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EXISTENCE OF POSITIVE BOUNDED SOLUTIONS FOR SOME NONLINEAR POLYHARMONIC ELLIPTIC SYSTEMS

SABRINE GONTARA, ZAGHARIDE ZINE EL ABIDINE

Abstract. We prove existence results for positive bounded continuous solutions of a nonlinear polyharmonic system by using a potential theory approach and properties of a large functional class $K_{m,n}$ called Kato class.

1. Introduction

The goal is to study the existence of positive continuous bounded solutions for the nonlinear elliptic higher order system

$$\begin{gathered}
(-\Delta)^m u+\lambda q g(v)=0 \quad \text{in } B,\\
(-\Delta)^m v+\mu p f(u)=0 \quad \text{in } B,\\
\lim_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}}=\varphi(\xi),\\
\lim_{x\to\xi\in\partial B}\frac{v(x)}{(1-|x|^2)^{m-1}}=\psi(\xi),
\end{gathered}\tag{1.1}$$

where m is a positive integer, $B=\{x\in\mathbb{R}^n:|x|<1\}$ is the unit ball of $\mathbb{R}^n$ $(n\geq 2)$, $\partial B=\{x\in\mathbb{R}^n:|x|=1\}$ is the boundary of B, λ, μ, are nonnegative constants and φ, ψ are two nontrivial nonnegative continuous functions on ∂B.

For the case $m=1$, the existence of solutions for nonlinear elliptic systems has been extensively studied for both bounded and unbounded $C^{1,1}$domain D in $\mathbb{R}^n$ $(n\geq 3)$ (see [8, 9, 11-13]).

The polyharmonic operator $(-\Delta)^m$, $m\in\mathbb{N}^*$, has been studied several years later. Indeed, Boggio [7] showed that the Green function $G_{m,n}$ of the operator $(-\Delta)^m$ on B with Dirichlet boundary conditions $u=\frac{\partial}{\partial\nu}u=\dots=\frac{\partial^{m-1}}{\partial\nu^{m-1}}u=0$ on ∂B, is given by:

$$G_{m,n}(x,y)=k_{m,n}|x-y|^{2m-n}\int_1^{\frac{[x,y]}{|x-y|}}\frac{(\nu^2-1)^{m-1}}{\nu^{n-1}}d\nu,\tag{1.2}$$

where $k_{m,n}$ is a positive constant, $\frac{\partial}{\partial\nu}$ is the outward normal derivative and for x, y in B, $[x,y]^2=|x-y|^2+(1-|x|^2)(1-|y|^2)$.

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From its expression, it is clear that $G_{m,n}$ is nonnegative in B^2. This does not hold for the Green function of $(-\Delta)^m$ in an arbitrary bounded domain (see for example [10]). It is well known that for $m = 1$, we do not have this restriction. In [2], the properties of the Green function $G_{m,n}$ of $(-\Delta)^m$ on B allowed the authors to introduce a large functional class called Kato class denoted by $K_{m,n}$ (see Definition 1.1 below). This class played a key role in the study of some nonlinear polyharmonic equation (see [2, 4, 14]). For the case $m = 1$, the Kato class has been introduced and studied for general domain possibly unbounded in [1, 3, 15] for $n \geq 3$ and [16] for $n = 2$.

Definition 1.1 ([2]). A borel measurable function q on B belongs to the Kato class $K_{m,n}$ if q satisfies the condition

$$\lim_{\alpha\to 0}\Big(\sup_{x\in B}\int_{B\cap B(x,\alpha)}\big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|q(y)|dy\Big) = 0.$$

Here and always $\delta(x) = 1 - |x|$, is the Euclidian distance between x and ∂B.

As typical example of functions belonging to the class $K_{m,n}$, we have

Example 1.2 ([4]). The function q defined in B by

$$q(x) = \frac{1}{(\delta(x))^\lambda(\log\frac{2}{\delta(x)})^\mu},$$

is in $K_{m,n}$ if and only if $\lambda < 2m$ and $\mu \in \mathbb{R}$ or $\lambda = 2m$ and $\mu > 1$.

Before presenting our main results, we lay out a number of potential theory tools and some notations which will be used throughout the paper. We are mainly concerned with the bounded continuous solution $H\varphi$ of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } B$$
$$u\big|_{\partial B} = \varphi,$$

where φ is a nonnegative continuous function on ∂B. We remark that the function defined on B and denoted by $H^m\varphi : x \to (1 - |x|^2)^{m-1}H\varphi(x)$ is a bounded continuous solution of the problem

$$\begin{gathered}(-\Delta)^m u = 0 \quad \text{in } B\\ \lim_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi).\end{gathered}\tag{1.3}$$

For simplicity, we denote

$$C_0(B) = \{w \text{ continuous on } B \text{ and } \lim_{x\to\xi\in\partial B} w(x) = 0\}$$

and

$$C(\overline{B}) = \{w \text{ continuous on } \overline{B}\}.$$

We also refer to $V_{m,n}f$ the m-potential of a nonnegative measurable function f on B by

$$V_{m,n}f(x) = \int_B G_{m,n}(x,y)\,f(y)dy, \quad \text{for } x \in B.$$

Recall that for each nonnegative measurable function f on B such that f and $V_{m,n}f$ are in $L^1_{\rm loc}(B)$, we have

$$(-\Delta)^m(V_{m,n}f) = f,$$

in the distributional sense.

The outline of this paper is as follows. In section 2, we collect some preliminary results about the Green function and the Kato class $K_{m,n}$. In section 3, a careful analysis about continuity is performed. In particular, we prove the following result.

Theorem 1.3. *Let $m-1 \le \beta \le m$, $q \in K_{m,n}$, then the function v defined on B by*

$$v(x) = \int_B \Big(\frac{\delta(y)}{\delta(x)}\Big)^{\beta} G_{m,n}(x,y)\,|q(y)|dy$$

is in $C(\overline{B})$ and if $m-1 \le \beta < m$, we have $\lim_{x \to \xi \in \partial B} v(x) = 0$.

