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ON THE PENALIZED OBSTACLE PROBLEM IN THE UNIT HALF BALL

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Abstract. We study the penalized obstacle problem in the unit half ball, i.e. an approximation of the obstacle problem in the unit half ball. The main result states that when the approximation parameter is small enough and when certain level sets are sufficiently close to the hyperplane $\{x_1 = 0\}$, then these level sets are uniformly C^1 regular graphs. As a by-product, we also recover some regularity of the free boundary for the limiting problem, i.e., for the obstacle problem.

1. Introduction

1.1. **Problem.** Given some non-negative boundary data $g \in L^\infty(B_1^+) \cap H^1(B_1^+)$ we study the penalized obstacle problem in the unit half ball; that is

$$\begin{gathered}
\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \quad \text{in } B_1^+ = B_1 \cap \{x_1 > 0\}, \\
u_\varepsilon = g \quad \text{on } \partial B_1 \cap \{x_1 > 0\}, \\
u_\varepsilon = 0 \quad \text{on } B_1 \cap \{x_1 = 0\}.
\end{gathered}$$

Here we assume that β_ε is any function satisfying the same assumptions as in [5] and [6], see section 1.3 for details. This would then imply by the maximum principle that $u_\varepsilon \geq 0$. We will refer to $B_1 \cap \{x_1 = 0\}$ as the fixed boundary.

1.2. **Known result.** The penalized obstacle problem has been well studied, see [4], [2], [5] and [6]. Brezis and Kinderlehrer proved the uniform local $C^{1,1}$ regularity. Redondo proved, using the result of Kinderlehrer and Stampacchia, that the level sets of u_ε are locally uniform $C^{1,\alpha}$-graphs in some direction. However, the behaviour of the level sets close to the fixed boundary is so far unknown for the penalized obstacle problem. The present paper contributes in this direction.

In the case of the obstacle problem (the limiting case when $\varepsilon \to 0$), the behaviour of the free boundary has been studied in [7]. There the authors prove that, close to the fixed boundary, the free boundary is a C^1-graph in directions close to e_1. In fact, they prove this in a more general setting, which includes the obstacle problem as a special case.

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1.3. **Main result.** Before mentioning the main result of this paper we must carefully define which class of solutions we will work with.

Definition 1.1. We say $\beta_\varepsilon \in \mathcal{C}_\varepsilon$ if

(1) For any $\varepsilon > 0$, β_ε is uniformly Lipschitz on $\mathbb{R}$,
(2) $\beta_\varepsilon(t) \le 0$ for all $t \le 0$ and all $\varepsilon > 0$,
(3) $\beta_\varepsilon(0) = 0$ and $\beta_\varepsilon(t) \le 1$,
(4) $\beta''_\varepsilon(t) \le 0$,
(5) There holds
$$0 \le \frac{1-\beta_\varepsilon(t)}{\varepsilon} \le \beta'_\varepsilon(t) \le \frac{1}{\varepsilon},$$
whenever $t \ge 0$.

One can deduce that $\Psi_\varepsilon \le \beta_\varepsilon \le \Phi_\varepsilon$, where
$$\Psi_\varepsilon(t) = \begin{cases} t/\varepsilon & t \le 0, \\ 1 - e^{-t/\varepsilon} & t > 0, \end{cases}$$
and
$$\Phi_\varepsilon(t) = \begin{cases} t/\varepsilon & t \le \varepsilon, \\ 1 & t > \varepsilon. \end{cases}$$
In addition, one can see that if $\beta_\varepsilon(\cdot) \in \mathcal{C}_\varepsilon$ then $\beta_\varepsilon(s\cdot) \in \mathcal{C}_{\varepsilon/s}$. This scaling property will be used frequently further on. In what follows we use the notation
$$B_r^+(x) = B_r(x) \cap \{x_1 > 0\}.$$
We define the local class of solutions as the following:

Definition 1.2. We say $u_\varepsilon \in P_r(M, \varepsilon)$ if

(1) $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ in B_r^+ where $\beta_\varepsilon \in \mathcal{C}_\varepsilon$.
(2) $u_\varepsilon \ge 0$ on $\partial B_r \cap \{x_1 > 0\}$.
(3) $u_\varepsilon = 0$ on $B_r \cap \{x_1 = 0\}$,
(4) $\sup_{B_r^+} u_\varepsilon \le M$.

The main result of this paper is the $C^{1,1}$-estimates and the uniform (in ε) C^1 regularity of the ε^γ-level sets for any $0 < \gamma < 1$, as stated below in Theorem 1 and 2. A simple consequence of Theorem 1.4 is Corollary 1.5.

