Electronic Journal of Differential Equations, Vol. 2010(2010), No. 07, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ORBITAL STABILITY OF PERIODIC TRAVELLING WAVES FOR COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

ADEMIR PASTOR

Abstract. This article addresses orbital stability of periodic travelling-wave solutions for coupled nonlinear Schrödinger equations. We prove the existence of smooth curves of periodic travelling-wave solutions depending on the dnoidal-type functions. Orbital stability analysis is developed in the context of Hamiltonian systems. We consider both the stability problem by periodic perturbations which have the same fundamental period as the corresponding periodic wave and the stability problem by periodic perturbations having two or more times the minimal period as the corresponding periodic wave.

1. Introduction

This paper is mainly concerned with the orbital stability of periodic travelling-wave solutions for the following coupled nonlinear Schrödinger equations

$$\begin{aligned} &iu_t + ru_{xx} + \Big(\eta|u|^2 + \sigma|w|^2\Big)u = 0 \\ &i\rho w_t + sw_{xx} + \Big(\frac{1}{\eta}|w|^2 + \sigma|u|^2\Big)w = 0, \end{aligned} \tag{1.1}$$

where u and w are complex-valued functions of the variables $x, t \in \mathbb{R}$, the parameters ρ, σ and η are positive real constants, and $r = \pm 1$, $s = \pm 1$.

In optics, (1.1) describes the interaction between two waves of different frequencies ω_1 and ω_2 or two waves of the same frequency ω but belonging to two different polarizations. The parameters η and ρ produce an effective asymmetry between the modes for the case of the interaction between waves of the different frequencies. Here, σ is the parameter of the cross-phase modulation, which may be determined in terms of the parameters of the corresponding physical problem (see [1, 18, 22]), and r, s describe the type of the group-velocity dispersion. For two waves of different frequencies ω_1 and ω_2, one usually has $\eta = \omega_1^2/\omega_2^2$ and $\sigma = 2$; for two waves of different polarizations in a birefringent optical medium one has $\eta = 1$ and $\sigma = 2/3$ (see e.g. [1]).

2000 *Mathematics Subject Classification.* 76B25, 35Q55, 35Q51.
Key words and phrases. Schrödinger equation; periodic travelling waves; orbital stability.

Submitted July 26, 2008. Published January 13, 2010.

More generally, (1.1) is a particular case of the system

$$\begin{aligned} &iu_t + ru_{xx} - \alpha_1 u + \Big(\eta|u|^2 + \sigma|w|^2\Big)u + \beta_1\overline{u}^2 w = 0 \\ &i\rho w_t + sw_{xx} - \alpha_2 w + \Big(\frac{1}{\eta}|w|^2 + \sigma|u|^2\Big)w + \beta_2 u^3 = 0, \end{aligned} \tag{1.2}$$

which arises in many physical situations. For instance, when $r = s = 1$, $\alpha_1 = 1$, $\eta = 1/9$, $\sigma = 2$, $\beta_1 = 1/3$, and $\beta_2 = 1/9$, the system (1.2) reads as

$$\begin{aligned} &iu_t + u_{xx} - u + \Big(\frac{1}{9}|u|^2 + 2|w|^2\Big)u + \frac{1}{3}\overline{u}^2 w = 0 \\ &i\rho w_t + w_{xx} - \alpha_2 w + \Big(9|w|^2 + 2|u|^2\Big)w + \frac{1}{9}u^3 = 0, \end{aligned} \tag{1.3}$$

which describes the resonant interaction between a linearly polarized beam of frequency ω and its third harmonic (see [27, 28]).

Here, we specialize the system (1.1) in the case where $\rho = 1$ and $r = s = 1$, but we permit all values of $\eta, \sigma > 0$. Thus, (1.1) reduces to

$$\begin{aligned} &iu_t + u_{xx} + \Big(\eta|u|^2 + \sigma|w|^2\Big)u = 0 \\ &iw_t + w_{xx} + \Big(\frac{1}{\eta}|w|^2 + \sigma|u|^2\Big)w = 0. \end{aligned} \tag{1.4}$$

From the mathematical viewpoint, (1.4) has been studied by many authors (see e.g. [2, 9, 19, 20, 30], [23]–[26], but only in the context of existence and stability of solitary-wave solutions. As far as we know, no results concerning the stability of periodic travelling-wave solutions have been shown.

As a matter of fact, only a few papers address orbital stability of periodic travelling-wave solutions for Schrödinger-type systems. We cite a few known. In [4], the authors considered a system arising in nonlinear optics (in a medium with quadratic nonlinearities, see [17]) and they showed the existence and stability/instability of periodic travelling waves depending on the Jacobian elliptic function of the cnoidal type. In [5, 24], the authors considered the system (1.3). The existence and stability/instability of periodic travelling waves depending on the *dnoidal* (in [5]) and *cnoidal* (in [24]) functions were shown. The techniques to obtain such results were the ones developed by Grillakis, Shatah, and Strauss [13], and Grillakis [12].

For the single cubic Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0, \tag{1.5}$$

Angulo [3] established the existence of periodic travelling waves based on the dnoidal-type functions. By combining the classical Lyapunov method and the Floquet theory associated to the Lamé equation

$$v'' + [\lambda - 6k^2 sn^2(x;k)]v = 0,$$

the author showed their orbital stability by periodic perturbations which have the same fundamental period as the corresponding dnoidal wave (note that one can also apply the theory in [13]), and orbital instability by periodic perturbations with twice the fundamental period of the dnoidal wave.