Based on these properties of the Green's function $G_{m,n}$ and Kato class $K_{m,n}$, we establish in section 4 the first existence result stated in Theorem 1.4 below. The following conditions are considered

- (H1) The functions f, $g : (0,\infty) \to [0,\infty)$ are nondecreasing and continuous.
- (H2) The functions p and q are measurable nonnegative in B such that the functions
$$x \mapsto \frac{p(x)}{(\delta(x))^{m-1}} \quad \text{and} \quad x \mapsto \frac{q(x)}{(\delta(x))^{m-1}}$$
belong to the Kato class $K_{m,n}$.
- (H3) We suppose that
$$\lambda_0 = \inf_{x\in B} \frac{H^m\varphi(x)}{V_{m,n}(qg(H^m\psi))(x)} > 0,$$
$$\mu_0 = \inf_{x\in B} \frac{H^m\psi(x)}{V_{m,n}(pf(H^m\varphi))(x)} > 0.$$

Theorem 1.4. *Assume* (H1)–(H3). *Then for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$, the problem* (1.1) *has a positive continuous solution (u, v) satisfying for each $x \in B$,*

$$\begin{gathered} (1-\frac{\lambda}{\lambda_0})H^m\varphi(x) \le u(x) \le H^m\varphi(x), \\ (1-\frac{\mu}{\mu_0})H^m\psi(x) \le v(x) \le H^m\psi(x). \end{gathered} \tag{1.4}$$

In section 5, we study the system (1.1) when the functions f and g are non-increasing and $\lambda = \mu = 1$. More precisely, we fix a nontrivial nonnegative continuous function Φ on ∂B and we suppose the following hypotheses

- (H4) The functions f, $g : (0,\infty) \to [0,\infty)$ are non-increasing and continuous.
- (H5) The functions p and q are measurable nonnegative in B such that the functions
$$\widetilde{p} : x \mapsto p(x)\frac{f(H^m\Phi(x))}{(\delta(x))^{m-1}H\Phi(x)}, \quad \widetilde{q} : x \mapsto q\,(x)\frac{g(H^m\Phi(x))}{(\delta(x))^{m-1}H\Phi(x)}$$
belong to the Kato class $K_{m,n}$.

Using a fixed point argument, we prove in section 5 the following second existence result.

Theorem 1.5. *Assume that $\lambda = \mu = 1$ and that* (H4)–(H5) *are satisfied. Suppose that there exists $\gamma > 1$ such that $\varphi \ge \gamma\Phi$ and $\psi \ge \gamma\Phi$ on ∂B. Then* (1.1) *has a* *positive continuous solution satisfying for each* $x \in B$

$$\begin{aligned} H^m\Phi(x) \le u(x) \le H^m\varphi(x), \\ H^m\Phi(x) \le v(x) \le H^m\psi(x). \end{aligned} \tag{1.5}$$

Note that for $m = 1$ we find again the result of [11] which was our original motivation for deriving our study. The last section is reserved to examples. We conclude this section by giving some notation.

(i) Let f and g be nonnegative functions on a set S. We write $f(x) \approx g(x)$ for $x \in S$ if there is $c > 0$ not depending on x such that

$$\frac{1}{c}g(x) \le f(x) \le cg(x), \quad \forall x \in S.$$

(ii) For $s,\, t \in \mathbb{R}$, we denote $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$.

(iii) For any measurable function f on B, we use the notation

$$\alpha_f := \sup_{x,y\in B} \int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)}|f(z)|dz.$$

Finally, we mention that the letter c will be a positive generic constant which may vary from line to line.

2. Properties of the Green function $G_{m,n}$ and class $K_{m,n}$

To make the paper self contained, this section is devoted to recall some results established in [2, 5] that will be useful in our study.

Proposition 2.1 (3G-Theorem)**.** *There exists* $C_{m,n} > 0$ *such that for each* x, y, $z \in B$

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \le C_{m,n}\Big[\Big(\frac{\delta(z)}{\delta(x)}\Big)^m G_{m,n}(x,z) + \Big(\frac{\delta(z)}{\delta(y)}\Big)^m G_{m,n}(y,z)\Big]. \tag{2.1}$$

Proposition 2.2. *On* B^2*, the following estimates hold*

(i) For $2m < n$*,*

$$G_{m,n}(x,y) \approx |x-y|^{2m-n}\Big(1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\Big). \tag{2.2}$$

(ii) For $2m = n$*,*

$$G_{m,n}(x,y) \approx \log\Big(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\Big). \tag{2.3}$$

(iii) For $2m > n$*,*

$$G_{m,n}(x,y) \approx (\delta(x)\delta(y))^{m-\frac{n}{2}}(1 \wedge \frac{(\delta(x)\delta(y))^{n/2}}{|x-y|^n}). \tag{2.4}$$

Proposition 2.3. *On* B^2 *there exists* $c > 0$ *such that*

$$c(\delta(x)\delta(y))^m \le G_{m,n}(x,y). \tag{2.5}$$

Moreover if $|x-y| \ge r$*, we have*

$$G_{m,n}(x,y) \le c\frac{(\delta(x)\delta(y))^m}{r^n}. \tag{2.6}$$

Proposition 2.4. *Let* q *be a function in* $K_{m,n}$*, then*

(i) The constant α_q *is finite.*

(ii) The function $x \mapsto (\delta(x))^{2m-1}q(x)$ *is in* $L^1(B)$*.*

Proposition 2.5. *For each nonnegative function $q \in K_{m,n}$ and h a nonnegative harmonic in B we have for $x \in B$*

$$\int_B G_{m,n}(x,y)(1-|y|^2)^{m-1}h(y)q(y)dy \le \alpha_q(1-|x|^2)^{m-1}h(x). \tag{2.7}$$

In particular,

$$\sup_{x\in B}\int_B \Big(\frac{\delta(y)}{\delta(x)}\Big)^{m-1} G_{m,n}(x,y)q(y)dy \le 2^{m-1}\alpha_q. \tag{2.8}$$

3. Modulus of Continuity

The objective of this section is to prove Theorem 1.3. Let q be the function defined in B by

$$q(x) = \frac{1}{(\delta(x))^\lambda}.$$

It is shown in [2] that the function $q \in K_{m,n}$ if and only if $\lambda < 2m$ and $V_{m,n}q$ is bounded if and only if $\lambda < m+1$. More precisely, we give in the following sharp estimates, on the m-potential $V_{m,n}q$, which improve the inequalities given in [2, Proposition 3.10].

Proposition 3.1. *On B, the following estimates hold:*

(i) $V_{m,n}q(x) \approx (\delta(x))^m$ if $\lambda < m$,

(ii) $V_{m,n}q(x) \approx (\delta(x))^m \log(\frac{2}{\delta(x)})$ if $\lambda = m$,

(iii) $V_{m,n}q(x) \approx (\delta(x))^{2m-\lambda}$ if $m < \lambda < m+1$.

