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MULTIPLE SOLUTIONS FOR A SINGULAR SEMILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENT AND SYMMETRIES

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Abstract. We consider the singular semilinear elliptic equation $-\Delta u - \frac{\mu}{|x|^2}u - \lambda u = f(x)|u|^{2^*-1}$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain, in $\mathbb{R}^N$, $N \geq 4$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, $f : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ in Ω and $0 < \mu < \overline{\mu} := (\frac{N-2}{2})^2$. We show that if Ω and f are invariant under a subgroup of $O(N)$, the effect of the equivariant topology of Ω will give many symmetric nodal solutions, which extends previous results of Guo and Niu [8].

1. Introduction

Much attention has been paid to the singular semilinear elliptic problem

$$\begin{gathered} -\Delta u - \mu\frac{u}{|x|^2} - \lambda u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a smooth bounded domain, $0 \in \Omega$, $0 \leq \mu < \overline{\mu} := ((N-2)/2)^2$, $\lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω and $2^* := 2N/(N-2)$ is the critical Sobolev exponent, and f is a continuous function. We state some related work here about this problem.

Brezis and Nirenberg [2] proved the existence of one positive solution for (1.1) with $\mu = 0$ and $f = 1$, with $\lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ on Ω and $N \geq 4$. Rey [13] and Lazzo [11] established a close relationship between the number of positive solutions for (1.1) with $\mu = 0$ and $f = 1$ and the domain topology if λ is positive and sufficiently small. Cerami, Solimini, and Struwe [6] proved that (1.1) with $\mu = 0$ and $f = 1$ has one solution changing sign exactly once for $N \geq 6$ and $\lambda \in (0, \lambda_1)$. In [5] Castro and Clapp proved that there is an effect of the domain topology on the number of minimal nodal solutions changing

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sign just once of (1.1) with $\mu = 0$ and $f = 1$, with λ positive sufficiently small. Recently Cano and Clapp [3] proved the multiplicity of sign changing solutions for (1.1) with $\lambda = a$ and $\mu = 0$, where a and f are continuous functions. The existence of non trivial positive solution for (1.1) with $f = 1$ and $\mu \in [0, \overline{\mu} - 1]$ and $\lambda \in (0, \lambda_1)$ where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω, was proved by Janelli [10]. Cao and Peng [4] proved the existence of a pair of sign changing solutions for (1.1) with $f = 1$, $N \geq 7$, $\mu \in [0, \overline{\mu} - 4]$, $\lambda \in (0, \lambda_1)$. Han and Liu [9] proved the existence of one non trivial solution for (1.1) with $\lambda > 0$, $f(x) > 0$ and some additional assumptions. Chen [7] proved the existence of one positive solution for (1.1) with $\lambda \in (0, \lambda_1)$ and f not necessarily positive but satisfying additional hypothesis. Guo and Niu [8] proved the existence of a symmetric nodal solution and a positive solution for $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω, with Ω and f invariant under a subgroup of $O(N)$.

2. Statement of results

Let Γ be a closed subgroup of the orthogonal transformations $O(N)$. We consider the problem

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} - \lambda u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(\gamma x) = u(x) \quad \forall x \in \Omega,\ \gamma \in \Gamma, \end{gathered} \tag{2.1}$$

where Ω is a smooth bounded domain, Γ-invariant in $\mathbb{R}^N$, $N \geq 4$, $2^* := (2N)/(N-2)$ is the critical Sobolev exponent, $f : \mathbb{R}^N \to \mathbb{R}$ is a Γ-invariant continuous function, $0 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω and $0 < \mu < \overline{\mu} := ((N-2)/2)^2$.

Note that a subset X of $\mathbb{R}^N$ is Γ-invariant if $\gamma x \in X$ for all $x \in X$ and $\gamma \in \Gamma$. A function $h : X \to \mathbb{R}$ is Γ-invariant if $h(\gamma x) = h(x)$ for all $x \in X$ and $\gamma \in \Gamma$. Let $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ be the Γ-orbit of a point $x \in \mathbb{R}^N$, and $\#\Gamma x$ its cardinality. Let $X/\Gamma := \{\Gamma x : x \in X\}$ denote the Γ-orbit space of $X \subset \mathbb{R}^N$ with the quotient topology.

Let us recall that the least energy solutions of

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \to 0 \quad \text{as } |x| \to \infty \end{gathered} \tag{2.2}$$

are the instantons

$$U_0^{\varepsilon,y}(x) := C(N)\Big(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2}\Big)^{(N-2)/2}, \tag{2.3}$$

where $C(N) = (N(N-2))^{(N-2)/2}$ (see [1], [15]). If the domain is not $\mathbb{R}^N$, there is no minimal energy solutions. These solutions minimize

$$S_0 := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{2/2^*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Also, for $0 < \mu < \overline{\mu}$ it is well known that the positive solutions to

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \tag{2.4}$$

are

$$U_\mu(x) := C_\mu(N)\Big(\frac{\varepsilon}{\varepsilon^2|x|^{(\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu})/\sqrt{\overline{\mu}}} + |x|^{(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})/\sqrt{\overline{\mu}}}}\Big)^{(N-2)/2},$$

where $\varepsilon > 0$ and $C_\mu(N) = (\frac{4N(\overline{\mu}-\mu)}{N-2})^{(N-2)/4}$ (see [16]). These solutions minimize

$$S_\mu := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\big)dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*}dx\big)^{2/2^*}}.$$

We denote

$$M := \big\{y \in \overline{\Omega} : \frac{\#\Gamma y}{f(y)^{(N-2/2}} = \min_{x \in \overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\big\}.$$

We shall assume that f satisfies:

(F1) $f(x) > 0$ for all $x \in \overline{\Omega}$.