As evidenced above, the abstract *Stability/Instability Theorem* in [13] can be applied for many dispersive equations. However, the main difficulty when one works with travelling waves for coupled systems, instead of one single equation, is that the spectral analysis for the "linearized Hamiltonian" turns out to be a more delicate matter. Indeed, in this case one needs to deal with a matrix having Schrödinger-type operators as components. As a consequence, in many examples, the Stability/Instability criterium in [13] turns out to be insufficient for a complete stability/instability analysis of the travelling waves. The main reason for this, is that in such approach one needs to know the exact number of negative eigenvalues of the linearized Hamiltonian.

Grillakis [11, 12] obtained others special instability theorems, which get orbital instability from the linear instability of the zero solution for the linearization of the system around the orbit generated by the corresponding travelling wave. The main advantage when one uses the Grillakis approach is that one does not need to know the exact number of negative eigenvalues of the linearized Hamiltonian, but only to have an estimate on a certain bound (see Subsection 4.3 for the details).

Now, we turn our attention to the structure of the paper. The periodic travelling-wave solutions we are interested in are of the form

$$u(x,t) = e^{i\gamma t}\phi_\gamma(x), \quad w(x,t) = e^{i\gamma t}\psi_\gamma(x), \tag{1.6}$$

where $\phi_\gamma, \psi_\gamma : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with the same fixed period $L > 0$ and γ is a real parameter. Substituting (1.6) into (1.4), we get the following system of ordinary differential equations

$$\begin{aligned} \phi_\gamma'' - \gamma\phi_\gamma + \Big(\eta\phi_\gamma^2 + \sigma\psi_\gamma^2\Big)\phi_\gamma &= 0 \\ \psi_\gamma'' - \gamma\psi_\gamma + \Big(\frac{1}{\eta}\psi_\gamma^2 + \sigma\phi_\gamma^2\Big)\psi_\gamma &= 0. \end{aligned} \tag{1.7}$$

It is well known that (1.7) admits solitary-wave solutions (for $\eta = 1$) of the form

$$\phi_\gamma(x) = \psi_\gamma(x) = \sqrt{\frac{2\gamma}{\sigma+1}}\,\operatorname{sech}(\sqrt{\gamma}x), \quad \gamma > 0. \tag{1.8}$$

In [25], the authors proved that the waves in (1.8) are linearly stable for $\sigma > 0$ and linearly unstable for $-1 < \sigma < 0$. Moreover, by using the concentration-compactness method, Ohta in [23] showed that those waves are orbitally stable in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for all $\sigma > -1$.

In the present paper, we consider two classes of periodic solutions. First, we suppose $\psi_\gamma \equiv 0$. Then, we can find a smooth curve of periodic solutions for (1.7) depending on the dnoidal type function, namely,

$$\gamma \in \Big(\frac{2\pi^2}{L^2}, +\infty\Big) \mapsto (\phi_\gamma, 0) \in H^m_{\rm per}([0,L]) \times H^n_{\rm per}([0,L]), \tag{1.9}$$

where

$$\phi_\gamma(x) = \eta_1 dn\Big(\frac{\sqrt{\eta}}{\sqrt{2}}\eta_1 x; k\Big), \quad k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \tag{1.10}$$

and η_1, η_2 are smooth functions depending on the parameter γ with $0 < \eta_2 < \eta_1$. Throughout the paper, we shall refer to the solutions in (1.9) as *semitrivial* solutions.

Second, we assume $\psi_\gamma = b\phi_\gamma$, for some real constant $b \neq 0$. Then, we can find another smooth curve of dnoidal waves,

$$\gamma \in \Big(\frac{2\pi^2}{L^2}, +\infty\Big) \mapsto (\phi_\gamma, b\phi_\gamma) \in H^m_{\rm per}([0,L]) \times H^n_{\rm per}([0,L]), \tag{1.11}$$

where, for $\theta = \eta + \sigma b^2$,

$$\phi_\gamma(x) = \theta_1 dn\Big(\frac{\sqrt{\theta}}{\sqrt{2}}\theta_1 x; k\Big), \quad k^2 = \frac{\theta_1^2 - \theta_2^2}{\theta_1^2}, \tag{1.12}$$

and θ_1, θ_2 depend smoothly on the parameter γ and satisfy $0 < \theta_2 < \theta_1$. Throughout the paper, we shall refer to the solutions in (1.11) as *non-semitrivial* solutions.

Concerning the orbital stability, for the semitrivial solutions in (1.9), we show that the orbit

$$\mathcal{O}_L = \{(e^{ir}\phi_\gamma(\cdot + s), 0); r, s \in \mathbb{R}\}$$

is stable in the energy space $H^1_{\rm per}([0,L]) \times H^1_{\rm per}([0,L])$ provided that $\eta > \sigma$, and the orbit $\mathcal{O}_{2L} = \{(e^{ir}\phi_\gamma(\cdot), 0);\ r \in \mathbb{R}\}$ is unstable in the space $H^1_{\rm per}([0,2L]) \times H^1_{\rm per}([0,2L])$. On the other hand, for the non-semitrivial solutions (1.11) (with $\eta = b = 1$), we show that the orbit $\widetilde{\mathcal{O}} = \{(e^{ir}\phi_\gamma(\cdot), e^{ir}\phi_\gamma(\cdot));\ r \in \mathbb{R}\}$ is spectrally stable with respect to periodic perturbations having the same fundamental period of ϕ_γ, and it is orbitally unstable in the space $H^1_{\rm per}([0,2L]) \times H^1_{\rm per}([0,2L])$. Moreover, if we assume $-1 < \sigma < 0$, then $\widetilde{\mathcal{O}}$ is orbitally unstable in the space $H^1_{\rm per}([0,L]) \times H^1_{\rm per}([0,L])$. To obtain the orbital stability result, we use the Grillakis, Shatah, and Strauss [13] theory. However, to get the instability results, we employ the theory developed by Grillakis [11, 12] (see also [26, 29]).