To prove Proposition 3.1, we need the next two lemmas. In what follows, for $x \in B$, we denote

$$D_1 = \{y \in B,\ |x-y|^2 \le \delta(x)\delta(y)\},$$
$$D_2 = \{y \in B,\ |x-y|^2 \ge \delta(x)\delta(y)\}.$$

Lemma 3.2 ([5]). *Let $x \in B$.*

(1) If $y \in D_1$, then

$$\frac{3-\sqrt{5}}{2}\delta(x) \le \delta(y) \le \frac{3+\sqrt{5}}{2}\delta(x) \quad \text{and} \quad |x-y| \le \frac{1+\sqrt{5}}{2}(\delta(x)\wedge\delta(y)).$$

(2) If $y \in D_2$, then

$$\delta(x)\vee\delta(y) \le \frac{\sqrt{5}+1}{2}|x-y|.$$

In particular, we have

$$B(x, \frac{\sqrt{5}-1}{2}\delta(x)) \subset D_1 \subset B(x, \frac{\sqrt{5}+1}{2}\delta(x)).$$

Lemma 3.3. *For each $x \in B$,*

$$\log(\frac{2}{\delta(x)}) \approx (1 + \int_{D_2}\frac{1}{|x-y|^n}dy).$$

Proof. In [6, Example 6], the authors showed that

$$\int_B \frac{G_{1,n}(x,y)}{\delta(y)}dy \underset{\delta(x)\to 0}{\sim} c\delta(x)\log(\frac{2}{\delta(x)}).$$

Then, since the functions $x \mapsto \int_B \frac{G_{1,n}(x,y)}{\delta(y)}dy$ and $x \mapsto \delta(x)\log(\frac{2}{\delta(x)})$ are positive continuous in B we deduce that

$$\int_B \frac{G_{1,n}(x,y)}{\delta(y)}dy \approx \delta(x)\log(\frac{2}{\delta(x)}) \text{ for all } x \in B. \tag{3.1}$$

Now for $x \in B$, we write

$$\int_B \frac{G_{1,n}(x,y)}{\delta(y)}dy = \int_{D_1} \frac{G_{1,n}(x,y)}{\delta(y)}dy + \int_{D_2} \frac{G_{1,n}(x,y)}{\delta(y)}dy.$$

So to prove the result, it is sufficient by (3.1) to show

$$\int_{D_1} \frac{G_{1,n}(x,y)}{\delta(y)}dy \approx \delta(x) \tag{3.2}$$

and

$$\int_{D_2} \frac{G_{1,n}(x,y)}{\delta(y)}dy \approx \delta(x)\int_{D_2} \frac{1}{|x-y|^n}dy. \tag{3.3}$$

To this end, we distinguish two cases.

Case 1: $n \geq 3$. Let $x \in B$. By using (2.2), we have

$$\int_{D_1} \frac{G_{1,n}(x,y)}{\delta(y)}dy \approx \frac{1}{\delta(x)}\int_{D_1} \frac{1}{|x-y|^{n-2}}dy. \tag{3.4}$$

On the other hand, by Lemma 3.2,

$$\int_{B(x,\frac{\sqrt{5}-1}{2}\delta(x))} \frac{1}{|x-y|^{n-2}}dy \leq \int_{D_1} \frac{1}{|x-y|^{n-2}}dy \leq \int_{B(x,\frac{\sqrt{5}+1}{2}\delta(x))} \frac{1}{|x-y|^{n-2}}dy,$$

which implies

$$\int_0^{\frac{\sqrt{5}-1}{2}\delta(x)} r\,dr \leq \int_{D_1} \frac{1}{|x-y|^{n-2}}dy \leq \int_0^{\frac{\sqrt{5}+1}{2}\delta(x)} r\,dr.$$

Hence, we deduce that

$$\int_{D_1} \frac{1}{|x-y|^{n-2}}dy \approx (\delta(x))^2. \tag{3.5}$$

By (3.4) and (3.5) we deduce (3.2). Furthermore, by (2.2) and the definition of D_2, we have for $x \in B$ and $y \in D_2$

$$G_{1,n}(x,y) \approx \frac{\delta(x)\delta(y)}{|x-y|^n}.$$

So we have clearly (3.3).

Case 2: $n = 2$. Let $y \in D_1$ and $x \in B$, then using that $\log(1+t) \leq ct^{1/2}$ for $t \geq 0$, we obtain

$$\log 2 \leq \log(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}) \leq c(\frac{\delta(x)\delta(y)}{|x-y|^2})^{1/2},$$

this together with (2.3) and Lemma 3.2 imply

$$\frac{1}{c\delta(x)}\int_{B(x,\frac{\sqrt{5}-1}{2}\delta(x))} dy \leq \int_{D_1} \frac{G_{1,n}(x,y)}{\delta(y)}dy \leq c\int_{B(x,\frac{\sqrt{5}+1}{2}\delta(x))} \frac{1}{|x-y|}dy.$$

So, we obtain

$$\frac{1}{c\delta(x)}\int_0^{\frac{\sqrt{5}-1}{2}\delta(x)} r\,dr \leq \int_{D_1} \frac{G_{1,n}(x,y)}{\delta(y)}dy \leq c\int_0^{\frac{\sqrt{5}+1}{2}\delta(x)} dr.$$

Hence, we obtain the claim (3.2). On the other hand, since $\frac{\delta(x)\delta(y)}{|x-y|^2} \in [0,1]$ for $x \in B$ and $y \in D_2$ and using the fact that $\log(1+t) \approx t$ for $t \in [0,1]$, we obtain

$$\int_{D_2} \frac{G_{1,n}(x,y)}{\delta(y)} dy \approx \delta(x) \int_{D_2} \frac{1}{|x-y|^2} dy,$$

which gives (3.3) for $n = 2$. This completes the proof. □

Proof of Proposition 3.1. In [2], the authors proved the result (i) and the upper estimates of $V_{m,n}q$ if $\lambda \in [m, m+1)$. Let us prove the lower estimates. First we need to show that

$$\int_{D_1} \frac{G_{m,n}(x,y)}{(\delta(y))^\lambda} dy \geq c(\delta(x))^{2m-\lambda} \quad \text{for } x \in B. \tag{3.6}$$

For this, we remark by Proposition 2.2 and the definition of D_1 that for each n, $m \in \mathbb{N}^*$

$$G_{m,n}(x,y) \geq c|x-y|^{2m-n}, \quad x \in B,\ y \in D_1.$$

It follows from Lemma 3.2, that

$$\begin{aligned}
\int_{D_1} \frac{G_{m,n}(x,y)}{(\delta(y))^\lambda} dy &\geq \frac{c}{(\delta(x))^\lambda} \int_{D_1} |x-y|^{2m-n} dy \\
&\geq \frac{c}{(\delta(x))^\lambda} \int_{B(x, \frac{\sqrt{5}-1}{2}\delta(x))} |x-y|^{2m-n} dy \\
&\geq \frac{c}{(\delta(x))^\lambda} \int_0^{\frac{\sqrt{5}-1}{2}\delta(x)} r^{2m-n} r^{n-1} dr \\
&\geq c(\delta(x))^{2m-\lambda}.
\end{aligned}$$

Then (3.6) is proved for each m and n and so (iii) holds.