Theorem 1.3. *Let $u_\varepsilon \in P_1(M, \varepsilon)$. Then there is a constant $C = C(n, M)$ such that*
$$\|u_\varepsilon\|_{C^{1,1}(B_{1/2}^+)} \le C.$$

Theorem 1.4. *Let $u_\varepsilon \in P_1(M, \varepsilon)$. Assume moreover that $y \in \partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_{1/2}^+$ for some $0 < \gamma < 1$. Then there are constants $\varepsilon_0, \rho > 0$ depending on M, γ and the dimension such that*
$$\max(\varepsilon, y_1) < \varepsilon_0$$
implies that $B_\rho^+(y) \cap \partial\{u_\varepsilon > \varepsilon^\gamma\}$ is a C^1-graph with the C^1-norm bounded independently of ε and y_1. Moreover, the normal of $\partial\{u_\varepsilon > \varepsilon^\gamma\}$ at y converges to e_1 as $y_1 \to 0$.

Sending $\varepsilon \to 0$ we obtain the following result for the obstacle problem, which is a slightly stronger result than the one in [7] since it only requires the free boundary to be near the fixed one.

Corollary 1.5. *Let $u_0 \in P_1(M, 0)$, that is u_0 is a solution to the obstacle problem, and suppose $y \in \partial\{u_0 > 0\}$. Then there is a constant ε_0 depending on M such that $y_1 < \varepsilon_0$ implies that $B_\rho^+(y) \cap \partial\{u_0 > 0\}$ is a C^1-graph with the C^1-norm bounded independently y_1.*

2. Non-degeneracy

Here we state the result that the solutions do not grow too slow. This is important when considering sequences as we will do later, since it implies the stability of the level sets under uniform convergence. For the sake of completeness, we give the standard proof.

Proposition 2.1. *Let $u_\varepsilon \in P_1(M, \varepsilon)$ and assume $u_\varepsilon(y) \geq \varepsilon$. Then there is $C = C(n) > 0$ such that*

$$\sup_{B_r(y)\cap B_1^+} (u_\varepsilon - u_\varepsilon(y)) \geq Cr^2$$

for $r < \operatorname{dist}(y, \partial B_1^+)$.

Proof. Take $y \in \{u_\varepsilon = \delta\}$ for some $\delta \geq \varepsilon$ and let

$$v = u_\varepsilon - \delta - \frac{|y - z|^2}{8n}.$$

Then $\Delta v \geq \beta_\varepsilon(u_\varepsilon) - 1/4 \geq 0$ in $\{u_\varepsilon > \delta/2\} \cap B_r^+(y)$ for any r such that $B_r^+(y) \subset B_1^+$. Therefore, v attains its maximum on the boundary of $\{u_\varepsilon > \delta/2\} \cap B_r^+(x_0)$. Furthermore, $v(y) = 0$, $v \leq 0$ on $\{u_\varepsilon = \delta\}$ and $v \leq 0$ on $\{x_1 = 0\}$. Hence v attains its positive maximum at some $z \in \partial B_r(y) \cap \{x_1 > 0\}$. This yields the desired result. □

3. $C^{1,1}$-estimates in $B_{1/2}^+$

In this section we prove that the functions u_ε are uniformly bounded in the space $C^{1,1}(B_{1/2}^+)$. The first step is to show that we have quadratic growth up to the fixed boundary.

Lemma 3.1. *Let $u_\varepsilon \in P_1(M, \varepsilon)$. Then there are constants $C, r_0 > 0$ depending on M and the dimension such that*

$$\sup_{B_r^+} |u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)| \leq Cr^2$$

for $r < r_0$.

Proof. We prove that for any r we have either

$$\sup_{B_r^+} |u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)| \leq Cr^2,$$

or there is a $k \in \mathbb{N}$ such that

$$\sup_{B_r^+} |u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)| \leq 4^{-k} \sup_{B_{2^k r}^+} |u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)|.$$

Assume that this statement fails, then there are sequences of $\varepsilon_j, r_j \to 0$ and $u_j = u_{\varepsilon_j}$ such that

$$S_j \geq Cjr_j^2$$

and

$$S_j \geq 4^{-k} S_{2^k r_j},$$

for all $k \in \mathbb{N}$. Here
$$S_j = \sup_{B^+_{r_j}} |u_j(x) - x \cdot \nabla u_j(0)|.$$
Let
$$v_j(x) = \frac{u_j(r_j x) - r_j x \cdot \nabla u_j(0)}{S_j}.$$
Then

(1) $v_j(0) = |\nabla v_j(0)| = 0$,
(2) $\sup_{B^+_1} |v_j| = 1$,
(3) $\sup_{B^+_{2^k}} |v_j| \le 4^k$ for all $k \in \mathbb{N}$,
(4) $\sup_{B^+_R} |\partial_e v_j| \le C/j$ for any $e \cdot e_1 = 0$ and any $R > 0$, since by the maximum principle $|\partial_e u_j(x)| \le C|x|$,
(5) $|\Delta v_j| = C/j$ in B^+_{1/r_j}.