Concerning the local well-posedness of the system (1.4), by introducing the Bourgain $X_{s,b}$-spaces and making use of the contraction principle, we can prove that the system (1.4) is locally well-posed in the spaces $H^s_{\rm per}([0,L]) \times H^s_{\rm per}([0,L])$, $s \geq 0$. Moreover, from the conserved quantity

$$\mathcal{F}(t) := \frac{1}{2}\int (|u|^2 + |w|^2)dx = \mathcal{F}(0) \tag{1.13}$$

the local solution can be extended for any interval of time.

The paper is organized as follows: in Section 2, we review the results concerning the well-posedness theory. In Section 3, we prove the existence of smooth curves of semitrivial and non-semitrivial solutions. Section 4 is devoted to the orbital stability/instability of the semitrivial solutions, whereas in Section 5 the results for the non-semitrivial solutions are provided.

Notation. For $s \in \mathbb{R}$, the Sobolev space $H^s_{\rm per} := H^s_{\rm per}([0,L])$ consists of all L-periodic distributions f such that

$$\|f\|^2_{H^s_{\rm per}} := L\sum_{k=-\infty}^{\infty}(1+k^2)^s|\widehat{f}(k)|^2 < \infty.$$

The symbols $sn(\cdot; k)$, $dn(\cdot; k)$, and $cn(\cdot; k)$ will denote, respectively, the Jacobian elliptic functions of snoidal, dnoidal, and cnoidal type.

2. Well-Posedness Theory

In this section, we review the well-posedness theory for the system (1.4). For additional details we refer the reader to [5], where a more general result can be found. We consider the Cauchy problem

$$\begin{gathered}
iu_t + u_{xx} + \Big(\eta|u|^2 + \sigma|w|^2\Big)u = 0, \quad x \in [0, L],\ t \in \mathbb{R}, \\
iw_t + w_{xx} + \Big(\frac{1}{\eta}|w|^2 + \sigma|u|^2\Big)w = 0, \quad x \in [0, L],\ t \in \mathbb{R}, \\
u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x).
\end{gathered} \tag{2.1}$$

Let $\{U(t)\}_{t=-\infty}^{\infty}$ be the unitary group corresponding to the linear Schrödinger equation.

Definition 2.1. Let $\mathcal{V}$ be the space of functions $f = f(x, t)$ such that

(i) $f : [0, L] \times \mathbb{R} \to \mathbb{C}$,
(ii) $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ (Schwartz space) for each $x \in [0, L]$,
(iii) $f(\cdot, t) \in C^{\infty}_{\rm per}([0, L])$ for each $t \in \mathbb{R}$.

For $s, b \in \mathbb{R}$, we define the Bourgain space $X_{s,b}$ to be the completion of $\mathcal{V}$ with respect to the norm

$$\|f\|_{X_{s,b}} = \Big(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \langle n\rangle^{2s} \langle \tau + n^2\rangle^{2b} |\widehat{f}(n, \tau)|^2 d\tau\Big)^{1/2},$$

where $\langle\cdot\rangle = 1 + |\cdot|$.

Remark 2.2. Notice that since $\|f\|_{X_{s,b}} = \|U(-t)f\|_{H^b(\mathbb{R}_t;H^s_{\rm per})}$, it follows from Sobolev's Lemma that if $b > 1/2$ then $X_{s,b} \hookrightarrow C(\mathbb{R}_t; H^s_{\rm per})$.

Let $\zeta \in C_0^{\infty}(\mathbb{R})$ be a cut off function such that $\operatorname{supp}\zeta \subset (-2, 2)$ and $\zeta \equiv 1$ on the interval $[-1, 1]$. For each $T > 0$, we define $\zeta_T(t) = \zeta(t/T)$.

Lemma 2.3. *Let $s \in \mathbb{R}$, $b \in (1/2, 1)$ and $T \in (0, 1]$. Then,*

(i) $\|\zeta_T U(t)v\|_{X_{s,b}} \le c\|v\|_{H^s_{\rm per}}$,
(ii) $\big\|\zeta_T \int_0^t U(t-t')f(t')dt'\big\|_{X_{s,b}} \le cT^{\gamma}\|f\|_{X_{s,b-1}}$, *where γ is a positive constant.*

For a proof of the above lemma, see for example Kenig, Ponce, and Vega [15, 16]. Next, we have a trilinear estimate, which may be proved following similar arguments as the ones in Bourgain [6]; see also [5, 7].

Lemma 2.4. *Let $s \ge 0$ and $b \in (3/8, 5/8)$. Then,*

$$\|u^{\alpha_1}\overline{u}^{\alpha_2}w^{\alpha_3}\overline{w}^{\alpha_4}\|_{X_{s,b-1}} \le c\|u\|^{\alpha_1+\alpha_2}_{X_{s,b}}\|w\|^{\alpha_3+\alpha_4}_{X_{s,b}},$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{0, 1, 2, 3\}$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3$.

With the above lemmas in hand, we are able to prove our local well-posedness result.

Theorem 2.5 (Local well-posedness)**.** *Let $s \ge 0$ and $b \in (1/2, 5/8)$. For any $(u_0, w_0) \in H^s_{\rm per}([0, L]) \times H^s_{\rm per}([0, L])$, there exist $T = T(\|(u_0, w_0)\|_{H^s_{\rm per}\times H^s_{\rm per}}) > 0$ and a unique solution $(u(t), w(t))$ of the initial-value problem* (2.1) *satisfying*

$$\begin{gathered}
(u, w) \in C([-T, T]; H^s_{\rm per}([0, L]) \times H^s_{\rm per}([0, L])), \\
(\zeta_T u, \zeta_T w) \in X_{s,b} \times X_{s,b}.
\end{gathered}$$

Moreover, given $T' \in (0, T)$, there exists a neighborhood $\mathcal{W}$ of (u_0, w_0) in $H^s_{\rm per} \times H^s_{\rm per}$ such that the map $(u_0, w_0) \mapsto (u(t), w(t))$ from $\mathcal{W}$ into $C([-T', T']; H^s_{\rm per} \times H^s_{\rm per})$ is Lipschitz continuous.