It remains to prove the lower estimate in (ii); i.e., for $\lambda = m$. Since $\frac{\delta(x)\delta(y)}{|x-y|^2} \in [0,1]$, for $y \in D_2$, $x \in B$ and using the fact that $\log(1+t) \approx t$ for $t \in [0,1]$, we obtain immediately by Proposition 2.2,

$$G_{m,n}(x,y) \approx \frac{(\delta(x)\delta(y))^m}{|x-y|^n}, \quad \text{for } y \in D_2,\ x \in B. \tag{3.7}$$

Now let $x \in B$, by writing

$$V_{m,n}q(x) = \int_{D_1} \frac{G_{m,n}(x,y)}{(\delta(y))^m} dy + \int_{D_2} \frac{G_{m,n}(x,y)}{(\delta(y))^m} dy,$$

it follows from (3.6) and (3.7) that

$$V_{m,n}q(x) \geq c(\delta(x))^m \Big(1 + \int_{D_2} \frac{1}{|x-y|^n} dy\Big).$$

Now, using Lemma 3.3, we deduce that

$$V_{m,n}q(x) \geq c(\delta(x))^m \log(\frac{2}{\delta(x)}).$$

This completes the proof. □

Proposition 3.4. *Let $x_0 \in \overline{B}$ and $q \in K_{m,n}$. Then we have*

$$\lim_{\alpha\to 0}\Big(\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\frac{G_{m,n}(x,y)G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|dy\Big)=0$$

uniformly in $z \in B$.

Proof. Let $\varepsilon > 0$, then by the definition of $K_{m,n}$, there is $r > 0$ such that

$$\sup_{x\in B}\int_{B\cap B(x,r)}\Big(\frac{\delta(y)}{\delta(x)}\Big)^m G_{m,n}(x,y)|q(y)|dy\le\varepsilon.$$

Now, let $x_0 \in \overline{B}$, $x, z \in B$ and $\alpha > 0$ then by (2.1)

$$\begin{aligned}&\int_{B\cap B(x_0,\alpha)}\frac{G_{m,n}(x,y)G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|dy\\&\le 2C_{m,n}\sup_{\xi\in B}\int_{B\cap B(x_0,\alpha)}\Big(\frac{\delta(y)}{\delta(\xi)}\Big)^m G_{m,n}(\xi,y)|q(y)|dy.\end{aligned}$$

Furthermore, from (2.6), for each $x \in B$, we have

$$\begin{aligned}&\int_{B\cap B(x_0,\alpha)}\Big(\frac{\delta(y)}{\delta(x)}\Big)^m G_{m,n}(x,y)|q(y)|dy\\&\le\int_{B\cap B(x_0,\alpha)\cap(|x-y|<r)}\Big(\frac{\delta(y)}{\delta(x)}\Big)^m G_{m,n}(x,y)|q(y)|dy\\&\quad+\int_{B\cap B(x_0,\alpha)\cap(|x-y|\ge r)}\Big(\frac{\delta(y)}{\delta(x)}\Big)^m G_{m,n}(x,y)|q(y)|dy\\&\le\varepsilon+\frac{c}{r^n}\int_{B\cap B(x_0,\alpha)}(\delta(y))^{2m}|q(y)|dy\\&\le\varepsilon+\frac{c}{r^n}\int_{B\cap B(x_0,\alpha)}(\delta(y))^{2m-1}|q(y)|dy.\end{aligned}$$

Using Proposition 2.4 (ii), we deduce the result by letting $\alpha \to 0$. □

Corollary 3.5. *Let $m-1 \le \beta \le m$, $x_0 \in \overline{B}$, then for each $q \in K_{m,n}$,*

$$\lim_{\alpha\to 0}\Big(\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)|q(y)|dy\Big)=0.$$

Proof. For $\beta = m-1$, the result was proved in [14]. For $\beta \in (m-1, m]$, we deduce from Proposition 3.1, that

$$h(x):=\int_B G_{m,n}(x,y)\frac{1}{(\delta(y))^\lambda}dy\approx(\delta(x))^\beta,\ x\in B, \tag{3.8}$$

where $\lambda = 2m-\beta$ if $\beta \in (m-1, m)$ and $\lambda < m$ if $\beta = m$. Let $\varepsilon > 0$, then by Proposition 3.4 there exists $\alpha > 0$ such that for each $z \in B$ we have

$$\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\frac{G_{m,n}(x,y)G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|dy\le\varepsilon.$$

By Fubini's theorem, we have

$$\begin{aligned}
&\int_{B\cap B(x_0,\alpha)} h(y)G_{m,n}(x,y)|q(y)|dy \\
&= \int_B \Big(\int_{B\cap B(x_0,\alpha)} \frac{G_{m,n}(x,y)G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|dy\Big)\frac{G_{m,n}(x,z)}{(\delta(z))^\lambda}dz \\
&\leq \varepsilon h(x).
\end{aligned}$$

Which together with (3.8) imply

$$\begin{aligned}
&\sup_{x\in B}\int_{B\cap B(x_0,\alpha)} \Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)|q(y)|dy \\
&\leq c \sup_{x\in B}\int_{B\cap B(x_0,\alpha)} \frac{h(y)}{h(x)}G_{m,n}(x,y)|q(y)|dy \leq c\varepsilon.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. Let $\beta\in[m-1,m]$, $x_0\in\overline{B}$ and $\varepsilon>0$. By Corollary 3.5, there exists $\alpha>0$ such that

$$\sup_{\xi\in B}\int_{B\cap B(x_0,2\alpha)} \Big(\frac{\delta(y)}{\delta(\xi)}\Big)^\beta G_{m,n}(\xi,y)|q(y)|dy \leq \varepsilon. \tag{3.9}$$

We distinguish following two cases.

Case 1: $\beta\in[m-1,m)$. First we prove that v is continuous on B. For this aim we fix $x_0\in B$ and x, $z\in B\cap B(x_0,\alpha)$. So we have

$$\begin{aligned}
|v(x)-v(z)| &\leq \int_B \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}-\frac{G_{m,n}(z,y)}{(\delta(z))^\beta}\Big|(\delta(y))^\beta|q(y)|dy \\
&\leq \int_{B\cap B(x_0,2\alpha)} \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}-\frac{G_{m,n}(z,y)}{(\delta(z))^\beta}\Big|(\delta(y))^\beta|q(y)|dy \\
&\quad+\int_{B\cap B^c(x_0,2\alpha)} \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}-\frac{G_{m,n}(z,y)}{(\delta(z))^\beta}\Big|(\delta(y))^\beta|q(y)|dy \\
&\leq 2\sup_{\xi\in B}\int_{B\cap B(x_0,2\alpha)} \Big(\frac{\delta(y)}{\delta(\xi)}\Big)^\beta G_{m,n}(,y)|q(y)|dy \\
&\quad+\int_{B\cap B^c(x_0,2\alpha)} \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}-\frac{G_{m,n}(z,y)}{(\delta(z))^\beta}\Big|(\delta(y))^\beta|q(y)|dy \\
&= I_1+I_2.
\end{aligned}$$

If $|y-x_0|\geq 2\alpha$ then $|y-x|\geq\alpha$ and $|y-z|\geq\alpha$.