Therefore, there is a subsequence, again labelled v_j, converging in $C^{1,\alpha}(\overline{B^+_R})$ for any R to a function v_0 satisfying

(1) $v_0(0) = |\nabla v_0(0)| = 0$,
(2) $\sup_{B^+_1} |v_0| = 1$,
(3) $\sup_{B^+_{2^k}} |v_0| \le 4^k$ for all $k \in \mathbb{N}$,
(4) $D_e v_0 = 0$ for any $e \cdot e_1 = 0$ in $\mathbb{R}^n_+$,
(5) $\Delta v_0 = 0$ in $\mathbb{R}^n_+$.

Clearly, (4) implies that v_0 is one-dimensional. Then (1) and (5) implies that $v_0 = 0$ which contradicts (2). □

This estimate implies, as we will now see, that u_ε is uniformly $C^{1,1}$. The proof is standard.

Proof of Theorem 1.3. Since interior estimates are already known (see Theorem 2 in [2] even though this is stated in a slightly different form, this can also be proved with methods similar to the ones in the proof of Lemma 3.1), it suffices to obtain estimates near the fixed boundary.

Take $y \in B^+_{1/2} \cap \{x_1 < r_0/2\}$ where r_0 is as in Lemma 3.1. Consider the function
$$v(x) = \frac{u_\varepsilon(|y|x + y) - (|y|x + y) \cdot \nabla u_\varepsilon(0)}{|y|^2}.$$
Then, by Lemma 3.1
$$\begin{aligned}\sup_{x \in B_1} |v(x)| &= \sup_{x \in B_{|y|}} \left| \frac{u_\varepsilon(x + y) - (x + y) \cdot \nabla u_\varepsilon(0)}{|y|^2} \right| \\ &\le \sup_{x \in B^+_{2|y|}} \left| \frac{u_\varepsilon(x) - x \cdot \nabla u_\varepsilon(0)}{|y|^2} \right| \le C.\end{aligned}$$
Moreover,
$$|\Delta v(x)| = |\beta_\varepsilon(y^2 v + (|y|x + y) \cdot \nabla u_\varepsilon(0))| \le 1.$$
Therefore $\|u\|_{C^{1,\alpha}(B_{1/2})} \le C$ by elliptic estimates. In particular, for any $x \in B_{\frac{1}{2}}$ there holds
$$C \ge |\nabla v(x)| = \left| \frac{\nabla u_\varepsilon(y) - \nabla u_\varepsilon(0)}{|y|} \right|,$$

and

$$C \geq |\nabla v(x)| = \left|\frac{\nabla u_\varepsilon(|y|x + y) - \nabla u_\varepsilon(0)}{|y|}\right|.$$

Combining the last to estimates gives

$$|\nabla u_\varepsilon(|y|x + y) - \nabla u_\varepsilon(y)| \leq C|y| \leq C|yx|.$$

Hence $D^2 u_\varepsilon(y)$ is bounded independently of y. □

4. Classification of certain global solutions

For the proofs in this section we will need the following theorem, which is proved in [1, Lemma 5.1] (even though slightly different form) and in [3, Theorem 9].

Theorem 4.1. *Let u and v be two subharmonic functions such that $u \cdot v = 0$. Then with*

$$\phi(r) = \frac{1}{r^4}\int_{B_r}\frac{|\nabla u|^2}{|x|^{n-2}}\,dx \int_{B_r}\frac{|\nabla v|^2}{|x|^{n-2}}\,dx$$

we have that ϕ is monotonically increasing. Moreover, if ϕ vanishes for all r then one of the functions vanishes.

In the forthcoming sections, we will need the following characterization of global solutions.

Proposition 4.2. *Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a function with the following properties:*

(1) $u \geq 0$,
(2) $e_1 \in \partial\{u > t\}$ *for some* $t \geq 0$,
(3) $u = 0$ *on* $x_1 = 0$,
(4) $\sup_{B_r+} u \leq Cr^2$ *for* $r > 1$,
(5) $\Delta u = \beta_k(u)$ *for some* $k \geq 0$, *with* $\beta_0(t) = \chi_{\{t>0\}}$ *and* $\beta_k \in \mathcal{C}_k$.

Then u is one-dimensional and monotone.

Proof. First we extend u to $x_1 < 0$ by odd reflection, that will give us

$$\Delta u = \beta_k(u)\chi_{\{x_1>0\}} - \beta_k(-u)\chi_{\{x_1<0\}}.$$

Then we observe that

$$\Delta(\partial_e u) = (\beta_k'(u)\chi_{\{x_1>0\}} + \beta_k'(-u)\chi_{\{x_1<0\}})\partial_e u + \beta_k(u)\partial_e(\chi_{\{x_1>0\}}) - \beta_k(-u)\partial_e(\chi_{\{x_1<0\}}).$$

Since $\beta(0) = 0$ and $\nabla\chi_{\{x_1>0\}}$ is a finite measure the last two terms vanish. This implies that $(\partial_e u)^\pm$ are both subharmonic and therefore Theorem 4.1 applies. Let

$$u_R(x) = \frac{u(Rx)}{R^2}.$$

Then

(1) $\Delta u_R = \beta_k(R^2 u_R)\chi_{\{x_1>0\}} - \beta_k(-R^2 u_R)\chi_{\{x_1<0\}}$ where $\beta_k(R^2 t) \in \mathcal{C}_{k/R^2}$,
(2) $|u_R(x)| \leq C|x|^2$ for $|x| > 1/R$,
(3) $e_1/R \in \partial\{u_R > t/R^2\}$,
(4) $-e_1/R \in \partial\{u_R < -t/R^2\}$,
(5) $|\nabla u_R(0)| \leq C/R$.

