39}$$
In fact, fix $l \in \mathbb{N}$. Let (u_{lm}) be the sequences obtained in Theorem 2.1 that approximates u_l. As g_{1l} is Lipschitzian and $g_{1l}(0)=0$, by [3], we have $g_{1l}(u_{lm}') \in V$. This fact, (3.37) and estimate $(3.11)_1$ give
$$(g_{1l}(u_{lm}')) \quad \text{is bounded in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)).$$
So
$$g_{1l}(u_{lm}') \to \chi_l \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)).$$

As in (3.16) we obtain

$$g_{1l}(u'_{lm}) \to g_{1l}(u'_l) \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)).$$

These two convergences imply

$$(m.\nu)g_{1l}(u'_l) \in L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)).$$

This result and boundary condition (3.23) give

$$\frac{\partial u_l}{\partial \nu} \in L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)).$$

Also, noting that u''_l and $\alpha u_l v_l^2$ belong to $L^\infty_{\rm loc}(0,\infty;L^2(\Omega))$ (see proof of Theorem 2.1), we obtain

$$-\Delta u_l = -u''_l - \alpha u_l v_l^2 \in L^\infty_{\rm loc}(0,\infty;L^2(\Omega)).$$

Applying results of regularity of elliptic problems to there two expressions, we obtain (3.39) for u_l. Analogously for v_l.

Regularity (3.39) allows us to obtain Rellich's identity for u_l, that is,

$$\begin{aligned} &2(\Delta u_l(t), m.\nabla u_l(t)) \\ &= (n-2)\|u_l(t)\|^2 - \int_{\Gamma_1} (m.\nu)|\nabla u_l(t)|^2 + 2\int_\Gamma \frac{\partial u_l(t)}{\partial \nu}\,[m.\nabla u_l(t)]\,d\Gamma \end{aligned} \tag{3.40}$$

(see [8] and [17]).

By (3.21) and boundary conditions (3.23), (3.24), we obtain

$$\frac{d}{dt}E_l(t) = -\int_{\Gamma_1} (m.\nu)g_{1l}(u'_l(t))u'_l(t)d\Gamma - \int_{\Gamma_1}(m.\nu)g_{2l}(v'_l(t))v'_l(t)d\Gamma$$

and by hypothesis (H5),

$$\frac{d}{dt}E_l(t) \le -d_1^*\int_{\Gamma_1}(m.\nu)u'^2_l(t)d\Gamma - d_2^*\int_{\Gamma_1}(m.\nu)v'^2_l(t)d\Gamma. \tag{3.41}$$

Introduce the perturbed energy

$$E_{l\varepsilon}(t) = E_l(t) + \varepsilon\psi_l(t), \quad \varepsilon > 0 \tag{3.42}$$

where

$$\psi_l(t) = \rho_l(t) + \theta_l(t), \tag{3.43}$$
$$\rho_l(t) = 2(u'_l(t), m.\nabla u_l(t)) + (n-1)(u'_l(t), u_l(t)), \tag{3.44}$$
$$\theta_l(t) = 2(v'_l(t), m.\nabla v_l(t)) + (n-1)(v'_l(t), v_l(t)). \tag{3.45}$$

By direct computations, we have $|\psi_l(t)| \le ME_l(t)$, where M were defined in (2.10). Then, for $\varepsilon \in (0, \frac{1}{2M})$,

$$\frac{1}{2}E_l(t) \le E_{l\varepsilon}(t) \le \frac{3}{2}E_l(t), \quad 0 < \varepsilon \le \frac{1}{2M}. \tag{3.46}$$