So applying (2.6), for all $x\in B\cap B(x_0,\alpha)$ and $y\in B\cap B^c(x_0,2\alpha)$, we have

$$\Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)\leq c(\delta(y))^{\beta+m}\leq c(\delta(y))^{2m-1}.$$

On the other hand, for $y\in B\cap B^c(x_0,2\alpha)$, $x\mapsto\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}$ is continuous in $B\cap B(x_0,\alpha)$. Hence since $x\mapsto(\delta(x))^{2m-1}q(x)$ is in $L^1(B)$ then by the dominated convergence theorem, we obtain

$$I_2=\int_{B\cap B^c(x_0,2\alpha)} \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^\beta}-\frac{G_{m,n}(z,y)}{(\delta(z))^\beta}\Big|(\delta(y))^\beta|q(y)|dy\to 0$$

as $|x-z|\to 0$. This together with (3.9) imply that v is continuous on B.

Next, we show that

$$v(x) \to 0 \quad \text{as } \delta(x) \to 0. \tag{3.10}$$

For this we consider $x_0 \in \partial B$ and $x \in B(x_0, \alpha) \cap B$, then

$$\begin{aligned} v(x) &= \int_{B \cap B(x_0, 2\alpha)} \Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)|q(y)|dy \\ &\quad + \int_{B \cap B^c(x_0, 2\alpha)} \Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)|q(y)|dy \\ &\le \sup_{\xi \in B} \int_{B \cap B(x_0, 2\alpha)} \Big(\frac{\delta(y)}{\delta(\xi)}\Big)^\beta G_{m,n}(\xi,y)|q(y)|dy \\ &\quad + \int_{B \cap B^c(x_0, 2\alpha)} \Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)|q(y)|dy \\ &= J_1 + J_2. \end{aligned}$$

For $y \in B \cap B^c(x_0, 2\alpha)$ we have $|y - x| \ge \alpha$. So from (2.6) we obtain

$$\Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y) \le c(\delta(x))^{m-\beta} \to 0 \text{ as } \delta(x) \to 0.$$

Then by the same arguments as above, we deduce that $J_2 \to 0$ as $\delta(x) \to 0$. This together with (3.9) gives (3.10).
Case 2: $\beta = m$. We point out that for $y \in B$, the function $x \mapsto \frac{G_{m,n}(x,y)}{(\delta(x))^m}$ is continuous in $\overline{B}$ outside the diagonal. So using similar arguments as in the case 1 we prove that $v \in C(\overline{B})$. This completes the proof. □

Proposition 3.6. *Let $m - 1 \le \beta < m$ and q be a nonnegative function in $K_{m,n}$. Then the family of functions*

$$\Big\{ \int_B \Big(\frac{\delta(y)}{\delta(x)}\Big)^\beta G_{m,n}(x,y)f(y)dy,\ |f| \le q \Big\}$$

is relatively compact in $C_0(B)$.

The proof of the above proposition is similar to the one of Theorem 1.3. So we omit it.

4. Proof of Theorem 1.4

Assume that the hypotheses (H1)–(H3) are satisfied. Then for $x \in B$ we have

$$\lambda_0 V_{m,n}(qg(H^m\psi))(x) \le H^m\varphi(x), \tag{4.1}$$
$$\mu_0 V_{m,n}(pf(H^m\varphi))(x) \le H^m\psi(x). \tag{4.2}$$

Let $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$. We define the sequences $(u_k)_{k\ge0}$ and $(v_k)_{k\ge0}$ by

$$\begin{aligned} v_0 &= H^m\psi \\ u_k &= H^m\varphi - \lambda V_{m,n}(qg(v_k)) \\ v_{k+1} &= H^m\psi - \mu V_{m,n}(pf(u_k)). \end{aligned}$$

We will prove that for all $k \in \mathbb{N}$,

$$0 < (1 - \frac{\lambda}{\lambda_0}) H^m \varphi \le u_k \le u_{k+1} \le H^m \varphi, \tag{4.3}$$

$$0 < (1 - \frac{\mu}{\mu_0}) H^m \psi \le v_{k+1} \le v_k \le H^m \psi. \tag{4.4}$$

From (4.1) we have that for each $x \in B$,

$$\begin{aligned} u_0(x) &= H^m\varphi(x) - \lambda V_{m,n}(qg(v_0))(x) \\ &\ge H^m\varphi(x) - \frac{\lambda}{\lambda_0} H^m\varphi(x) \\ &= (1 - \frac{\lambda}{\lambda_0}) H^m\varphi(x) > 0. \end{aligned}$$

So

$$v_1(x) - v_0(x) = -\mu V_{m,n}(pf(u_0))(x) \le 0.$$

On the other hand, since g is nondecreasing we have

$$u_1(x) - u_0(x) = \lambda V_{m,n}[q(g(v_0) - g(v_1))](x) \ge 0.$$

Since f is nondecreasing and using that

$$u_0(x) \le H^m\varphi(x), \tag{4.5}$$

we deduce from (4.2) that

$$v_1(x) = H^m\psi(x) - \mu V_{m,n}(pf(u_0))(x) \ge (1 - \frac{\mu}{\mu_0}) H^m\psi(x) > 0.$$

This implies that

$$u_1(x) \le H^m\varphi(x).$$

Finally, we obtain

$$\begin{aligned} 0 &< (1 - \frac{\lambda}{\lambda_0}) H^m \varphi \le u_0 \le u_1 \le H^m \varphi, \\ 0 &< (1 - \frac{\mu}{\mu_0}) H^m \psi \le v_1 \le v_0 \le H^m \psi. \end{aligned}$$

This implies that (4.3) and (4.4) hold for $k = 0$ and we conclude for any $k \in \mathbb{N}$ by induction.

Therefore, the sequences $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ converge respectively to two functions u and v satisfying

$$\begin{aligned} 0 &< (1 - \frac{\lambda}{\lambda_0}) H^m \varphi \le u \le H^m \varphi, \\ 0 &< (1 - \frac{\mu}{\mu_0}) H^m \psi \le v \le H^m \psi. \end{aligned} \tag{4.6}$$

Now, since g is nondecreasing continuous, we obtain by (4.4) that for each $(x, y) \in B^2$

$$0 \le G_{m,n}(x, y) q(y) g(v_k) \le ||g(H^m\psi)||_\infty G_{m,n}(x, y) q(y).$$

Moreover, since $x \mapsto \frac{q(x)}{(\delta(x))^{m-1}} \in K_{m,n}$ then by (2.8), we have for each $x \in B$,

$$y \mapsto G_{m,n}(x, y) q(y) \in L^1(B).$$

So using the continuity of g and the dominated convergence theorem we deduce that

$$\lim_{k\to\infty} V_{m,n}(qg(v_k)) = V_{m,n}(qg(v)),$$

and so we have that for each $x \in B$,

$$u(x) = H^m\varphi(x) - \lambda V_{m,n}(qg(v))(x). \tag{4.7}$$

Similarly we prove that for each $x \in B$,

$$v(x) = H^m\psi(x) - \mu V_{m,n}(pf(u))(x). \tag{4.8}$$

Next, we claim that (u, v) satisfies

$$\begin{gathered}(-\Delta)^m u = -\lambda qg(v),\\ (-\Delta)^m v = -\mu pf(u).\end{gathered}$$