We see that Δu_R will be uniformly bounded from above and from below. Moreover, $\sup_{B_1}|u_R|$ is uniformly bounded. Hence, there is by standard elliptic estimates, a subsequence $u_j = u_{R_j}$ with $R_j \to \infty$, converging in $W^{2,p}_{\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}}$. Invoking Proposition 2.1 we can conclude that $u_j \to u_\infty$ where u_∞ satisfies

(1) $\Delta u_\infty = \chi_{\{u_\infty>0\}}\chi_{\{x_1>0\}} - \chi_{\{u_\infty<0\}}\chi_{\{x_1<0\}} = \chi_{\{u_\infty>0\}} - \chi_{\{u_\infty<0\}}$,
(2) $|u_\infty(x)| \le C|x|^2$ for all x,
(3) $0 \in \partial\{u_\infty > 0\} \cap \partial\{u_\infty < 0\} \cap \{|\nabla u_\infty| = 0\}$.

By applying the classification of global solutions in [8], we obtain that up to rotations

$$u_\infty = \frac{(x_1^+)^2}{2} - \frac{(x_1^-)^2}{2}.$$

Again, Theorem 4.1 applies to $\partial_e u_j^\pm$ for any direction e. Thus,

$$0 = \phi(\partial_e u_\infty, 1) = \lim \phi(\partial_e u_j, 1) = \lim \phi(\partial_e u, R_j) = \phi(\partial_e u, \infty).$$

Since ϕ is non-decreasing this implies that $\phi(\partial_e u, r) = 0$ for all r. Hence, either $\partial_e u \ge 0$ or $\partial_e u \le 0$, for any direction e. This implies that u must be one-dimensional. □

Remark 4.3. It seems plausible that a proof of this can be given without the use of the ACF monotonicity formula. However, such a proof would be much longer and more technically involved.

5. The C^1 regularity of certain level sets close to the fixed boundary

In this section we will prove that if the ε^γ-level set is close to the fixed boundary then, due to the results in [5] and [6], the ε^γ-level set is a uniform (with respect to ε) C^1-graph.

The first step is to prove that the set $\{u_\varepsilon \le \varepsilon^\gamma\}$ is not going to be too small in this case, see Figure 1. This is needed in order to apply the results in [5] and [6], which are of the type "exterior flatness implies regularity". In the rest of this paper $\gamma \in (0,1)$ is fixed, so all estimates and results will depend on γ.

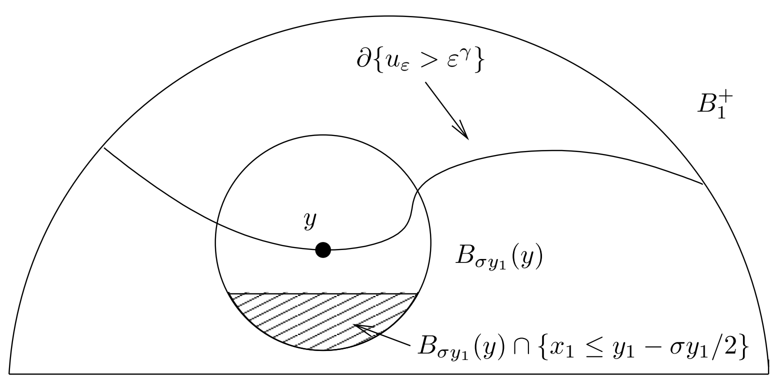

Figure 1. The situation when $\partial\{u_\varepsilon > \varepsilon^\gamma\}$ is close to the fixed boundary.

Proposition 5.1. *Let $u_\varepsilon \in P_1(M,\varepsilon)$ with $|\nabla u_\varepsilon(y)| < y_1$ for some $y \in B_1^+$. Then for any $\sigma > 0$ there is $\tau = \tau(\sigma, M, \gamma, n)$ such that $y \in \{u_\varepsilon > \varepsilon^\gamma\}$, $y \cdot e_1 = y_1 < \tau$ and $\varepsilon < \tau$ imply that*

$$B_{y_1\sigma}(y) \cap \{x_1 \le y_1 - \frac{y_1\sigma}{2}\} \subset \{u_\varepsilon \le \varepsilon^\gamma\}. \tag{5.1}$$