Sketch of the proof. We define the metric space of functions

$$\mathcal{X}_M = \{(u,w) \in X_{s,b} \times X_{s,b}; \|(u,w)\|_{X_{s,b}\times X_{s,b}} := \|u\|_{X_{s,b}} + \|w\|_{X_{s,b}} \le M\},$$

and the map $\Phi = (\Phi_1, \Phi_2)$, where

$$\begin{aligned} \Phi_1(u,w)(t) &= \zeta_T(t)U(t)u_0 + i\zeta_T(t)\int_0^t U(t-t')\Big(\eta|u|^2u + \sigma|w|^2u\Big)(t')dt', \\ \Phi_2(u,w)(t) &= \zeta_T(t)U(t)w_0 + i\zeta_T(t)\int_0^t U(t-t')\Big(\frac{1}{\eta}|w|^2w + \sigma|u|^2w\Big)(t')dt', \end{aligned} \tag{2.2}$$

where $M > 0$ and $T \in (0,1]$. By choosing M and T suitably, and using Lemmas 2.3 and 2.4, we can prove that $\Phi : \mathcal{X}_M \to \mathcal{X}_M$ is a contraction. Hence, the contraction principle implies the existence of a unique fixed point for the integral equations (2.2), which solves our problem. The rest of the proof follows standard arguments, which will be omitted. □

Finally, we can establish our global well-posedness result.

Theorem 2.6 (Global well-posedness). *For $s \ge 0$ and $(u_0, w_0) \in H^s_{\mathrm{per}}([0,L]) \times H^s_{\mathrm{per}}([0,L])$, the solution $(u(t), w(t))$ given in Theorem 2.5 can be extend to any interval of time.*

Proof. This follows from the conserved quantity

$$\int (|u(x,t)|^2 + |w(x,t)|^2)dx = \int (|u_0(x)|^2 + |w_0(x)|^2)dx,$$

and *a priori* estimates (see e.g. [7]). □

Remark 2.7. The same ideas used to prove Theorems 2.5 and 2.6 can be applied to show local and global well-posedness results for the system (1.2) (see [5]).

3. Existence of smooth curves of dnoidal waves

This section is devoted to establishing the existence of smooth curves of periodic travelling-wave solutions for the system (1.4) having the form

$$u(x,t) = e^{i\gamma t}\phi_\gamma(x), \quad w(x,t) = e^{i\gamma t}\psi_\gamma(x), \tag{3.1}$$

where $\phi_\gamma, \psi_\gamma : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with the same fixed period $L > 0$ and γ is a real parameter. Thus, $\psi_\gamma = \psi$ and $\phi_\gamma = \phi$ must satisfy the system of ordinary differential equations

$$\begin{aligned} \phi'' - \gamma\phi + \Big(\eta\phi^2 + \sigma\psi^2\Big)\phi &= 0 \\ \psi'' - \gamma\psi + \Big(\frac{1}{\eta}\psi^2 + \sigma\phi^2\Big)\psi &= 0. \end{aligned} \tag{3.2}$$

To solve this system, we consider $\psi = b\phi$, for some real constant b, and analyze two cases.

Case 1 (semitrivial solutions): $b = 0$. In this case, (3.2) reduces to the differential equation

$$\phi'' - \gamma\phi + \eta\phi^3 = 0. \tag{3.3}$$

It is well known that this equation has a positive solution of the form

$$\phi(x) = \eta_1 dn\Big(\frac{\sqrt{\eta}}{\sqrt{2}}\eta_1 x; k\Big), \quad k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \tag{3.4}$$

where η_1, η_2 are real constants satisfying $\eta_1 > \eta_2 > 0$.

In the sequel, we prove that the parameters η_1, η_2 can be chosen such that the function ϕ in (3.4) has fundamental period L (here we give only the main ingredients, for details see [3],[5]). Indeed, it is easy to see that parameters η_1, η_2 must satisfy

$$\begin{gathered} \eta_1^2 + \eta_2^2 = \frac{2\gamma}{\eta} = 2\omega \\ -\eta_1^2\eta_2^2 = \frac{4}{\eta} B_\phi, \end{gathered} \tag{3.5}$$

where $\omega := \gamma/\eta$ and B_ϕ is an integration constant. Therefore, for a fixed $\omega > 0$, we get from (3.5) that $0 < \eta_2 < \sqrt{\omega} < \eta_1 < \sqrt{2\omega}$. Since dn has fundamental period $2K(k)$, it follows that the function ϕ in (3.4) has fundamental period

$$T_\phi = \frac{2\sqrt{2}}{\eta_1\sqrt{\eta}} K(k).$$

From (3.5), we can see T_ϕ as a function depending only on η_2 (since $\omega > 0$ is fixed), namely,

$$T_\phi(\eta_2) = \frac{2\sqrt{2}}{\sqrt{\eta}\sqrt{2\omega - \eta_2^2}} K(k(\eta_2)), \quad k^2(\eta_2) = \frac{2\omega - 2\eta_2^2}{2\omega - \eta_2^2}.$$

But, since $T_\psi(\eta_2) \to \infty$ as $\eta_2 \to 0^+$, $T_\psi(\eta_2) \to \frac{\pi\sqrt{2}}{\sqrt{\omega\eta}}$ as $\eta_2 \to \sqrt{\omega}$, and the function $\eta_2 \in (0, \sqrt{\omega}) \mapsto T_\phi(\eta_2)$ is strictly decreasing, we conclude that $T_\phi > \frac{\pi\sqrt{2}}{\sqrt{\omega\eta}}$.