(F2) f is *locally flat* at M, that is, there exist $r > 0$, $\nu > N$ and $A > 0$ such that

$$|f(x) - f(y)| \le A|x-y|^\nu \quad \text{if } y \in M \text{ and } |x-y| < r.$$

For all $0 < \mu < \overline{\mu}$ and $0 < \lambda < \lambda_1$ we define the bilinear operator $\langle \cdot,\cdot \rangle_{\lambda,\mu}$: $H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle u, v\rangle_{\lambda,\mu} := \int_\Omega (\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} - \lambda uv)dx$$

which is an inner product in $H_0^1(\Omega)$. Its induced norm

$$\|u\|_{\lambda,\mu} := \sqrt{\langle u,u\rangle_{\lambda,\mu}} = \Big(\int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2} - \lambda|u|^2)dx\Big)^{1/2}$$

is equivalent to the usual norm $\|u\| := \|u\|_{0,0}$ in $H_0^1(\Omega)$. This fact is a direct consequence of the Hardy inequality

$$\int_\Omega \frac{u^2}{|x|^2}dx \le \frac{1}{\overline{\mu}} \int_\Omega |\nabla u|^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{2.5}$$

Since λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω,

$$\int_\Omega \lambda |u|^2 dx \le \frac{\lambda}{\lambda_1}\int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\Big)dx. \tag{2.6}$$

Therefore, by (2.5),

$$\begin{aligned} \|u\|_{\lambda,\mu}^2 &:= \int_\Omega \Big(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda|u|^2\Big)dx \\ &\ge (1-\frac{\lambda}{\lambda_1})\int_\Omega \Big(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\Big)dx, \\ &\ge (1-\frac{\lambda}{\lambda_1})(1-\frac{\mu}{\overline{\mu}})\int_\Omega |\nabla u|^2 dx \\ &= (1-\frac{\lambda}{\lambda_1})(1-\frac{\mu}{\overline{\mu}})\|u\|^2. \end{aligned} \tag{2.7}$$

The other inequality follows from the Sobolev imbedding theorem.

It is easy to see that, if $f \in C(\overline{\Omega})$ satisfies (F1) then the norms

$$|u|_{2^*} := (\int_\Omega |u|^{2^*} dx)^{1/2^*}, \quad \text{and} \quad |u|_{f,2^*} := (\int_\Omega f(x)|u|^{2^*} dx)^{1/2^*}$$

are equivalent. We denote

$$\ell_f^\Gamma := \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) S_0^{N/2}.$$

Our multiplicity results will require the following non existence assumption.

(A1)) The problem

$$\begin{gathered} -\Delta u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(\gamma x) = u(x) \quad \forall x \in \Omega,\ \gamma \in \Gamma \end{gathered} \tag{2.8}$$

does not have a positive solution u which satisfies $\|u\|^2 \leq \ell_f^\Gamma$.

2.1. **Multiplicity of positive solutions.** Our next result generalizes the work of Guo and Niu [8] for the problem (2.1) and establishes a relationship between the topology of the domain and the multiplicity of positive solutions. For $\delta > 0$ let

$$M_\delta^- := \{y \in M : \operatorname{dist}(y, \partial\Omega) \geq \delta\},\ B_\delta(M) := \{z \in \mathbb{R}^N : \operatorname{dist}(z, M) \leq \delta\}. \tag{2.9}$$

Theorem 2.1. *Let $N \geq 4$, Ω and f be Γ-invariant, and* (F1), (F2), (A1) *and $\ell_f^\Gamma \leq S_\mu^{N/2}$ hold. For each $\delta, \delta' > 0$ there exist $\lambda^* \in (0, \lambda_1)$, $\mu^* \in (0, \overline{\mu})$ such that for all $\lambda \in (0, \lambda^*)$, $\mu \in (0, \mu^*)$ the problem* (2.1) *has at least*

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$$

positive solutions which satisfy

$$\ell_f^\Gamma - \delta' \leq \|u\|_{\lambda,\mu}^2 < \ell_f^\Gamma.$$

2.2. **Multiplicity of nodal solutions.** We assume that Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2 := \{-1, 1\}$, where G is a closed subgroup of $O(N)$ for which, Ω is G-invariant and $f : \mathbb{R}^N \to \mathbb{R}$ is a G-invariant function.

A real valued function u defined in Ω will be called τ-equivariant if

$$u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G.$$

In this section we study the problem

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} - \lambda u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G \end{gathered} \tag{2.10}$$

Note that all τ-equivariant functions u are Γ-invariant; i.e., $u(gx) = u(x)$ for all $x \in \Omega$, $g \in \Gamma$. If u is a τ-equivariant function then $u(gx) = -u(x)$ for all $x \in \Omega$ and $g \in \tau^{-1}(-1)$. Thus all non trivial τ-equivariant solution of (2.1) change sign.

Definition 2.2. We call a Γ-invariant subset X of $\mathbb{R}^N$ Γ-connected if cannot be written as the union of two disjoint open Γ-invariant subsets. A real valued function $u : \Omega \to \mathbb{R}$ is $(\Gamma, 2)$-nodal if the sets

$$\{x \in \Omega : u(x) > 0\} \quad \text{and} \quad \{x \in \Omega : u(x) < 0\}$$

are nonempty and Γ-connected.

For each G-invariant subset X of $\mathbb{R}^N$, we define

$$X^\tau := \{x \in X : Gx = \Gamma x\}.$$

Let $\delta > 0$, define

$$M^-_{\tau,\delta} := \{y \in M : \operatorname{dist}(y, \partial\Omega \cap \Omega^\tau) \geq \delta\},$$

and $B_\delta(M)$ as in (2.9).

The next theorem is a multiplicity result for τ-equivariant $(\Gamma, 2)$-nodal solutions for the problem (2.1).

Theorem 2.3. *Let $N \geq 4$, and* (F1), (F2), (A1) *and $\ell_f^\Gamma \leq S_\mu^{N/2}$ hold. If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ defined on a closed subgroup G of $O(N)$ for which Ω and f are G-invariant. Given $\delta, \delta' > 0$ there exists $\lambda^* \in (0, \lambda_1)$, $\mu^* \in (0, \overline{\mu})$ such that for all $\lambda \in (0, \lambda^*)$, $\mu \in (0, \mu^*)$ the problem* (2.1) *has at least*

$$\operatorname{cat}_{(B_\delta(M) \setminus B_\delta(M)^\tau)/G}(M^-_{\tau,\delta}/G)$$

pairs $\pm u$ of τ-equivariants $(\Gamma, 2)$-nodal solutions which satisfy

$$2\ell_f^\Gamma - \delta' \leq \|u\|^2_{\lambda,\mu} < 2\ell_f^\Gamma.$$

2.3. **Non symmetric properties for solutions.** Let $\Gamma \subset \widetilde{\Gamma} \subset O(N)$. Next we give sufficient conditions for the existence of many solutions which are Γ-invariant but are not $\widetilde{\Gamma}$-invariant.