Indeed, since $g(v)$ is bounded and $x \mapsto \frac{q(x)}{(\delta(x))^{m-1}} \in K_{m,n}$, we deduce by Proposition 2.4 that

$$qg(v) \in L^1_{\rm loc}(B).$$

On the other hand by Theorem 1.3, we have

$$x \mapsto \frac{1}{(\delta(x))^{m-1}} \int_B G_{m,n}(x,y)q(y)dy \in C_0(B).$$

Therefore, using that $g(v)$ is bounded we get

$$V_{m,n}(qg(v)) \in C_0(B), \tag{4.9}$$

which implies

$$V_{m,n}(qg(v)) \in L^1_{\rm loc}(B).$$

So we have in the distributional sense

$$(-\Delta)^m V_{m,n}(qg(v)) = qg(v) \quad \text{in } B.$$

Similarly,

$$(-\Delta)^m V_{m,n}(pf(u)) = pf(u) \quad \text{in } B.$$

Now, applying the operator $(-\Delta)^m$ in (4.7) and (4.8), it follows by (4.6) that (u, v) is a positive bounded solution of

$$\begin{gathered}(-\Delta)^m u + \lambda qg(v) = 0 \quad \text{in } B,\\ (-\Delta)^m v + \mu pf(u) = 0 \quad \text{in } B.\end{gathered}$$

From (4.7) and (4.9), we deduce that u is continuous in B. Similarly v is continuous.

Finally, by (1.3), (4.7) and Theorem 1.3, we obtain

$$\lim_{x\to\xi\in\partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi).$$

Similarly,

$$\lim_{x\to\xi\in\partial B} \frac{v(x)}{(1-|x|^2)^{m-1}} = \psi(\xi).$$

This completes the proof.

5. Proof of Theorem 1.5

Assume that $\lambda = \mu = 1$ and the hypotheses (H4) and (H5) are satisfied. Let $\widetilde{p}$ and $\widetilde{q}$ be the functions in $K_{m,n}$ given by hypothesis (H5). Put $\gamma = 1 + \alpha_{\widetilde{p}} + \alpha_{\widetilde{q}}$, where $\alpha_{\widetilde{p}}$ and $\alpha_{\widetilde{q}}$ are the constants associated respectively to the functions $\widetilde{p}$ and $\widetilde{q}$.

Let us consider two nonnegative continuous functions φ and ψ on ∂B such that $\varphi \geq \gamma\Phi$ and $\psi \geq \gamma\Phi$. It follows that for each $x \in B$,

$$H^m\varphi(x) \geq \gamma H^m\Phi(x), \quad H^m\psi(x) \geq \gamma H^m\Phi(x). \tag{5.1}$$

Let S be the non-empty closed convex set given by

$$S = \{w \in C_0(B) : H^m\Phi \leq w \leq H^m\psi\}.$$

We define the operator T on S by

$$Tw = H^m\psi - V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(w))]).$$

We aim to prove that T has a fixed point in S. First, we shall prove that TS is relatively compact in $C_0(B)$. Let $w \in S$, then since $w \geq H^m\Phi$ we deduce from hypothesis (H4) that

$$V_{m,n}(qg(w)) \leq V_{m,n}(qg(H^m\Phi)) = V_{m,n}((\delta(.))^{m-1}\widetilde{q}H\Phi).$$

Which implies by (H5) and (2.7) that

$$V_{m,n}(qg(w)) \leq \alpha_{\widetilde{q}}H^m\Phi. \tag{5.2}$$

This together with (5.1) imply

$$\begin{aligned} H^m\varphi - V_{m,n}(qg(w)) &\geq \gamma H^m\Phi - \alpha_{\widetilde{q}}H^m\Phi \\ &= (1+\alpha_{\widetilde{p}})H^m\Phi \\ &\geq H^m\Phi. \end{aligned}$$

Hence, using (H4), we have

$$pf[H^m\varphi - V_{m,n}(qg(w))] \leq pf(H^m\Phi) = (\delta(.))^{m-1}\widetilde{p}H\Phi. \tag{5.3}$$

This yields

$$pf[H^m\varphi - V_{m,n}(qg(w))] \leq \|H\Phi\|_\infty(\delta(.))^{m-1}\widetilde{p}. \tag{5.4}$$

Then using Proposition 3.6 with $\beta = m-1$, we deduce that the family of functions

$$\{V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(w))]) : w \in S\}$$

is relatively compact in $C_0(B)$. So since $H^m\psi \in C_0(B)$, we conclude that the family TS is relatively compact in $C_0(B)$.

Next, we shall prove that $T(S) \subset S$. For all $w \in S$, we have obviously

$$Tw(x) \leq H^m\psi(x), \quad \forall x \in B.$$

On the other hand, by (5.3), we have

$$\begin{aligned} V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(w))] &\leq V_{m,n}((\delta(.))^{m-1}\widetilde{p}H\Phi) \\ &\leq V_{m,n}(\widetilde{p}H^m\Phi). \end{aligned}$$

Then, by (H5) and (2.7) we have

$$V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(w))] \leq \alpha_{\widetilde{p}}H^m\Phi. \tag{5.5}$$

Which implies by (5.1), that for each $x \in B$

$$\begin{aligned} Tw(x) &\geq H^m\psi(x) - \alpha_{\widetilde{p}} H^m\Phi(x) \\ &\geq (\gamma - \alpha_{\widetilde{p}}) H^m\Phi(x) \\ &\geq (1 + \alpha_{\widetilde{q}}) H^m\Phi(x) \\ &\geq H^m\Phi(x), \end{aligned}$$

which proves that $T(S) \subset S$.

Now, we prove the continuity of the operator T in S for the supremum norm. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in S which converges uniformly to a function w in S. Since g is nonincreasing we deduce by (H_5) that

$$qg(w_k) \leq qg(H^m\Phi) \leq \|H\Phi\|_\infty (\delta(.))^{m-1}\widetilde{q}.$$

Now, it follows from (H5) and (2.8), that for each $x \in B$,

$$y \mapsto (\delta(y))^{m-1} G_{m,n}(x, y)\widetilde{q}(y) \in L^1(B).$$

We conclude by the dominated convergence theorem that for all $x \in B$,

$$\lim_{k\to\infty} V_{m,n}(qg(w_k))(x) = V_{m,n}(qg(w))(x) \tag{5.6}$$

and so from the continuity of f, we have

$$\lim_{k\to\infty} p(x)f[H^m\varphi(x) - V_{m,n}(qg(w_k))(x)] = p(x)f[H^m\varphi(x) - V_{m,n}(qg(w))(x)].$$

By (5.4), for each x, y in B,

$$G_{m,n}(x, y)p(y)f[H^m\varphi(y) - V_{m,n}(qg(w_k))(y)] \leq c(\delta(y))^{m-1}\widetilde{p}(y)G_{m,n}(x, y).$$