Proof. If this is not true then there exist $\{u_{\varepsilon_j} > \varepsilon_j^\gamma\} \ni y_1^j \to 0$, $\varepsilon_j \to 0$, $u_{\varepsilon_j} \in P_1(M, \varepsilon_j)$ and

$$x^j \in B_{y_1^j\sigma}(y) \cap \{x_1 \leq y_1^j - \frac{y_1^j\sigma}{2}\} \cap \{u_{\varepsilon_j} > \varepsilon_j^\gamma\}, \quad \text{for } j = 1, 2, \dots$$

For

$$v_j(x) = \frac{u_{\varepsilon_j}(y_1^j x + y^j)}{(y_1^j)^2}$$

we have

(1) $\Delta v_j = \beta_{\varepsilon_j}(v_j(y_1^j)^2) \in \mathcal{C}_{\varepsilon_j/(y_1^j)^2}$ for $x_1 > -1$,
(2) $v_j \geq 0$,
(3) $\sup_{B_\rho \cap \{x_1 > -1\}} v_j \leq C(M)\rho^2 + C|\nabla u_{\varepsilon_j}(y^j)|\rho/y_1^j + \varepsilon_j^\gamma/(y_1^j)^2 + v_j(0)$ for $\rho < r_0/y_1^j$,
(4) $0 \in \partial\{v_j > \varepsilon_j^\gamma/(y_1^j)^2\}$,
(5) $|\nabla v_j(0)| = |\nabla u_{\varepsilon_j}(y^j)|/|y^j| \leq 1$,
(6) $v_j = 0$ on $x_1 = -1$,
(7) $z_j = (x_j - y_j)/y_1^j \in B_\sigma \cap \{x_1 \leq -\frac{\sigma}{2}\} \cap \partial\{v_{\varepsilon_j} > \varepsilon_j^\gamma/(y_1^j)^2\}$.

All these properties together with Proposition 2.1 allow us to pass to the limit for a subsequence and obtain a limit function v_0 and a limit point $z = \lim_j z_j$. We observe that the $C^{1,1}$-estimates give

$$0 = u_{\varepsilon_j}(y^j - y_1^j e_1) \geq u_{\varepsilon_j}(y^j) - \nabla u_{\varepsilon_j}(y^j) \cdot e_1 y_1^j - C(y_1^j)^2,$$

which implies $\varepsilon_j^\gamma < C(y_1^j)^2$. Clearly, this gives $\varepsilon_j/(y_1^j)^2 \to 0$. We split this into two cases, depending on the limit of $\varepsilon_j^\gamma/(y_1^j)^2$.

Case 1: $\varepsilon_j^\gamma/(y_1^j)^2 \to 0$. Then

(1) $\Delta v_0 = \chi_{\{v_0 > 0\}}$ for $x_1 > -1$,
(2) $v_0 \geq 0$,
(3) $\sup_{B_\rho \cap \{x_1 > -1\}} v_0 \leq C(M)\rho^2$ for $\rho > 1$,
(4) $0 \in \partial\{v_0 > 0\}$,
(5) $|\nabla v_0(0)| = 0$,
(6) $v_0 = 0$ on $x_1 = -1$,
(7) $z \in B_\sigma \cap \{x_1 \leq -\sigma/2\} \cap (\{v_0 > 0\} \cup \partial\{v_0 > 0\})$.

Note that(1) follows from the fact that $\Delta v_j \in \mathcal{C}_{\varepsilon_j/(y_1^j)^2}$ where $\varepsilon_j/(y_1^j)^2 \to 0$. Moreover, (5) is a simple consequence of (4). Theorem B in [7] now implies that $v_0 = (x_1^+)^2/2$. This contradicts (7) whenever $1 > \sigma > 0$.

Case 2: $\varepsilon_j^\gamma/(y_1^j)^2 \to t > 0$. This is similar to case 1. The only difference is that the origin will be in the t-level set instead of in the zero level set which also means that we do not know that the gradient vanishes at the origin. Instead of using the result from [7] we use Proposition 4.2 to say that v_0 must be one-dimensional and monotone. This yields a contradiction for $1 > \sigma > 0$. □

Corollary 5.2. *Let $u_\varepsilon \in P_1(M, \varepsilon)$. Then for any $0 < \gamma < 1$ there is an $\varepsilon_0(M, \gamma, n)$ such that $y \in \partial\{u_\varepsilon > \varepsilon^\gamma\}$ and $y_1 < \varepsilon_0$ imply that $\partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_{ry_1}(y)$ is a uniform (with respect to ε) C^1-graph for some $r > 0$.*

Proof. If $|\nabla u(y)| \geq y_1$ then, by the implicit function theorem, $\partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_{ry_1}(y)$ is a $C^{1,\alpha}$-graph for some $r > 0$.

In the other case, $|\nabla u_\varepsilon(y)| < y_1$, let

$$v(x) = \frac{u_\varepsilon(y_1 x + y)}{y_1^2}.$$

Moreover, take $y_1 < \tau(\sigma, M, \gamma, n)$ as in Proposition 5.1 with σ small enough and so that ε/y_1^2 is sufficiently small. This is possible since we have $\varepsilon^\gamma \leq C y_1^2$.