Theorem 2.4. *Let $N \geq 4$ and assume that f satisfies* (F1), (F2), (A1) *and $\ell_f^\Gamma \leq S_\mu^{N/2}$. Let $\widetilde{\Gamma}$ be a closed subgroup of $O(N)$ containing Γ, for which Ω and f are $\widetilde{\Gamma}$-invariant and*

$$\min_{x \in \overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} < \min_{x \in \overline{\Omega}} \frac{\#\widetilde{\Gamma} x}{f(x)^{(N-2)/2}}.$$

Given $\delta, \delta' > 0$ there exist $\lambda^* \in (0, \lambda_1)$, $\mu^* \in (0, \overline{\mu})$ such that for all $\lambda \in (0, \lambda^*)$, $\mu \in (0, \mu^*)$ the problem (2.1) *has at least*

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M^-_\delta/\Gamma)$$

positive solutions which are not $\widetilde{\Gamma}$-invariant and satisfy

$$2\ell_f^\Gamma - \delta' \leq \|u\|^2_{\lambda,\mu} < 2\ell_f^\Gamma.$$

3. The variational problem

Let $\tau : G \to \mathbb{Z}/2$ be a homomorphism defined on a closed subgroup G of $O(N)$, and $\Gamma := \ker \tau$. Consider the problem

$$\begin{gathered}
-\Delta u - \mu \frac{u}{|x|^2} - \lambda u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega \\
u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G,
\end{gathered} \tag{3.1}$$

where Ω is a G-invariant bounded smooth subset of $\mathbb{R}^N$, and $f : \mathbb{R}^N \to \mathbb{R}$ is a G-invariant continuous function which satisfies (F1).

If $\tau \equiv 1$ then the problems (2.10) and (2.1) coincide. If τ is an epimorphism then a solution of (2.10) is a solution of (2.1) with the additional property $u(gx) = -u(x)$ for all $x \in \Omega$ and $g \in \tau^{-1}(-1)$. So every non trivial solution of (2.10) is a sign changing solution for (2.1).

The homomorphism τ induces the action of G on $H_0^1(\Omega)$ given by

$$(gu)(x) := \tau(g)u(g^{-1}x).$$

The fixed point space of the action is given by

$$\begin{aligned} H_0^1(\Omega)^\tau &:= \{u \in H_0^1(\Omega) : gu = u \quad \forall g \in G\} \\ &= \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x) \quad \forall g \in G, \quad \forall x \in \Omega\}, \end{aligned}$$

is the space of τ-equivariant functions. The fixed point space of the restriction of this action to Γ

$$H_0^1(\Omega)^\Gamma = \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x), \forall g \in \Gamma,\ \forall x \in \Omega\}$$

are the Γ-invariant functions of $H_0^1(\Omega)$. The norms $\|\cdot\|_{\lambda,\mu}$, $\|\cdot\|$ on $H_0^1(\Omega)$ and $|\cdot|_{2^*}$, $|\cdot|_{f,2^*}$ on $L^{2^*}(\Omega)$ are G-invariant with respect to the action induced by τ; therefore, the functional

$$\begin{aligned} E_{\lambda,\mu,f}(u) &:= \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda|u|^2)dx - \frac{1}{2^*}\int_\Omega f(x)|u|^{2^*}dx \\ &= \frac{1}{2}\|u\|_{\lambda,\mu}^2 - \frac{1}{2^*}|u|_{f,2^*}^{2^*} \end{aligned}$$

is G-invariant, with derivative

$$DE_{\lambda,\mu,f}(u)v = \int_\Omega \Big(\nabla u \cdot \nabla v - \mu\frac{uv}{|x|^2} - \lambda uv\Big)dx - \int_\Omega f(x)|u|^{2^*-2}uv dx.$$

By the principle of symmetric criticality [12], the critical points of its restriction to $H_0^1(\Omega)^\tau$ are the solutions of (2.10), and all non trivial solutions lie on the Nehari manifold

$$\begin{aligned} \mathcal{N}_{\lambda,\mu,f}^\tau &:= \{u \in H_0^1(\Omega)^\tau : u \neq 0, DE_{\lambda,\mu,f}(u)u = 0\} \\ &= \{u \in H_0^1(\Omega)^\tau : u \neq 0, \|u\|_{\lambda,\mu}^2 = |u|_{f,2^*}^{2^*}\}. \end{aligned}$$

which is of class C^2 and radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\tau$ by the radial projection

$$\pi_{\lambda,\mu,f} : H_0^1(\Omega)^\tau \setminus \{0\} \to \mathcal{N}_{\lambda,\mu,f}^\tau \quad \pi_{\lambda,\mu,f}(u) := \Big(\frac{\|u\|_{\lambda,\mu}^2}{|u|_{f,2^*}^{2^*}}\Big)^{(N-2)/4} u.$$

Therefore, the nontrivial solutions of (2.10) are precisely the critical points of the restriction of $E_{\lambda,\mu,f}$ to $\mathcal{N}_{\lambda,\mu,f}^\tau$. If $\tau \equiv 1$ we write $\mathcal{N}_{\lambda,\mu,f}^\Gamma$ and if G is a trivial group $\mathcal{N}_{\lambda,\mu,f}$. Note that

$$E_{\lambda,\mu,f}(u) = \frac{1}{N}\|u\|_{\lambda,\mu}^2 = \frac{1}{N}|u|_{f,2^*}^{2^*} \quad \forall u \in \mathcal{N}_{\lambda,\mu,f}^\tau. \tag{3.2}$$

and

$$E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)) = \frac{1}{N}\Big(\frac{\|u\|_{\lambda,\mu}^2}{|u|_{f,2^*}^2}\Big)^{N/2} \quad \forall u \in H_0^1(\Omega)^\tau \setminus \{0\}.$$