Then since $\widetilde{p} \in K_{m,n}$, we get by (2.8) and the dominated convergence theorem that for each $x \in B$,

$$Tw_k(x) \to Tw(x) \quad \text{as } k \to +\infty.$$

Consequently, since $T(S)$ is relatively compact in $C_0(B)$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|Tw_k - Tw\|_\infty \to 0 \quad \text{as } k \to +\infty.$$

Therefore, T is a continuous mapping of S to itself. So, since $T(S)$ is relatively compact in $C_0(B)$, it follows that T is a compact mapping on S. Finally, the Schauder fixed-point theorem implies the existence of a function $w \in S$ such that $w = Tw$. We put for $x \in B$

$$u(x) = H^m\varphi(x) - V_{m,n}(qg(w))(x) \tag{5.7}$$

and $v(x) = w(x)$. Then

$$v(x) = H^m\psi(x) - V_{m,n}(pf(u))(x).$$

It is clear that (u, v) satisfies (1.5) and it remains to prove that (u, v) satisfies (1.1) with $\lambda = \mu = 1$.

Since $0 \leq qg(v) \leq c(\delta(.))^{m-1}\widetilde{q}$ then by Proposition 2.4, it follows that $qg(v) \in L^1_{\rm loc}(B)$ and from (5.2), we have $V_{m,n}(qg(v)) \in L^1_{\rm loc}(B)$. Hence u satisfies (in the distributional sense)

$$(-\Delta)^m u = -(-\Delta)^m V_{m,n}(qg(w)) = -qg(v).$$

On the other hand,

$$(-\Delta)^m v = -(-\Delta)^m V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(v))]).$$

Using (5.4) and Proposition 2.4 we deduce that $pf[H^m\varphi - V_{m,n}(qg(v))] \in L^1_{\rm loc}(B)$. Moreover, by (5.5) we get

$$V_{m,n}(pf(u)) = V_{m,n}(pf[H^m\varphi - V_{m,n}(qg(v))]) \in L^1_{\rm loc}(B).$$

Hence, we have in the distibutional sense

$$(-\Delta)^m v = -pf(u).$$

Finally, let $\xi \in \partial B$, then since $qg(v) \leq c(\delta(.))^{m-1}\widetilde{q}$, we deduce by Theorem 1.3 for $\beta = m-1$, that

$$\lim_{x\to\xi} \frac{V_{m,n}(qg(v))(x)}{(1-|x^2|)^{m-1}} = 0.$$

Hence by (1.3) and (5.7) we have

$$\lim_{x\to\xi} \frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi) - \lim_{x\to\xi} \frac{V_{m,n}(qg(v))(x)}{(1-|x^2|)^{m-1}} = \varphi(\xi).$$

Similarly,

$$\lim_{x\to\xi} \frac{v(x)}{(1-|x|^2)^{m-1}} = \psi(\xi) - \lim_{x\to\xi} \frac{V_{m,n}(pf(u))}{(1-|x|^2)^{m-1}} = \psi(\xi).$$

This completes the proof.

6. Examples

In this section, we give examples that illustrate the existence results for (1.1). In the following two examples (H3) is satisfied.

Example 6.1. Let φ be a continuous function on ∂B such that there exists $c_0 > 0$ satisfying $\varphi(x) \geq c_0$ for all $x \in \partial B$. Let p be a nonnegative function on B such that $p_0 = \frac{p}{(\delta(.))^{m-1}}$ is in $K_{m,n}$ and q be a nonnegative measurable function satisfying for each $x \in B$, $q(x) \leq \frac{c}{(\delta(x))^\lambda}$ with $\lambda < m$. We consider $f,\ g : (0,\infty) \to [0,\infty)$ nondecreasing and continuous functions. Then (H3) is satisfied. Indeed, let $x \in B$, by (2.8), we have

$$V_{m,n}(p)(x) \leq 2^{m-1}\alpha_{p_0}(\delta(x))^{m-1}.$$

So

$$\begin{aligned}\frac{H^m\varphi(x)}{V_{m,n}(pf(H^m\psi))(x)} &\geq \frac{(1-|x|^2)^{m-1}c_0}{2^{m-1}\alpha_{p_0}\|f(H\psi)\|_\infty(\delta(x))^{m-1}}\\ &\geq \frac{c_0}{2^{m-1}\alpha_{p_0}\|f(H\psi)\|_\infty} > 0,\end{aligned}$$

which implies that $\lambda_0 > 0$.

Now since ψ is a nonnegative continuous function, then there exists $c > 0$ such that for all $x \in B$, $H\psi(x) \geq c\delta(x)$. So we have

$$\frac{H^m\psi(x)}{V_{m,n}(qg(H^m\varphi))(x)} \geq \frac{c\delta(x)(1-|x|^2)^{m-1}}{\|g(H\varphi)\|_\infty V_{m,n}q(x)}.$$

Since $q(x) \leq \frac{c}{(\delta(x))^\lambda}$, $\lambda < m$, we have by Proposition 3.1 that

$$V_{m,n}(q)(x) \approx (\delta(x))^m.$$

So

$$\frac{(1-|x|^2)^{m-1}H\psi(x)}{V_{m,n}(qg(H^m\varphi))(x)} \geq \frac{c\delta(x)(1-|x|^2)^{m-1}}{\|g(H\varphi)\|_\infty(\delta(x))^m} \geq \frac{c}{\|g(H\varphi)\|_\infty} > 0.$$

This proves that $\mu_0 > 0$.

Example 6.2. Let φ and ψ two nonnegative continuous functions on ∂B. We consider f, $g : (0,\infty) \to [0,\infty)$ nondecreasing and continuous functions. Since the functions $H^m\varphi$ and $H^m\psi$ are nonnegative bounded, then there exist $a_1 \geq 0$, $a_2 \geq 0$ such that $a_1 + a_2 > 0$ and for each $x \in B$,

$$f(H^m\varphi(x)) \leq a_1 H^m\varphi(x) + a_2, \quad g(H^m\psi(x)) \leq a_1 H^m\psi(x) + a_2.$$

We assume

- (A1) $a_1\varphi \approx a_1\psi$;
- (A2) $a_2 p \leq a_2 \frac{c}{(\delta(x))^\sigma}$ $a_2 q \leq a_2 \frac{c}{(\delta(x))^\sigma}$ with $\sigma < m$.