If $|\nabla u_\varepsilon(y)| < y_1$ we know by Proposition 5.1 that (5.1) holds. Thus,

$$\frac{|\{v \leq \varepsilon^\gamma / y_1^2\} \cap B_\sigma|}{|B_\sigma|} \geq c_0.$$

From [5, Proposition 19] and with [6, Theorem 16] it follows that, for σ sufficiently small, $\partial\{v > \varepsilon^\gamma/y_1^2\} \cap B_r(y)$ is a $C^{1,\alpha}$-graph for some $r > 0$.

By rescaling and combining the two cases, we obtain the desired result. $\square$

Remark 5.3. Noteworthy is that in the proof of Corollary 5.2 we apply the result in [5, 6] for the rescaled function v. For this we need ε/y_1^2 to be small. This is assured by the assumption $\gamma < 1$. To extend this result to $\gamma = 1$ we need to come up with something different. In addition we note that in Corollary 5.2 we obtain that the level sets are C^1 but that the C^1-norm might blow up as $y_1 \to 0$.

6. Tangential touching property of the ε^γ-level sets

Here we prove that the ε^γ-level set can be approximated with planes, provided that ε is small enough and that the ε^γ-level set is close enough to the fixed boundary, see Figure 2. In what follows, we will use the notation

$$K_\delta(z) = \{|x_1 - z_1| < \delta\sqrt{(x_2 - z_2)^2 + \cdots + (x_n - z_n)^2}\}.$$

To avoid cumbersome notation we will also write $K_\delta(0) = K_\delta$.

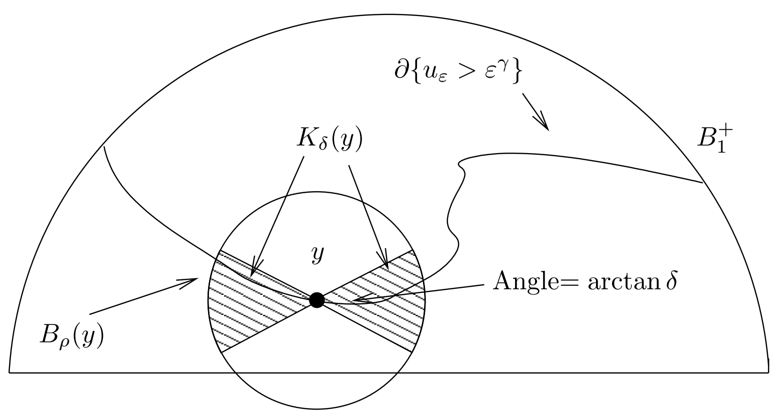

Figure 2. The level set $\partial\{u_\varepsilon > \varepsilon^\gamma\}$ is inside $K_\delta(y)$ when y is close to the fixed boundary.

Proposition 6.1. *Let $u \in P_1(\varepsilon, M)$. Then for any $\delta > 0$ there are $\varepsilon_0(M, \gamma, n, \delta)$ and $\rho(M, \gamma, n, \delta)$ such that $y \in \partial\{u_\varepsilon > \varepsilon^\gamma\}$ and $\max(\varepsilon, y_1, |\nabla u_\varepsilon(y)|) < \varepsilon_0$ imply*

$$\partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_\rho(y) \cap B_1^+ \subset K_\delta(y) \cap B_\rho(y) \cap B_1^+. \tag{6.1}$$

Proof. If this is not true then for $\delta > 0$ there are $\varepsilon_j \to 0$, $y^j \in \partial\{u_{\varepsilon_j} > \varepsilon_j^\gamma\}$ with $\max(|y_1^j|, |\nabla u_{\varepsilon_j}(y^j)|)| \to 0$ and x^j with $r_j = |x^j - y^j| \to 0$, such that

$$x^j \in K_\delta(y^j)^c \cap \partial\{u_{\varepsilon_j} > \varepsilon_j^\gamma\}. \tag{6.2}$$

We want to prove that this cannot be true. The proof is divided into two different cases with some sub-cases depending on the rate of convergence of y_1^j, ε_j, $|\nabla u_{\varepsilon_j}(y^j)|$ and r_j.

Case A: $|\nabla u_{\varepsilon_j}(y^j)| < y_1^j$. Then we have cases A1-A3 as follows. By the $C^{1,1}$-estimates,

$$0 = u(y^j - y_1^j e_1) \ge u(y^j) - |\nabla u(y^j) \cdot e_1 y_1^j| - C(y_1^j)^2 \ge \varepsilon_j^\gamma - C|y_1^j|^2.$$