We define

$$\begin{aligned} m(\lambda,\mu,f) &:= \inf_{\mathcal{N}_{\lambda,\mu,f}} E_{\lambda,\mu,f}(u) = \inf_{\mathcal{N}_{\lambda,\mu,f}} \frac{1}{N}\|u\|_{\lambda,\mu}^2 \\ &= \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{1}{N}\Big(\frac{\|u\|_{\lambda,\mu}^2}{|u|_{f,2^*}^2}\Big)^{N/2}. \end{aligned}$$

In particular, $E_{\lambda,\mu,f}$ are bounded below on $\mathcal{N}_{\lambda,\mu,f}$. We denote by

$$m^{\Gamma}(\lambda,\mu,f) := \inf_{\mathcal{N}^{\Gamma}_{\lambda,\mu,f}} E_{\lambda,\mu,f}, \quad m^{\tau}(\lambda,\mu,f) := \inf_{\mathcal{N}^{\tau}_{\lambda,\mu,f}} E_{\lambda,\mu,f}.$$

3.1. **Estimates for the infimum.**

Proposition 3.1. $m^{\Gamma}(\lambda,\mu,f) > 0$.

Proof. Assume that $m^{\Gamma}(\lambda,\mu,f) = 0$. Then there exist a sequence (u_n) on $\mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ such that

$$E_{\lambda,\mu,f}(u_n) \to m^{\Gamma}(\lambda,\mu,f) = 0.$$

So $E_{\lambda,\mu,f}(u_n) = \frac{1}{N}\|u_n\|^2_{\lambda,\mu}$. Since $\|\cdot\|_{\lambda,\mu}$ and $\|\cdot\|$ are equivalent norms of $H^1_0(\Omega)$ we have that $u_n \to 0$ strongly in $H^1_0(\Omega)$; but $\mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ is closed in $H^1_0(\Omega)$ then $0 \in \mathcal{N}^{\Gamma}_{\lambda,\mu,f}$ which is a contradiction. $\square$

Proposition 3.2. *Let $0 < \lambda \le \lambda' < \lambda_1$, $0 < \mu \le \mu' < \overline{\mu}$ and $f : \mathbb{R}^N \to \mathbb{R}$ a continuous function Σ-invariant, such that f satisfies (F1), and Σ is a closed subgroup of $O(N)$. Then $\|u\|^2_{\lambda',\mu'} \le \|u\|^2_{\lambda,\mu}$,*

$$m(\lambda',\mu',f) \le m(\lambda,\mu,f) \quad \text{and} \quad m^{\Sigma}(\lambda',\mu',f) \le m^{\Sigma}(\lambda,\mu,f).$$

Proof. By definition of $\|\cdot\|_{\lambda,\mu}$ we obtain the first inequality. Let $u \in H^1_0(\Omega) \setminus \{0\}$, then

$$\begin{aligned} m(\lambda',\mu',f) &\le E_{\lambda',\mu',f}(\pi_{\lambda',\mu',f}(u)) \\ &= \frac{1}{N}\Big(\frac{\|u\|^2_{\lambda',\mu'}}{|u|^2_{f,2^*}}\Big)^{N/2} \\ &\le \frac{1}{N}\Big(\frac{\|u\|^2_{\lambda,\mu}}{|u|^2_{f,2^*}}\Big)^{N/2} \\ &= E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)). \end{aligned}$$

From this inequality there proof follows. $\square$

We denote by λ_1 the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ in $H^1_0(\Omega)$.

Lemma 3.3. *For all $\lambda \in (0,\lambda_1)$, $\mu \in (0,\overline{\mu})$, $u \in H^1_0(\Omega)^{\tau}$, it follows that*

$$E_{0,0,f}(\pi_{0,0,f}(u)) \le (\frac{\bar{\mu}}{\bar{\mu}-\mu})^{\frac{N}{2}}(\frac{\lambda_1}{\lambda_1-\lambda})^{\frac{N}{2}} E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)).$$

Proof. Since

$$E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)) = \frac{1}{N}\Big(\frac{\|u\|^2_{\lambda,\mu}}{|u|^2_{f,2^*}}\Big)^{N/2} = \frac{1}{N}\Big(\frac{\|u\|^N_{\lambda,\mu}}{|u|^N_{f,2^*}}\Big),$$

and by (2.7)

$$(1-\frac{\mu}{\bar{\mu}})(1-\frac{\lambda}{\lambda_1})\|u\|^2 \le \|u\|^2_{\lambda,\mu},$$

then

$$(1-\frac{\mu}{\bar{\mu}})^{\frac{N}{2}}(1-\frac{\lambda}{\lambda_1})^{\frac{N}{2}}\|u\|^N \le \|u\|^N_{\lambda,\mu}$$

$$(1-\frac{\mu}{\bar{\mu}})^{\frac{N}{2}}(1-\frac{\lambda}{\lambda_1})^{\frac{N}{2}}\frac{1}{N}\frac{\|u\|^N}{|u|^N_{f,2^*}} \le E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u))$$

so

$$E_{0,0,f}(\pi_{0,0,f}(u)) \le \big(\frac{\bar\mu}{\bar\mu-\mu}\big)^{\frac{N}{2}}\big(\frac{\lambda_1}{\lambda_1-\lambda}\big)^{\frac{N}{2}} E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(u)),$$

which concludes the proof. □

As a immediately consequence we have the following result.

Corollary 3.4.

$$m^\tau(0,0,f) \le \big(\frac{\bar\mu}{\bar\mu-\mu}\big)^{\frac{N}{2}}\big(\frac{\lambda_1}{\lambda_1-\lambda}\big)^{\frac{N}{2}} m^\tau(\lambda,\mu,f).$$

For the proof of the next lemma we refer the reader to [3].

Lemma 3.5. *If $\Omega\cap M \ne \emptyset$ then*

(a) $m^\Gamma(0,0,f) \le \frac{1}{N}\ell_f^\Gamma$.