Hence, given $L > 0$, by fixing $\omega > 0$ such that $\sqrt{\omega} > \frac{\pi\sqrt{2}}{L\sqrt{\eta}}$, there is a unique $\eta_2 = \eta_2(\omega) \in (0, \sqrt{\omega})$ such that the dnoidal wave ϕ in (3.4) has fundamental period $L = T_\phi$.

In addition, we can construct, for each $L > 0$ fixed, a smooth curve of dnoidal-wave solutions depending on the parameter ω (and hence on γ) such that each element of the curve has fundamental period L. More precisely, we prove the following.

Theorem 3.1. *Let $L > 0$ be fixed. The following statements hold.*

(i) *There is a smooth function $\Gamma : \big(\frac{2\pi^2}{\eta L^2}, +\infty\big) \to \mathbb{R}$ such that*

$$\frac{2\sqrt{2}}{\sqrt{\eta}\sqrt{2\omega - \eta_2^2}} K(k) = L,$$

where $\eta_2 = \Gamma(\omega)$ and $k^2 = k^2(\omega) = \frac{2\omega - 2\eta_2^2}{2\omega - \eta_2^2}$.

(ii) *The function Γ in (i) is strictly decreasing and the modulus $k = k(\omega)$ satisfies $\frac{dk}{d\omega} > 0$.*

(iii) *For every $\gamma \in \big(\frac{2\pi^2}{L^2}, +\infty\big)$ and $\omega = \omega(\gamma) = \gamma/\eta$, the dnoidal wave defined in* (3.4)*,*

$$\phi_\gamma(x) := \phi_{\omega(\gamma)}(x) = \sqrt{2\omega - \eta_2^2}\, dn\Big(\frac{\sqrt{\eta(2\omega - \eta_2^2)}}{\sqrt{2}} x; k\Big)$$

has fundamental period L and satisfies (3.3)*. Moreover, the mapping*

$$\gamma \in \Big(\frac{2\pi^2}{L^2}, +\infty\Big) \mapsto \phi_\gamma \in H^n_{\rm per}([0, L])$$

is a smooth function.

Proof. The proof is an application of the Implicit Function Theorem. We refer to [3] or [5] for the details. □

Case 2 (non-semitrivial solutions): $b \neq 0$. In this case, from (3.2), we get the system

$$\begin{gathered}\phi'' - \gamma\phi + \big(\eta + \sigma b^2\big)\phi^3 = 0 \\ \phi'' - \gamma\phi + \big(\frac{b^2}{\eta} + \sigma\big)\phi^3 = 0.\end{gathered} \tag{3.6}$$

To solve this system, we make the following assumptions:

(H1) $\eta + \sigma b^2 = \frac{b^2}{\eta} + \sigma$,
(H2) $(\sigma - \eta)(\sigma - \frac{1}{\eta}) > 0$.

These two assumptions allows us to reduce (3.6) to the single equation

$$\phi'' - \gamma\phi + \theta\phi^3 = 0, \tag{3.7}$$

where $\theta = \eta + \sigma b^2 > 0$ is a real constant. As in Case 1, the equation (3.7) admits a dnoidal-wave solution

$$\phi(x) = \theta_1 dn\Big(\frac{\theta_1\sqrt{\theta}}{\sqrt{2}}x; k\Big), \quad k^2 = \frac{\theta_1^2 - \theta_2^2}{\theta_1^2}, \tag{3.8}$$

where θ_1, θ_2 are real constants and satisfy $\theta_1^2 + \theta_2^2 = 2\omega$, where in this case we have written $\omega = \gamma/\theta$.

By similar arguments as in Case 1, we may verify that for every $L > 0$ and $\omega > 0$ such that $\sqrt{\omega} > \frac{\pi\sqrt{2}}{\sqrt{\theta}L}$, there exists a unique $\theta_2 = \theta_2(\omega) \in (0, \sqrt{\omega})$ such that the dnoidal wave $\phi = \phi(\cdot; \theta_1(\omega); \theta_2(\omega))$, given in (3.8), has fundamental period $L = T_\phi$.

Remark 3.2. Formally, the periodic-wave solution (3.8) contains the solitary-wave solution (1.8). Indeed, as $\theta_2 \to 0^+$ it follows that $\theta_1 \to \sqrt{2\gamma/(\eta + \sigma b^2)}$ and $dn(\cdot; 1^-) \sim \operatorname{sech}(\cdot)$. Thus,

$$\phi(x) \sim \sqrt{\frac{2\gamma}{\eta + \sigma b^2}}\operatorname{sech}(\sqrt{\gamma}x),$$

As in Theorem 3.1, we can prove the following.

Theorem 3.3. *Let $L > 0$ be fixed. The following statements hold.*

(i) *There exists a smooth function $\Lambda : \big(\frac{2\pi^2}{\theta L^2}, +\infty\big) \to \mathbb{R}$ such that*

$$\frac{2\sqrt{2}}{\sqrt{\theta}\sqrt{2\omega - \theta_2^2}}K(k) = L,$$

where $\theta_2 = \Lambda(\omega)$ and $k^2 = k^2(\omega) = \frac{2\omega - 2\theta_2^2}{2\omega - \theta_2^2}$.