Then (H3) is satisfied. Indeed for each $x \in B$, we have

$$V_{m,n}(qg(H^m\psi)(x) \leq a_1 V_{m,n}(qH^m\psi)(x) + a_2 V_{m,n}(q)(x).$$

By (2.7), we have

$$V_{m,n}(qH^m\psi)(x) \leq \alpha_q H^m\psi(x),$$

and by Proposition 3.1,

$$V_{m,n}(q)(x) \leq c(\delta(x))^m.$$

Then

$$\begin{aligned} V_{m,n}(qg(H^m\psi))(x) &\leq a_1\alpha_q H^m\psi(x) + a_2 c(\delta(x))^m \\ &\leq c(\delta(x))^{m-1}(a_1 H\psi(x) + a_2\delta(x)). \end{aligned}$$

So using that there exists $c > 0$ such that for all $x \in B$, $H\varphi(x) \geq c\delta(x)$, we obtain

$$\begin{aligned} \frac{H^m\varphi(x)}{V_{m,n}(qg(H^m\psi))(x)} &\geq c\frac{(a_1 + a_2)H\varphi(x)}{a_1 H\psi(x) + a_2\delta(x)} \\ &\geq c\frac{a_1 H\psi(x) + a_2\delta(x)}{a_1 H\psi(x) + a_2\delta(x)} = c > 0. \end{aligned}$$

Hence $\lambda_0 > 0$. Similarly we have $\mu_0 > 0$. Note that if $a_1 = 0$ then hypothesis (A1) is satisfied for each φ and ψ and if $a_2 = 0$ then the hypothesis (A2) is satisfied for each p and q.

Now, as an application of Theorem 1.4, we give the following example.

Example 6.3. Let λ, μ be nonnegative constants, and φ, ψ be two nontrivial nonnegative continuous functions on ∂B. Let $f(t) = t^\alpha$ and $g(t) = t^\beta$, where α, $\beta > 0$. Now, let $\sigma < m$. We take p and q two nonnegative measurable functions satisfying for each $x \in B$,

$$p(x) \leq \frac{c}{(\delta(x))^\sigma}, \quad q(x) \leq \frac{c}{(\delta(x))^\sigma}.$$

Using similar arguments as above in Example 6.1, we show that (H3) is satisfied. Then for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$, the problem

$$\begin{gathered} (-\Delta)^m u + \lambda q v^\alpha = 0 \quad \text{in } B, \\ (-\Delta)^m v + \mu p u^\beta = 0 \quad \text{in } B, \\ \lim_{x \to \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi), \\ \lim_{x \to \xi \in \partial B} \frac{v(x)}{(1 - |x|^2)^{m-1}} = \psi(\xi), \end{gathered}$$

has positive continuous solution (u, v) satisfying (1.4).

We end this section by giving an example as application of Theorem 1.5.

Example 6.4. Let $\alpha > 0$, $\beta > 0$, $f(t) = t^{-\alpha}$ and $g(t) = t^{-\beta}$. Let p and q two nonnegative measurable functions such that

$$p(x) \le \frac{c}{(\delta(x))^{\lambda}} \quad \text{with } \lambda < m(1-\alpha),$$

and

$$q(x) \le \frac{c}{(\delta(x))^{\mu}} \quad \text{with } \mu < m(1-\beta).$$

Let φ, ψ and Φ nontrivial nonnegative continuous functions on ∂B. Then there exists a constant $\gamma > 1$ such that if $\varphi \ge \gamma\Phi$ and $\psi \ge \gamma\Phi$ on ∂B, the problem

$$\begin{gathered}
(-\Delta)^m u + q v^{-\alpha} = 0 \quad \text{in } B, \\
(-\Delta)^m v + p u^{-\beta} = 0 \quad \text{in } B, \\
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi), \\
\lim_{x \to \xi \in \partial B} \frac{v(x)}{(1-|x|^2)^{m-1}} = \psi(\xi),
\end{gathered}$$

has a positive continuous solution satisfying (1.5).

References

[1] I. Bachar, H. Mâagli; *Estimates on the Green's function and existence of positive solutions of nonlinear singular elliptic equations in the half space*, Positivity 9 (2003), 153-192.

[2] I. Bachar, H. Mâagli, S. Masmoudi, M. Zribi; *Estimates for the Green function and singular solutions for polyharmonic nonlinear equation*, Abstract and Applied Analysis 12 (2003), 715-741.

[3] I. Bachar, H. Mâagli, N. Zeddini; *Estimates on the Green function and existence of positive solutions of nonlinear singular elliptic equations*, Commun. Contemp. Math. 53 (2003), 401-434.

[4] S. Ben Othman; *On a singular sublinear polyharmonic problem*, Abstract and Applied Analysis 2006 (2006), 1-14.

[5] S. Ben Othman, H. Mâagli, M. Zribi; *Existence results for polyharmonic boundary value problems in the unit ball*, Abstract and Applied Analysis 2007 (2007), 1-17.

[6] S. Ben Othman, H. Mâagli, S. Masmoudi, M. Zribi; *Exact asymptotic behavior near the boundary to the solution for singular nonlinear Dirichlet problems*, Nonlinear Anal. 71 (2009), 4173-4150.

[7] T. Boggio; *Sulle funzioni di Green d'ordine m*, Rend. Circ. Math. Palermo, 20 (1905), 97-135.

[8] F. C. Cirstea, V. D. Radulescu; *Entire solutions blowing up at infinity for semilinear elliptic systems*, J. Math. Pures. Appl. 81 (2002), 827-846.

[9] F. David; *Radial solutions of an elliptic system*, Houston J. Math. 15 (1989), 425-458.

[10] P. R. Garabedian; *A partial differential equation arising in conformal mapping*, Pacific J. Math. 1 (1951), 485-524 .

[11] A. Ghanmi, H. Mâagli, S. Turki, N. Zeddini; *Existence of positive bounded solutions for some nonlinear elliptic systems*, J. Math. Anal. Appl. 352 (2009), 440-448.

[12] M. Ghergu, V. D. Radulescu; *On a class of singular Gierer-Meinhart systems arising in morphogenesis*, C. R. Acad. Sci. Paris. Ser.I 344 (2007), 163-168.

[13] A. V. Lair , A. W. Wood; *Existence of entire large positive solutions of semilinear elliptic systems*, Journal of Differential Equations 164 No.2 (2000), 380-394.

[14] H. Mâagli, F. Toumi, M. Zribi; *Existence of positive solutions for some polyharmonic nonlinear boundary-value problems*, Electronic Journal of Differential Equations Vol. 2003 (2003), No. 58, 1-19.

[15] H. Mâagli, M. Zribi; *On a new Kato class and singular solutions of a nonlinear elliptic equation in bounded domains*, Positivity 9 (2005), 667-686.

[16] N. Zeddini; *Positive solutions for a singular nonlinear problem on a bounded domain in* $\mathbb{R}^2$, Potential Analysis 18 (2003), 97-118.

Sabrine Gontara
Département de mathématiques, Faculte des sciences de Tunis, campus universitaire, 2092 Tunis, Tunisia
E-mail address: sabrine-28@hotmail.fr

Zagharide Zine El Abidine
Département de mathématiques, Faculte des sciences de Tunis, campus universitaire, 2092 Tunis, Tunisia
E-mail address: Zagharide.Zinelabidine@ipeib.rnu.tn