(b) *if there exists $y\in\Omega\cap M$ with $\Gamma x \ne Gy$, then $m^\tau(0,0,f)\le \frac{2}{N}\ell_f^\Gamma$.*

3.2. **A compactness result.**

Definition 3.6. A sequence $\{u_n\}\subset H_0^1(\Omega)$ satisfying

$$E_{\lambda,\mu,f}(u_n)\to c \quad\text{and}\quad \nabla E_{\lambda,\mu,f}(u_n)\to 0.$$

is called a Palais-Smale sequence for $E_{\lambda,\mu,f}$ at c. We say that $E_{\lambda,\mu,f}$ satisfies the Palais-Smale condition $(PS)_c$ if every Palais-Smale sequence for $E_{\lambda,\mu,f}$ at c has a convergent subsequence. If $\{u_n\}\subset H_0^1(\Omega)^\tau$ then $\{u_n\}$ is a τ-equivariant Palais-Smale sequence and $E_{\lambda,\mu,f}$ satisfies the τ-equivariant Palais-Smale condition, $(PS)_c^\tau$. If $\tau\equiv 1$ $\{u_n\}$ is a Γ-invariant Palais-Smale sequence and $E_{\lambda,\mu,f}$ satisfies the Γ-invariant Palais-Smale condition $(PS)_c^\Gamma$.

The next theorem, proved by Guo-Niu [8], describes the τ-equivariant Palais-Smale sequence for $E_{\lambda,\mu,f}$.

Theorem 3.7. *Let (u_n) be a Palais-Smale in $H_0^1(\Omega)^\tau$, for $E_{\lambda,\mu,f}$ at $c\ge 0$. Then there exist a solution u of (2.10), $m,l\in\mathbb{N}$; a closed subgroup G^i of finite index in G, sequences $\{y_n^i\}\subset\Omega$, $\{r_n^i\}\subset(0,+\infty)$; a solution $\widehat{u}_0^i$ of (2.2), for $i=1,\dots,m$; and $\{R_n^j\}\subset\mathbb{R}^+$, a solution $\widehat{u}_\mu^j$ of (2.4) for $j=1,\dots,l$. Such that*

(i) $G_{y_n^i} = G^i$

(ii) $(r_n^i)^{-1}dist(y_n^i,\partial\Omega)\to\infty$, $y_n^i\to y^i$, *if $n\to\infty$, for $i=1,\dots,m$.*

(iii) $(r_n^i)^{-1}|gy_n^i - g'y_n^i|\to\infty$, *if $n\to\infty$, and $[g]\ne[g']\in G/G^i$, for $i=1,\dots,m$,*

(iv) $\widehat{u}_0^i(gx)=\tau(g)\widehat{u}_0^i(x)$ $\forall z\in\mathbb{R}^N$ *and* $g\in G^i$,

(v) $\widehat{u}_\mu^j(gx)=\tau(g)\widehat{u}_\mu^j(x)$ $\forall z\in\mathbb{R}^N$ *and* $g\in G$, $R_n^j\to 0$ *for* $j=1,\dots,l$

(vi)
$$\begin{aligned} u_n(x) &= u(x) + \sum_{i=1}^m \sum_{[g]\in G/G^i} (r_n^i)^{\frac{2-N}{2}} f(y^i)^{\frac{2-N}{4}}\tau(g)\widehat{u}_0^i\big(g^{-1}\big(\frac{x-gy_n^i}{r_n^i}\big)\big) \\ &\quad + \sum_{j=1}^l (R_n^j)^{\frac{2-N}{2}}\widehat{u}_\mu^i\big(\frac{x}{R_n^j}\big) + o(1), \end{aligned}$$

(vii) $E_{\lambda,\mu,f}(u_n)\to E_{\lambda,\mu,f}(u) + \sum_{i=1}^m\big(\frac{\#(G/G^i)}{f(y^i)^{\frac{N-2}{2}}}\big)E_{0,0,1}^\infty(\widehat{u}_0^i) + \sum_{j=1}^l E_{0,\mu,1}^\infty(\widehat{u}_\mu^j)$, *as $n\to\infty$*

Corollary 3.8. *$E_{\lambda,\mu,f}$ satisfies $(PS)_c^\tau$ at every*
$$c < \min\big\{\#(G/\Gamma)\frac{\ell_f^\Gamma}{N}, \frac{\#(G/\Gamma)}{N}S_\mu^{N/2}\big\}.$$

4. The bariorbit map

We will assume the nonexistence condition

(NE) The infimum of $E_{0,0,f}$ is not achieved in $\mathcal{N}_{0,0,f}^\Gamma$.

Corollary 3.8 and Lemma 3.5 imply
$$m^\Gamma(0,0,f) := \inf_{\mathcal{N}_{0,0,f}^\Gamma} E_{0,0,f} = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{1}{N}S^{N/2}. \tag{4.1}$$
if (NE) is assumed. It is well known that (NE) holds, if $\Gamma=\{1\}$ and f is constant (see [14, Cap. III, Teorema 1.2]). Set
$$M := \big\{y\in\overline{\Omega} : \frac{\#\Gamma y}{f(y)^{(N-2)/2}} = \min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\big\}.$$
For every $y\in\mathbb{R}^N$, $\gamma\in\Gamma$, the isotropy subgroups satisfy $\Gamma_{\gamma y}=\gamma\Gamma_y\gamma^{-1}$. Therefore the set of isotropy subgroups of Γ-invariant subsets consists of complete conjugacy classes. We choose $\Gamma_i\subset\Gamma$, $i=1,\dots,m$, one in each conjugacy class of an isotropy subgroup of M. Set
$$V^i := \big\{z\in V : \gamma z = z \quad \forall\gamma\in\Gamma_i\big\}$$
the fixed point space of $V\subset\mathbb{R}^N$ under the action of Γ_i. Set
$$M^i := \{y\in M : \Gamma_y = \Gamma_i\},$$
$$\Gamma M^i := \{\gamma y : \gamma\in\Gamma,\ y\in M^i\} = \{y\in M : (\Gamma_y) = (\Gamma_i)\}.$$
By definition of M it follows that f is constant on each ΓM^i. Set
$$f_i := f(\Gamma M^i)\in\mathbb{R}.$$
Fix $\delta_0>0$ such that
$$\begin{gathered}|y-\gamma y|\ge 3\delta_0 \quad \forall y\in M,\ \gamma\in\Gamma \text{ if } \gamma y\ne y,\\ \operatorname{dist}(\Gamma M^i,\Gamma M^j)\ge 3\delta_0 \quad \forall i,j=1,\dots,m \text{ if } i\ne j,\end{gathered} \tag{4.2}$$
and such that the isotropy subgroup of each point in $M_{\delta_0}^i := \{z\in V^i : \operatorname{dist}(z,M^i)\le \delta_0\}$ is precisely Γ_i. Define
$$W_{\varepsilon,z} := \sum_{[g]\in\Gamma/\Gamma_i} f_i^{\frac{2-N}{4}} U_{\varepsilon,gz} \quad \text{if } z\in M_{\delta_0}^i,$$
where $U_{\varepsilon,y} := U_0^{\varepsilon,y}$ as in (2.3). For each $\delta\in(0,\delta_0)$ define
$$M_\delta := M_\delta^1\cup\dots\cup M_\delta^m,$$
$$B_\delta := \{(\varepsilon,z) : \varepsilon\in(0,\delta),\ z\in M_\delta\},$$
$$\Theta_\delta := \{\pm W_{\varepsilon,z} : (\varepsilon,z)\in B_\delta\}, \quad \Theta_0 := \Theta_{\delta_0}.$$
For the proof of next proposition see [3].