(ii) *The function Λ in (i) is strictly decreasing and the modulus $k = k(\omega)$ satisfies $\frac{dk}{d\omega} > 0$.*

(iii) *For every $\gamma \in \big(\frac{2\pi^2}{L^2}, +\infty\big)$ and $\omega = \omega(\gamma) = \gamma/\theta$, the dnoidal wave defined in* (3.8),

$$\phi_\gamma(x) = \phi_{\omega(\gamma)}(x) = \sqrt{2\omega - \theta_2^2}\, dn\Big(\frac{\sqrt{\theta(2\omega - \theta_2^2)}}{\sqrt{2}}x; k\Big)$$

has fundamental period L and satisfies (3.7)*. Moreover, the mapping*

$$\gamma \in \Big(\frac{2\pi^2}{L^2}, +\infty\Big) \mapsto \phi_\gamma \in H^n_{\rm per}([0,L])$$

is a smooth function.

4. Orbital Stability/Instability of semitrivial solutions

In this section, we prove our results concerning the orbital stability/instability of the semitrivial solutions given in Theorem 3.1. First, we note that the system (1.4) may be written as a Hamiltonian system. Indeed, by writing $u = P + iQ$, $w = R + iS$, and $U = (P, R, Q, S)$, we rewrite (1.4) as

$$\frac{\partial U}{\partial t}(t) = J\mathcal{H}'(U(t)), \tag{4.1}$$

where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and $\mathcal{H}$ is the energy functional

$$\begin{aligned}\mathcal{H}(P, R, Q, S) = \frac{1}{2}\int \Big\{&(P_x^2 + Q_x^2) + (R_x^2 + S_x^2) - \frac{\eta}{2}(P^2 + Q^2)^2 - \frac{1}{2\eta}(R^2 + S^2)^2 \\ &- \sigma(P^2 + Q^2)(R^2 + S^2)\Big\}dx.\end{aligned} \tag{4.3}$$

In (4.1), $\mathcal{H}'$ denotes the Fréchet derivative of $\mathcal{H}$.

Recall that by using this notation, the functional $\mathcal{F}$ in (1.13) reads as

$$\mathcal{F}(P, R, Q, S) = \frac{1}{2}\int \big\{(P^2 + Q^2) + (R^2 + S^2)\big\}dx.$$

4.1. **Spectral Analysis.** We consider $L > 0$ fixed and define $\Phi = (\phi_\gamma, 0, 0, 0)$, where $\phi = \phi_\gamma$ is the dnoidal wave given by Theorem 3.1. Next, we consider the linearized operator

$$\mathcal{L}_\gamma = \mathcal{H}''(\Phi) + \gamma\mathcal{F}''(\Phi) = \begin{pmatrix} \mathcal{L}_R & 0 \\ 0 & \mathcal{L}_I \end{pmatrix}, \tag{4.4}$$

where $\mathcal{L}_R$ and $\mathcal{L}_I$ are 2×2 matrix operators defined by

$$\mathcal{L}_R = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_3 \end{pmatrix}, \quad \mathcal{L}_I = \begin{pmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{L}_3 \end{pmatrix}, \tag{4.5}$$

with

$$\mathcal{L}_1 = -\frac{d^2}{dx^2} + \gamma - 3\eta\phi^2, \quad \mathcal{L}_2 = -\frac{d^2}{dx^2} + \gamma - \eta\phi^2, \quad \mathcal{L}_3 = -\frac{d^2}{dx^2} + \gamma - \sigma\phi^2. \tag{4.6}$$

In the sequel, we study the spectrum of the diagonal operator $\mathcal{L}_\gamma$.

Theorem 4.1. *Let $\phi = \phi_\gamma$ be the dnoidal wave given by Theorem 3.1. Consider the operator $\mathcal{L}_\gamma$ in* (4.4) *defined in $L^2_{\rm per}([0,L])$ with domain $H^2_{\rm per}([0,L])$. The following statements hold.*

(i) *If* $n(\mathcal{L}_\gamma)$ *denotes the number of negative eigenvalues of* $\mathcal{L}_\gamma$ *(counting multiplicities), then* $n(\mathcal{L}_\gamma) = 2k+1$*, for some* $k \in \mathbb{N} \cup \{0\}$*. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

(ii) *The kernel of* $\mathcal{L}_\gamma$*,* $\ker(\mathcal{L}_\gamma)$*, is at least two-dimensional and contains the space spanned by* $(\phi', 0, 0, 0)$ *and* $(0, 0, \phi, 0)$*.*

To prove Theorem 4.1, the following lemma is fundamental.

Lemma 4.2. *Let* $\phi = \phi_\gamma$ *be the dnoidal wave given by Theorem 3.1. Then the following spectral properties hold.*

(i) *The operator* $\mathcal{L}_1$ *in* (4.6) *defined in* $L^2_{\rm per}([0,L])$ *with domain* $H^2_{\rm per}([0,L])$ *has exactly one negative eigenvalue which is simple; zero is an eigenvalue which is simple with eigenfunction* ϕ'*. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

(ii) *The operator* $\mathcal{L}_2$ *in* (4.6) *defined in* $L^2_{\rm per}([0,L])$ *with domain* $H^2_{\rm per}([0,L])$ *has no negative eigenvalues; zero is a simple eigenvalue with eigenfunction* ϕ*. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. The proof is essentially the same one as in [3, Theorems 3.1 and 3.2] and [5, Theorem 4.1], with obvious modifications. Note that part (ii) is an immediate consequence of the Floquet theory (see e.g. [21]), since $\mathcal{L}_2\phi = 0$ and ϕ has no zeros in $[0, L]$. □

Proof of Theorem 4.1. Part (ii) follows immediately from Lemma 4.2. To prove part (i), we first note that, from Lemma 4.2, $\mathcal{L}_\gamma$ always has a negative eigenvalue (which comes from the operator $\mathcal{L}_1$). Moreover, from the definition of $\mathcal{L}_\gamma$, we see that if λ is an eigenvalue of $\mathcal{L}_3$ (defined in $L^2_{\rm per}([0,L])$ with domain $H^2_{\rm per}([0,L])$) then it is a double eigenvalue of $\mathcal{L}_\gamma$. Therefore, since $\mathcal{L}_2$ has no negative eigenvalues, we have proved the theorem. □

Corollary 4.3. *Let* $\phi = \phi_\gamma$ *be the dnoidal wave given by Theorem 3.1. Consider the operator* $\mathcal{L}_\gamma$ *in* (4.4) *defined in* $L^2_{\rm per}([0,L])$ *with domain* $H^2_{\rm per}([0,L])$ *and suppose that* $\eta > \sigma$*. Then,*

(i) $n(\mathcal{L}_\gamma) = 1$.