Case A1: $|y_1^j| = o(r_j)$. Let

$$v_j(x) = \frac{u_{\varepsilon_j}(r_j x + y^j)}{r_j^2}.$$

Then

1. $\Delta v_j = \beta_{\varepsilon_j}(v_j r_j^2) \in \mathcal{C}_{\varepsilon_j / r_j^2}$ for $x_1 > -y_1^j / r_j$,
2. $v_j \ge 0$,
3. $\sup_{B_\rho \cap \{x_1 > -y_1^j/r_j\}} v_j \le C(M)\rho^2 + C|\nabla u_{\varepsilon_j}(y^j)|\rho/r_j + \varepsilon_j^\gamma / r_j^2 + v_j(0)$ for $\rho < r_0/r_j$ for some $r_0 > 0$,
4. $0 \in \partial\{v_j > \varepsilon_j^\gamma / r_j^2\}$ with $\varepsilon_j^\gamma / r_j^2 \le C(M)|y^j|^2/r_j^2 \to 0$,
5. $|\nabla v_j(0)| = |\nabla u_{\varepsilon_j}(y^j)/r_j| \le |y_1^j|/r_j \to 0$,
6. $v_j = 0$ on $x_1 = -y_1^j / r_j$,
7. $z_j = (x^j - y^j)/r_j \in \partial B_1 \cap K_\delta^c \cap \partial\{v_j > \varepsilon_j^\gamma / r_j^2\}$.

We can, by standard compactness arguments, pass to the limit for a subsequence $v_{j_k} \to v_0$. Using Proposition 2.1, we see that v_0 satisfies

1. $\Delta v_0 = \chi_{\{v_0 > 0\}}$ in $x_1 > 0$,
2. $v_0 \ge 0$,
3. $\sup_{B_\rho^+} v_0 \le C(M)\rho^2$ for all $\rho > 1$,
4. $0 \in \partial\{v_0 > 0\}$,
5. $|\nabla v_0(0)| = 0$,
6. $v_0 = 0$ for $x_1 = 0$,
7. $z_0 \in \partial B_1 \cap K_\delta^c \cap \partial\{v_0 > 0\}$.

[7, Theorem B] implies together with (1)-(6) that

$$v_0 = \frac{(x_1^+)^2}{2},$$

which contradicts (7).

Case A2: $0 < 1/A < |y_1^j|/r_j < A < \infty$. This case is very similar to the ones in the proof of of Proposition 5.1. We obtain that the limit will be one-dimensional via either [7] or Proposition 4.2. This leads to a contradiction.

Case A3: $y_1^j / r_j \to \infty$. We recall that, as in the proof of Proposition 5.1, we have $\varepsilon_j^\gamma < C(y_1^j)^2$. This implies $(y_1^j)^2/\varepsilon_j^\gamma \not\to 0$ and $(y_1^j)^2/\varepsilon_j \to \infty$. Let

$$v_j(x) = \frac{u_{\varepsilon_j}(y_1^j x + y^j)}{(y_1^j)^2}.$$

Then

(1) $\Delta v_j = \beta_{\varepsilon_j}(v_j|y_1^j|^2) \in \mathcal{C}_{\varepsilon_j/(y_1^j)^2}$ on $v_j > 0$ for $x_1 > -1$,
(2) $v_j \geq 0$,
(3) $\sup_{B_\rho \cap \{x_1 > -1\}} v_j \leq C(M)\rho^2 + C|\nabla u_{\varepsilon_j}(y^j)|\rho/y_1^j + \varepsilon_j^\gamma/(y_1^j)^2 + v_j(0)$ for $\rho < r_0/y_1^j$,
(4) $0 \in \partial\{v_j > \varepsilon_j^\gamma/(y_1^j)^2\}$,
(5) $|\nabla v_j(0)| = \nabla u_{\varepsilon_j}(y^j)/|y^j| \leq 1$,
(6) $v_j = 0$ on $x_1 = -1$,
(7) $z_j = (x^j - y^j)/y_1^j \in K_\delta^c \cap \{v_j > \varepsilon_j^\gamma/(y_1^j)^2\}$.

When y_1^j is small enough, Corollary 5.2 assures that $\partial\{v_j > \varepsilon_j^\gamma/(y_1^j)^2\} \cap B_{c_0}$ is a uniform C^1-graph for c_0 small. Since $|z_j| \to 0$, (7) implies that $\partial\{v_j > \varepsilon_j^\gamma/(y_1^j)^2\}$ has a tangential direction in K_δ^c. All these properties allow us, together with Proposition 2.1, to pass to the limit for a subsequence and obtain a function v_0 satisfying

(1) $\Delta v_0 = \chi_{\{v_0 > 0\}}$ for $x_1 > -1$,
(2) $v_0 \geq 0$,
(3) $\sup_{B_\rho \cap \{x_1 > -1\}} v_0 \leq C(M)\rho^2 + t$ for all $\rho > 1$,
(4) $0 \in \partial\{v_0 > t\}$ for some bounded t,
(5) $|\nabla v_0(0)| \leq 1$,
(6) $v_0 = 0$ on $x_1 = -1$,
(7) The set $\partial\{v_0 > t\} \cap B_{c_0}$ has a tangential direction in K_δ^c, for some small c_0.