Proposition 4.1. *Let $\delta \in (0,\delta_0)$, and assume that (NE) holds. There exists $\eta > m^\Gamma(0,0,f)$ with following properties: For each $u \in \mathcal{N}^\Gamma_{0,0,f}$ such that $E_{0,0,f}(u) \le \eta$ we have*

$$\inf_{W\in\Theta_0} \|u-W\| < \sqrt{\frac{1}{2}Nm^\Gamma(0,0,f)},$$

and there exist precisely one $\nu \in \{-1,1\}$, one $\varepsilon \in (0,\delta_0)$ and one Γ-orbit $\Gamma z \in M_{\delta_0}$ such that

$$\|u-\nu W_{\varepsilon,z}\| = \inf_{W\in\Theta_0}\|u-W\|.$$

Moreover $(\varepsilon,z) \in B_\delta$.

4.1. **Definition of the bariorbit map.** Fix $\delta \in (0,\delta_0)$ and choose $\eta > m^\Gamma(0,0,f)$ as in Proposition 4.1. Define

$$E^\eta_{0,0,f} := \{u \in H^1_0(\Omega) : E_{0,0,f}(u) \le \eta\},$$
$$B_\delta(M) := \{z \in \mathbb{R}^N : \operatorname{dist}(z,M) \le \delta\},$$

and the space of Γ-orbits of $B_\delta(M)$ by $B_\delta(M)/\Gamma$.

From Proposition 4.1 we can define

Definition 4.2. The bariorbit map

$$\beta^\Gamma : \mathcal{N}^\Gamma_{0,0,f} \cap E^\eta_{0,0,f} \to B_\delta(M)/\Gamma,$$

is defined by

$$\beta^\Gamma(u) = \Gamma y \overset{def}{\Longleftrightarrow} \|u \pm W_{\varepsilon,y}\| = \min_{W\in\Theta_0}\|u-W\|.$$

This map is continuous and $\mathbb{Z}/2$-invariant by the compactness of M_δ.

If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, choose $g_\tau \in \tau^{-1}(-1)$. Let $u \in \mathcal{N}^\tau_{0,0,f}$ then u changes sign and $u^-(x) = -u^+(g_\tau^{-1}x)$. Therefore, $\|u^+\|^2 = \|u^-\|^2$ and $|u^+|^{2^*}_{f,2^*} = |u^-|^{2^*}_{f,2^*}$. So

$$u \in \mathcal{N}^\tau_{0,0,f} \Longrightarrow u^\pm \in \mathcal{N}^\Gamma_{0,0,f} \quad \text{and} \quad E_{0,0,f}(u) = 2E_{0,0,f}(u^\pm). \tag{4.3}$$

Lemma 4.3. *If $E_{0,0,f}$ does not achieve its infimum at $\mathcal{N}^\tau_{0,0,f}$, then*

$$m^\tau(0,0,f) := \inf_{\mathcal{N}^\tau_{0,0,f}} E_{0,0,f} = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{2}{N}S^{N/2} = 2m^\Gamma(0,0,f).$$

Proof. By contradiction. Suppose that there exists $u \in \mathcal{N}^\tau_{0,0,f}$ such that $E_{0,0,f}(u) = m^\tau(0,0,f)$. Then $u^+ \in \mathcal{N}^\Gamma_{0,0,f}$ and

$$m^\tau(0,0,f) \le \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{2}{N}S^{N/2}.$$

Hence

$$m^\Gamma(0,0,f) \le E_{0,0,f}(u^+) = \frac{1}{2}m^\tau(0,0,f) \le \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}}\Big)\frac{1}{N}S^{N/2} = m^\Gamma(0,0,f).$$

Thus u^+ is a minimum of $E_{0,0,f}$ on $\mathcal{N}^\Gamma_{0,0,f}$, which contradicts (NE). The corollary 3.8 implies

$$m^\tau(0,0,f) = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{2}{N}S^{N/2}.$$

□

Then property (4.3) implies

$$u^{\pm} \in \mathcal{N}^{\Gamma}_{0,0,f} \cap E^{\eta}_{0,0,f} \quad \forall u \in \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f},$$

so

$$\|u^+ - \nu W_{\varepsilon,y}\| = \min_{W \in \Theta_0} \|u^+ - W\| \Leftrightarrow \|u^- + \nu W_{\varepsilon,g_\tau y}\| = \min_{W \in \Theta_0} \|u^- - W\|. \tag{4.4}$$

Therefore,

$$\beta^{\Gamma}(u^+) = \Gamma y \Longleftrightarrow \beta^{\Gamma}(u^-) = \Gamma(g_\tau y), \tag{4.5}$$

and

$$\beta^{\Gamma}(u^+) \neq \beta^{\Gamma}(u^-) \quad \forall u \in \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f}. \tag{4.6}$$