(ii) $\ker(\mathcal{L}_\gamma)$ *is two-dimensional and spanned by* $(\phi', 0, 0, 0)$ *and* $(0, 0, \phi, 0)$*.*

Proof. Since $\eta > \sigma$, we have $\gamma - \eta\phi^2(x) < \gamma - \sigma\phi^2(x)$ in $[0, L]$. Thus, because zero is the first eigenvalue of $\mathcal{L}_2$, an application of the Comparison Theorem (see e.g. [10]) gives us that $\mathcal{L}_3$ is a strictly positive operator. Hence, Theorem 4.1 yields the desired result. □

Remarks.

(1) If $\sigma = \eta$ then $n(\mathcal{L}_\gamma) = 1$ but $\ker(\mathcal{L}_\gamma)$ is 4-dimensional.

(2) If $\sigma > \eta$ then, from Lemma 4.2 and the Comparison Theorem, we obtain that $n(\mathcal{L}_\gamma) \geq 3$. Moreover, if $3\eta > \sigma > \eta$ then $n(\mathcal{L}_\gamma) = 3$, and $\ker(\mathcal{L}_\gamma)$ is spanned by $(\phi', 0, 0, 0)$ and $(0, 0, \phi, 0)$.

With the goal of proving a stability/instability result by periodic perturbation having twice the wavelength of ϕ_γ, we now prove the following.

Theorem 4.4. *Let* $\phi_\gamma = \phi$ *be the dnoidal wave given by Theorem* 3.1.

(i) *Consider the operator* $\mathcal{L}_R$ *in* (4.5) *defined in* $L^2_{\rm per}([0,2L])$ *with domain* $H^2_{\rm per}([0,2L])$. *If* $\widetilde{n}(\mathcal{L}_R)$ *denotes the number of negative eigenvalues of* $\mathcal{L}_R$ *(counting multiplicities), then* $\widetilde{n}(\mathcal{L}_R) = k_0 + 3$ *for some* $k_0 \in \mathbb{N}$. *Moreover, there exist* k_0+2 *eigenvalues such that the corresponding eigenfunctions are orthogonal to* $\ker(\mathcal{L}_I)$.

(ii) *Consider the operator* $\mathcal{L}_I$ *in* (4.5) *defined in* $L^2_{\rm per}([0,2L])$ *with domain* $H^2_{\rm per}([0,2L])$. *If* $\widetilde{n}(\mathcal{L}_I)$ *denotes the number of negative eigenvalues of* $\mathcal{L}_I$ *(counting multiplicities), then* $\widetilde{n}(\mathcal{L}_I) = k_0$. *Moreover, all the eigenfunctions corresponding to negative eigenvalues are orthogonal to* $\ker(\mathcal{L}_R)$.

Proof. (i) Let k_0 be the number of negative eigenvalues of the operator $\mathcal{L}_3$ (defined in $L^2_{\rm per}([0,2L])$ with domain $H^2_{\rm per}([0,2L])$). So, for the first part, it suffices to show that the operator $\mathcal{L}_1$ (defined in $L^2_{\rm per}([0,2L])$ with domain $H^2_{\rm per}([0,2L])$) has exactly 3 negative eigenvalues. To do this, we consider the semi-periodic eigenvalue problem

$$\begin{gathered} \mathcal{L}_1\chi = \mu\chi \\ \chi(0) = -\chi(L), \quad \chi'(0) = -\chi'(L). \end{gathered} \tag{4.7}$$

which is equivalent (under the transformation $\Lambda(x) = \chi(\alpha x)$, $\alpha = \frac{\sqrt{2}}{\eta_1\sqrt{\eta}}$) to the following semi-periodic eigenvalue problem associated to the Lamé equation:

$$\begin{gathered} \Lambda'' + [\widetilde{\mu} - 6k^2 sn^2(x;k)]\Lambda = 0 \\ \Lambda(0) = -\Lambda(2K), \quad \Lambda'(0) = -\Lambda'(2K), \end{gathered} \tag{4.8}$$

where

$$\widetilde{\mu} = \frac{2}{\eta\eta_1^2}[\mu - \gamma + 3\eta\eta_1^2]. \tag{4.9}$$

Now, a straightforward calculation shows us that $\widetilde{\mu}_0 = 1 + k^2$ and $\widetilde{\mu}_1 = 1 + 4k^2$ are the first two eigenvalues to (4.8) (see [14]), which are simple with eigenfunctions given, respectively, by

$$\Lambda_{1,sm}(x) = cn(x;k)dn(x;k), \quad \Lambda_{2,sm}(x) = sn(x;k)dn(x;k).$$

Thus, from (4.9) and the Floquet theory, we obtain the first part.