To facilitate the writing we omit the argument t in $\rho_l(t)$. By identity (3.40), Green formulae, boundary condition (3.23) and noting that $u''_l = \Delta u_l - \alpha u_l v_l^2$, it follows from (3.44)

$$\begin{aligned}\rho_l' &= (n-2)\|u_l\|^2 - \int_\Gamma (m.\nu)|\nabla u_l|^2 + 2\int_\Gamma \frac{\partial u_l}{\partial \nu}(m.\nabla u_l)d\Gamma \\ &\quad - 2\alpha(u_l v_l^2, m.\nabla u_l) + 2(u_l', m.\nabla u_l') + (n-1)|u_l'|^2 \\ &\quad - (n-1)\|u_l\|^2 - (n-1)\int_{\Gamma_1}(m.\nu)g_{1l}(u_l')u_l d\Gamma - \alpha(n-1)|u_l v_l|^2 \\ &= I_1 + I_2 + \dots + I_9.\end{aligned} \tag{3.47}$$

The idea is to obtain

$$\rho_l' \le -\eta E_{1l} - \eta|u_l v_l| + C\int_{\Gamma_1}(m.\nu)u'^2{}_l d\Gamma,$$

where $\eta > 0$, $C > 0$ and

$$E_{1l}(t) = \frac{1}{2}\left[|u_l'(t)|^2 + \|u_l(t)\|^2\right].$$

We have

$$\frac{\partial u_l}{\partial x_i} = \nu_i \frac{\partial u_l}{\partial \nu}, \quad |\nabla u_l|^2 = \big(\frac{\partial u_l}{\partial \nu}\big)^2 \quad \text{on } \Gamma_0 \tag{3.48}$$

By (3.48), we find

$$I_2 = -\int_\Gamma (m.\nu)|\nabla u_l|^2 d\Gamma = -\int_{\Gamma_0}(m.\nu)\left(\frac{\partial u_l}{\partial \nu}\right)^2 d\Gamma - \int_{\Gamma_1}(m.\nu)|\nabla u_l|^2 d\Gamma. \tag{3.49}$$

- Analysis of $I_3 = 2\int_\Gamma \frac{\partial u_l}{\partial \nu}(m.\nabla u_l)d\Gamma$.

By (3.48) and boundary condition (3.23), we derive

$$I_3 = 2\int_{\Gamma_0}(m.\nu)\big(\frac{\partial u_l}{\partial \nu}\big)^2 d\Gamma - 2\int_{\Gamma_1}(m.\nu)g_{1l}(u_l')(m.\nabla u_l)d\Gamma.$$

Recall R defined in (2.6). By (3.37), we have

$$\begin{aligned}-2\int_{\Gamma_1}(m.\nu)g_{1l}(u_l')(m.\nabla u_l)d\Gamma &\le R^2\int_{\Gamma_1}(m.\nu)[g_{1l}(u_l')]^2 d\Gamma + \int_{\Gamma_1}(m.\nu)|\nabla u_l|^2 d\Gamma \\ &\le R^2\Big(\frac{3}{2}k_1^*\Big)^2\int_{\Gamma_1}(m.\nu)u_l'^2 d\Gamma + \int_{\Gamma_1}(m.\nu)|\nabla u_l|^2 d\Gamma.\end{aligned}$$

So

$$I_3 \le 2\int_{\Gamma_0}(m.\nu)\big(\frac{\partial u_l}{\partial \nu}\big)^2 d\Gamma + R^2\big(\frac{3}{2}k_1^*\big)^2\int_{\Gamma_1}(m.\nu)u_l'^2 d\Gamma + \int_{\Gamma_1}(m.\nu)|\nabla u_l|^2 d\Gamma. \tag{3.50}$$

Simplifying similar terms in (3.49), (3.50) and noting that $\int_{\Gamma_0}(m.\nu)\big(\frac{\partial u_l}{\partial \nu}\big)^2 d\Gamma \le 0$, we obtain

$$I_2 + I_3 \le R^2\big(\frac{3}{2}k_1^*\big)^2\int_{\Gamma_1}(m.\nu)u_l'^2 d\Gamma$$