Set

$$B_\delta(M)^\tau := \{z \in B_\delta(M) : Gz = \Gamma z\}.$$

Proposition 4.4. *The map*

$$\beta^\tau : \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f} \to (B_\delta(M) \setminus B_\delta(M)^\tau)/\Gamma, \quad \beta^\tau(u) := \beta^{\Gamma}(u^+),$$

is well defined, continuous and $\mathbb{Z}/2$-equivariant; i.e.,

$$\beta^\tau(-u) = \Gamma(g_\tau y) \Longleftrightarrow \beta^\tau(u) = \Gamma y.$$

Proof. If $u \in \mathcal{N}^{\tau}_{0,0,f} \cap E^{2\eta}_{0,0,f}$ and $\beta^\tau(u) = \Gamma y \in B_\delta(M)^\tau/\Gamma$ then $\beta^{\Gamma}(u^+) = \Gamma y = \Gamma(g_\tau y) = \beta^{\Gamma}(u^-)$, this is a contradiction to (4.6). We conclude that $\beta^\tau(u) \notin B_\delta(M)^\tau/\Gamma$. The continuity and $\mathbb{Z}/2$-equivariant properties follows by β^{Γ} ones. ☐

5. Multiplicity of solutions

5.1. **Lusternik-Schnirelmann theory.** An involution on a topological space X is a map $\varrho_X : X \to X$, such that $\varrho_X \circ \varrho_X = id_X$. Given an involution we can define an action of $\mathbb{Z}/2$ on X and viceversa. The trivial action is given by the identity $\varrho_X = id_X$, the action of $G/\Gamma \simeq \mathbb{Z}/2$ on the orbit space $\mathbb{R}^N/\Gamma$ where $G \subset O(N)$ and Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, and the antipodal action $\varrho(u) = -u$ on $\mathcal{N}^{\tau}_{\lambda,\mu,f}$. A map $f : X \to Y$ is called $\mathbb{Z}/2$-equivariant (or a $\mathbb{Z}/2$-map) if $\varrho_Y \circ f = f \circ \varrho_X$, and two $\mathbb{Z}/2$-maps, $f_0, f_1 : X \to Y$, are said to be $\mathbb{Z}/2$-homotopic if there exists a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\varrho_X x, t) = \varrho_Y \Theta(x,t)$ for every $x \in X$, $t \in [0,1]$. A subset A of X is $\mathbb{Z}/2$-equivariant if $\varrho_X a \in A$ for every $a \in A$.

Definition 5.1. The $\mathbb{Z}/2$-category of a $\mathbb{Z}/2$-map $f : X \to Y$ is the smallest integer $k := \mathbb{Z}/2\text{-cat}(f)$ with following properties

(i) There exists a cover of $X = X_1 \cup \cdots \cup X_k$ by k open $\mathbb{Z}/2$-invariant subsets,
(ii) The restriction $f|_{X_i} : X_i \to Y$ is $\mathbb{Z}/2$-homotopic to the composition $\kappa_i \circ \alpha_i$ of a $\mathbb{Z}/2$-map $\alpha_i : X_i \to \{y_i, \varrho_Y y_i\}$, $y_i \in Y$, and the inclusion $\kappa_i : \{y_i, \varrho_Y y_i\} \hookrightarrow Y$.

If not such covering exists, we define $\mathbb{Z}/2\text{-cat}(f) := \infty$.

If A is a $\mathbb{Z}/2$-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion we write

$$\mathbb{Z}/2\text{-}cat_X(A) := \mathbb{Z}/2\text{- cat}(\iota), \quad \mathbb{Z}/2\text{-}cat_X(X) := \mathbb{Z}/2\text{- cat}(X).$$

Note that if $\varrho_x = id_X$ then

$$\mathbb{Z}/2\text{-}cat_X(A) := cat_X(A), \quad \mathbb{Z}/2\text{- cat}(X) := \text{cat}(X),$$

are the usual Lusternik-Schnirelmann category (see [17, definition 5.4]).

Theorem 5.2. *Let $\phi : M \to \mathbb{R}$ be an even functional of class C^1, and M a submanifold of a Hilbert space of class C^2, symmetric with respect to the origin. If ϕ is bounded below and satisfies $(PS)_c$ for each $c \le d$, then ϕ has at least $\mathbb{Z}/2$-*$\operatorname{cat}(\phi^d)$ *pairs critical points such that $\phi(u) \le d$.*

5.2. **Proof of Theorems.** We prove Theorem 2.3 only; the proof of Theorem 2.1 is analogous. Recall that if τ is the identity or an epimorphism then $\#(G/\Gamma)$ is 1 or 2.

Proof of Theorem 2.3. By Corollary 3.8, $E_{\lambda,\mu,f}$ satisfies $(PS)^\tau_\theta$ for

$$\theta < \min\{\#(G/\Gamma)\frac{\ell_f^\Gamma}{N}, \frac{\#(G/\Gamma)}{N} S_\mu^{N/2}\}.$$

By Lusternik-Schnirelmann theory $E_{\lambda,\mu,f}$ has at least $\mathbb{Z}/2$-$\operatorname{cat}(\mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f})$ pairs $\pm u$ of critical points in $\mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}$. We are going to estimate this category for an appropriate value of θ.

Without lost of generality we can assume that $\delta \in (0, \delta_0)$, with δ_0 as in (4.2). Let $\eta > \frac{\ell_f^\Gamma}{N}$, $\mu^* \in (0, \overline{\mu})$ and $\lambda^* \in (0, \lambda_1)$ such that

$$(\frac{\bar\mu}{\bar\mu - \mu^*})^{N/2} (\frac{\lambda_1}{\lambda_1 - \lambda^*})^{N/2} = \min\{2, \frac{N\eta}{\#(G/\Gamma)\ell_f^\Gamma}, \frac{\ell_f^\Gamma}{\ell_f^\Gamma - \delta'}\}.$$

By Lemma 3.3, if $u \in \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}$, $\mu \in (0, \mu^*)$, $\lambda \in (0, \lambda^*)$ we have

$$\begin{aligned} E_{0,0,f}(\pi_{0,0,f}(u)) &\le (\frac{\bar\mu}{\bar\mu - \mu})^{\frac{N}{2}} (\frac{\lambda_1}{\lambda_1 - \lambda})^{\frac{N}{2}} E_{\lambda,\mu,f}(u) \\ &< (\frac{\bar\mu}{\bar\mu - \mu})^{\frac{N}{2}} (\frac{\lambda_1}{\lambda_1 - \lambda})^{\frac{N}{2}} \#(G/\Gamma)\frac{\ell_f^\Gamma}{N} \\ &\le \#(G/\Gamma)\eta. \end{aligned}$$

Let β^τ be the τ-bariorbit function, defined in Proposition 4.4. Hence the composition map

$$\beta^\tau \circ \pi_{0,0,f} : \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f} \to (B_\delta(M) \setminus B_\delta(M)^\tau)/\Gamma,$$

is a well defined $\mathbb{Z}/2$-invariant continuous function.