As in cases 1 and 2 in the proof of Proposition 5.1, (1)-(7) imply that v_0 is one-dimensional. Therefore it can only depend on x_1, which contradicts (7), since (7) says that the level set of v_0 has a tangent lying in K_δ^c.
Case B: $|\nabla u_{\varepsilon_j}(y^j)| > y_1^j$. This implies by the $C^{1,1}$-estimates that

$$0 = u(y^j - y_1^j e_1) \geq u(y^j) - |\nabla u(y^j) \cdot e_1 y_1^j| - C(y_1^j)^2 \geq \varepsilon_j^\gamma - C|\nabla u_{\varepsilon_j}(y^j)|^2,$$

which gives $\varepsilon_j^\gamma \leq C|\nabla u(y^j)|^2$. Let $s_j = |\nabla u(y^j)|$. For the first two cases B1 and B2, let us define

$$v_j(x) = \frac{u_{\varepsilon_j}(s^j x + y^j)}{s_j^2}.$$

Then

(1) $\Delta v_j = \beta_{\varepsilon_j}(v_j s_j^2) \in \mathcal{C}_{\varepsilon_j/s_j^2}$ on $v_j > 0$ for $x_1 > -y_1^j/s_j$,
(2) $v_j \geq 0$,
(3) $\sup_{B_\rho \cap \{x_1 > -y_1^j/s_j\}} |v_j| \leq C(M)\rho^2 + \varepsilon_j^\gamma/(s_j)^2 + v_j(0)$ for $\rho < r_0/s_j$,
(4) $0 \in \partial\{v_j > \varepsilon_j^\gamma/(s_j)^2\}$,
(5) $|\nabla v_j(0)| = 1$,
(6) $v_j = 0$ on $x_1 = -y_1/s_j$,
(7) With $z_j = (x^j - y^j)/s_j \in K_\delta^c \cap \partial\{v_j > \varepsilon_j^\gamma/(s_j)^2\}$.

For this, we consider two different cases.

Case B1: $|z_j| = r_j/s_j \to 0$. This yields a situation similar to case A3. Via Corollary 5.2 and Proposition 4.2 we obtain a contradiction.

Case B2: $A < |z_j| < 1/A$. Then the situation is similar to the one in case A2. Thus, we obtain, either by Proposition 4.2 or Theorem B in [7], that the limit is one dimensional. This leads to a contradiction.

Case B3: $|z_j| \to \infty$, i.e. $s_j = o(r_j)$. Then $\varepsilon_j^\gamma \le Cs_j^2 = o(r_j^2)$. By the assumptions in this case, $y_1^j = o(r_j)$. Let

$$v_j(x) = \frac{u_{\varepsilon_j}(r_j x + y^j)}{r_j^2}.$$

Then similar arguments as in case A1 leads to contradiction. Since all cases lead to a contradiction, the theorem is proved. □

6.1. **The C^1 regularity of the ε^γ-level set.** Now we are ready to prove the uniform C^1 regularity of the ε^γ-level sets.

Proof of Theorem 1.4. We apply Corollary 5.2 and obtain that for $y_1 < \varepsilon_0$, the level set $\partial\{u_\varepsilon > \varepsilon^\gamma\} \cap B_{ry_1}(y)$ is differentiable. In addition, we know that the normal at a point y, denoted by n_y, is continuous with modulus of continuity $\sigma(\cdot/y_1)$, where σ is a modulus of continuity. Hence, it might blow up when $y_1 \to 0$. We need to prove that n_y is uniformly continuous.

Take two points $y, z \in \partial\{u_\varepsilon > \varepsilon^\gamma\} \cap \{x_1 < \varepsilon_0\}$. From Proposition 6.1 it follows that for any $\tau > 0$ there is δ_τ such that $y_1 < \delta_\tau$ implies $\|n_y - e_1\| < \tau/2$. We split the proof into three cases:
Case 1: $y_1, z_1 < \delta_\tau/2$. Then obviously $\|n_y - n_z\| \le \tau$.
Case 2: $y_1 < \delta_\tau/2$ and $z_1 > \delta_\tau/2$. Then $\|z-y\| < \delta_\tau/2$ implies $\|n_y - n_z\| \le \tau$.
Case 3: $y_1, z_1 > \delta_\tau/2$. From the arguments above,

$$\|n_y - n_z\| \le \sigma(2\|y-z\|/\delta_\tau),$$

which implies that $\|n_y - n_z\| \le \tau$ if $\|y-z\|$ is small enough. □

Finally we give the proof of Corollary 1.5.

Proof of Corollary 1.5. By Theorem 1.3 we have uniform $C^{1,1}$-estimates and by Theorem 1.4 we have uniform C^1-estimates of the sets $\partial\{u_\varepsilon > \varepsilon^\gamma\}$ when y_1 and ε are smaller than ε_0. Letting $\varepsilon \to 0$ we have $u_\varepsilon \to u_0$. Observing that Proposition 2.1 implies that the limiting set is indeed the free boundary, $\partial\{u_0 > 0\}$, the result follows. □

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