- Analysis of $I_4 = -2\alpha(u_l v_l^2, m.\nabla u_l)$. We recall N given by (2) and the embedding constant K given by (2.5). By (3.3), we have

$$\|u_l(t)\|^2 + \|v_l(t)\|^2 \le N, \quad \forall t \ge 0,\ \forall l \ge l_0. \tag{3.51}$$

By (3.51) and Holder inequality, we deduce

$$I_4 \le 2\alpha R K^3 N\|u_l\|^2.$$

• Analysis of $I_5 = 2(u_l', m.\nabla u_l')$. By Green formulae and noting that $\frac{\partial m_j}{\partial x_j} = 1$ and $u_l' = 0$ on Γ_0, we obtain

$$I_5 = -n|u_l'|^2 + \int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma.$$

• Analysis of $I_8 = -(n-1)\int_{\Gamma_1}(m.\nu) g_{1l}(u_l') u_l d\Gamma$. Recall the embedding constant K^* given by (2.5) and the constant L_1 given by (2.8). By (3.37) and usual inequalities, we get

$$I_8 \le \frac{1}{2}(n-1)^2\big(\frac{3}{2}k_1^*\big)^2 R(K^*)^2 \int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma + \frac{1}{4}\|u_l\|^2;$$

that is,

$$I_8 \le L_1 \int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma + \frac{1}{4}\|u_l\|^2.$$

By (3.47), using estimates for $I_2 + I_3, I_4, I_5, I_8$ and cancelling equal terms with different sign, we obtain

$$\begin{aligned} \rho_l' &\le -|u_l'|^2 - \|u_l\|^2 + 2\alpha R K^3 N(\alpha)\|u_l\|^2 + \frac{1}{4}\|u_l\|^2 \\ &\quad + \big[R^2\big(\frac{3}{2}k_1^*\big)^2 + L_1 + 1\big] \int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma - \alpha(n-1)|u_l v_l|^2. \end{aligned}$$

Recall L defined by (2.9). Hypothesis (H7) implies

$$\rho_l' \le -\frac{1}{2}|u_l'|^2 - \frac{1}{2}\|u_l\|^2 - \frac{\alpha}{4}|u_l v_l|^2 + L\int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma, \quad 0 \le \alpha \le \alpha_0.$$

Similarly, θ_l given by (3.45), satisfies

$$\theta_l' \le -\frac{1}{2}|v_l'|^2 - \frac{1}{2}\|v_l\|^2 - \frac{\alpha}{4}|u_l v_l|^2 + L\int_{\Gamma_1} (m.\nu) v_l'^2 d\Gamma, \quad 0 \le \alpha \le \alpha_0.$$

Combining these two inequalities with (3.42), (3.43) and using inequality (3.41), we have

$$E_{l\varepsilon}' \le -\varepsilon E_l - (d_1^* - \varepsilon L)\int_{\Gamma_1} (m.\nu) u_l'^2 d\Gamma - (d_2^* - \varepsilon L)\int_{\Gamma_1} (m.\nu) v_l'^2 d\Gamma.$$

This implies

$$E_{l\varepsilon}'(t) \le -\varepsilon E_l(t), \quad \forall t \ge 0,\ 0 < \varepsilon \le \min\big\{\frac{{d_1}^*}{L}, \frac{{d_2}^*}{L}\big\},\ 0 \le \alpha \le \alpha_0. \tag{3.52}$$

Take ω given by the theorem. Then (3.46) and (3.52) hold with $\varepsilon = \omega$. By (3.46) and (3.52), we deduce

$$E_{l\varepsilon}'(t) \le -\frac{2}{3}\omega E_{l\varepsilon}(t), \quad \forall t \ge 0,\ 0 \le \alpha \le \alpha_0.$$

This inequality and (3.46) give (2.11) with $E_l(t)$. Inequality (2.11) for the solution $\{u, v\}$ follows by taking the lim inf of both sides of the preceding inequality. □

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