By the [3, Proposition 3] using (F2) we can choose $\varepsilon > 0$ small enough and $\theta := \theta_\varepsilon < \#(G/\Gamma)\frac{\ell_f^\Gamma}{N}$ such that

$$E_{\lambda,\mu,f}(\pi_{\lambda,\mu,f}(w^\tau_{\varepsilon,y})) \le \theta < \#(G/\Gamma)\frac{\ell_f^\Gamma}{N}, \quad \forall\, y \in M_\delta^-,$$

where $w^\tau_{\varepsilon,y} = w^\Gamma_{\varepsilon,y} - w^\Gamma_{\varepsilon,g_\tau y}$, $\tau(g_\tau) = -1$, and

$$w^\Gamma_{\varepsilon,y}(x) = \sum_{[\gamma]\in\Gamma/\Gamma_y} f(y)^{(2-N)/4} U_{\varepsilon,\gamma y}(x)\varphi_{\gamma y}(x).$$

Thus the map

$$\begin{gathered} \alpha^\tau_\delta : M^-_{\tau,\delta}/\Gamma \to \mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}, \\ \alpha^\tau_\delta(\Gamma y) := \pi_{\lambda,\mu,f}(w^\tau_{\varepsilon,y}), \end{gathered}$$

is a well defined $\mathbb{Z}/2$-invariant continuous function. Moreover, $\beta^\tau(\pi_{0,0,f}(\alpha_\delta^\tau(\Gamma y))) = \Gamma y$ for all $y \in M^-_{\tau,\delta}$. Therefore,

$$\mathbb{Z}/2\text{-}\operatorname{cat}(\mathcal{N}^\tau_{\lambda,\mu,f} \cap E^\theta_{\lambda,\mu,f}) \geq \operatorname{cat}_{((B_\delta(M)\setminus B_\delta(M)^\tau)/\Gamma)}(M^-_{\tau,\delta}/\Gamma).$$

So (2.10) has at least

$$\operatorname{cat}_{((B_\delta(M)\setminus B_\delta(M)^\tau)/G)}(M^-_{\tau,\delta}/G)$$

pairs $\pm u$ solution which satisfy

$$E_{\lambda,\mu,f}(u) < \#(G/\Gamma)\frac{\ell_f^\Gamma}{N}.$$

By the choice of λ^* and μ^* we have

$$(\frac{\bar\mu}{\bar\mu - \mu^*})^{N/2}(\frac{\lambda_1}{\lambda_1 - \lambda^*})^{N/2} \leq \frac{\ell_f^\Gamma}{\ell_f^\Gamma - \delta'}.$$

Then

$$\begin{aligned}
\#(G/\Gamma)\frac{\ell_f^\Gamma - \delta'}{N} &\leq (\frac{\bar\mu - \mu}{\bar\mu})^{N/2}(\frac{\lambda_1 - \lambda}{\lambda_1})^{N/2}\#(G/\Gamma)\frac{\ell_f^\Gamma}{N} \\
&\leq m^\tau(\lambda,\mu,f) \leq E_{\lambda,\mu,f}(u) \\
&= \frac{1}{N}\|u\|^2_{\lambda,\mu} < \#(G/\Gamma)\frac{\ell_f^\Gamma}{N}
\end{aligned}$$

therefore

$$\#(G/\Gamma)\ell_f^\Gamma - \delta'' \leq \|u\|^2_{\lambda,\mu} < \#(G/\Gamma)\ell_f^\Gamma.$$

□

Proof of Theorem 2.4. By Theorem 2.1 there exist λ and μ sufficiently close to zero such that the problem (2.1) has at least $\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$ positive solutions such that $E_{\lambda,\mu,f}(u) < \frac{\ell_f^\Gamma}{N}$.

We will prove that $\frac{\ell_f^\Gamma}{N} < m^{\widetilde\Gamma}(0,0,f)$. First suppose that $m^{\widetilde\Gamma}(0,0,f)$ does not achieve then by the hypothesis $m^{\widetilde\Gamma}(0,0,f) = \frac{\ell_f^{\widetilde\Gamma}}{N} > \frac{\ell_f^\Gamma}{N}$. If $m^{\widetilde\Gamma}(0,0,f)$ is achieved there exists $u \in \mathcal{N}^{\widetilde\Gamma}_{0,0,f} \subset \mathcal{N}^\Gamma_{0,0,f}$ and

$$\frac{\ell_f^\Gamma}{N} = m^\Gamma(0,0,f) < m^{\widetilde\Gamma}(0,0,f) = E_{0,0,f}(u).$$

By (3.4) there exist $\widehat\lambda \in (0,\lambda_1)$ and $\widehat\mu \in (0,\bar\mu)$ such that for each $\lambda \in (0,\widehat\lambda)$ and $\mu \in (0,\widehat\mu)$ such that

$$\frac{\ell_f^\Gamma}{N} < m^{\tilde\Gamma}(0,0,f) \leq (\frac{\lambda_1}{\lambda_1 - \lambda})^{N/2}(\frac{\overline{\mu}}{\overline{\mu} - \mu})^{N/2} m^{\tilde\Gamma}(\lambda,\mu,f).$$

Then

$$E_{\lambda,\mu,f}(u) < \frac{\ell_f^\Gamma}{N} < m^{\tilde\Gamma}(\lambda,\mu,f).$$

Therefore, u is not $\tilde\Gamma$-invariant solution. □

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