For the second part, we note that if μ_1, μ_2 denote the corresponding negative eigenvalues for (4.7), via (4.9), then the unique (up to a constant) eigenfunction for $\mathcal{L}_1$ associated to $\mu_i, i = 1,2$ is given by $\chi_i(x) = \Lambda_{i,sm}(\frac{1}{\alpha}x)$, and so $(\chi_i, 0)$ is an eigenfunction for $\mathcal{L}_R$ associated to $\mu_i, i = 1,2$. Next, let $(u,v) \in \ker(\mathcal{L}_I)$, with $u \neq 0$. It follows from Lemma 4.2 that $u = c\phi$ for some real constant c. Since

$$\int_0^{4K} dn^2(x;k)cn(x;k)dx = \int_0^{4K} dn^2(x;k)sn(x;k)dx = 0,$$

it is easy to see, from the explicit form of ϕ and χ_i, that

$$\langle(\chi_i,0),(u,v)\rangle_{L^2_{\rm per}([0,2L])} = c\int_0^{2L} \chi_i(x)\phi(x)dx = 0, \quad i = 1,2.$$

Furthermore, since $\mathcal{L}_3$ is a self-adjoint operator all its eigenfunctions are two-to-two orthogonal, and so, all the eigenfunctions of $\mathcal{L}_R$ corresponding to negative eigenvalues, which come from $\mathcal{L}_3$ (if they exist) are orthogonal to $\ker(\mathcal{L}_I)$.

(ii) Since $\mathcal{L}_2$ has no negative eigenvalues, all the negative eigenvalues of $\mathcal{L}_I$ come from the operator $\mathcal{L}_3$ (if there exists any negative eigenvalue). Therefore, the statement follows because $\mathcal{L}_3$ is a self-adjoint operator. This completes the proof of the theorem. $\square$

4.2. **Orbital Stability.** In this subsection, we establish the stability result for the periodic travelling waves $(e^{i\gamma t}\phi_\gamma(x), 0)$, where ϕ_γ is a dnoidal wave given by Theorem 3.1. To make clear our notion of orbital stability, we note that the system (1.1) has phase and translation symmetries, i.e., if $(u(x,t), w(x,t))$ is a solution of (1.1) so are

$$(e^{is}u(x,t), e^{is}w(x,t)) \quad \text{and} \quad (u(x+r,t), w(x+r,t)),$$

for any $r, s \in \mathbb{R}$ (we denote these symmetries by $T_p(s)$ and $T_{tr}(r)$, respectively). Therefore, by orbital stability we mean stability modulo phase and space translation. More precisely.

Definition 4.5. Let $X_1 = H^1_{\rm per}([0,L]) \times H^1_{\rm per}([0,L])$. A travelling-wave solution for (1.1), $\Phi(x,t) = (e^{i\gamma t}\phi_\gamma(x), e^{i\gamma t}\psi_\gamma(x))$, is said to be orbitally stable in X_1 (or X_1-stable) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z_0 \in X_1$ and $\|z_0 - (\phi_\gamma, \psi_\gamma)\|_{X_1} < \delta$, then the solution $z(t) = (u(t), w(t))$ of (1.1) with $z(0) = z_0$ exists for all t and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{s,r\in\mathbb{R}} \|z(t) - T_p(s)T_{tr}(r)(\phi_\gamma, \psi_\gamma)\|_{X_1} < \varepsilon.$$

Otherwise, we say that $\Phi(x,t)$ is orbitally unstable in X_1 (or X_1-unstable).

Our stability result is as follows.

Theorem 4.6. *Let $\gamma \in \left(\frac{2\pi^2}{L^2}, \infty\right)$, and assume that $\sigma, \eta > 0$ satisfy $\eta > \sigma$. Then, for ϕ_γ given by Theorem* 3.1*, the periodic travelling waves $\Phi_\gamma(x,t) = (e^{i\gamma t}\phi_\gamma(x), 0)$ are orbitally stable in X_1.*

Proof. The idea is to apply the theory developed by Grillakis, Shatah, and Strauss [13] for abstract Hamiltonian system. To do so, we first note that from Corollary 4.3, the real Hilbert space $X_\mathbb{R} := [H^1_{\rm per}([0,L])]^4$ can be orthogonally decomposed as

$$X_\mathbb{R} = \mathcal{N} \oplus \ker(\mathcal{L}_\gamma) \oplus \mathcal{P},$$

where $\mathcal{N}$ denotes the negative eigenspace of $\mathcal{L}_\gamma$ and $\mathcal{P}$ is a closed subspace such that $\langle \mathcal{L}_\gamma p, p\rangle \geq \vartheta_0 \|p\|^2_{X_1}$, for all $p \in \mathcal{P}$ and some $\vartheta_0 > 0$.

Next, for $\gamma \in I = \left(\frac{2\pi^2}{L^2}, \infty\right)$ and $\Phi_\gamma = (\phi_\gamma, 0, 0, 0)$, we define the real function

$$d(\gamma) = \mathcal{H}(\Phi_\gamma) + \gamma\mathcal{F}(\Phi_\gamma), \tag{4.10}$$

Hence, since Φ_γ is a critical point of the functional $\mathcal{H} + \gamma\mathcal{F}$, $\ker(\mathcal{L}_\gamma)$ is two-dimensional, and $\mathcal{N}$ is one-dimensional, from the abstract Stability Theorem in [13], we just need to prove that $d''(\gamma) > 0$. To this end, from (4.10), we immediately obtain

$$d'(\gamma) = \mathcal{F}(\Phi_\gamma) = \frac{1}{2}\|\phi_\gamma\|^2_{L^2_{\rm per}([0,L])}. \tag{4.11}$$

But, from the explicit form of ϕ_γ, we calculate

$$\|\phi_\gamma\|^2_{L^2_{\rm per}([0,L])} = \frac{8}{L}K(k)\int_0^{K(k)} dn^2(x;k)dx = \frac{8}{L}K(k)E(k), \tag{4.12}$$

where in the last equality we have used that $\int_0^{K(k)} dn^2(x;k)dx = E(k)$ (see [8, pg. 10], here $E(k)$ denotes the complete elliptic integral of the second kind). So, from (4.11) and (4.12), we deduce that

$$d''(\gamma) = \frac{4}{L}\frac{d}{dk}[K(k)E(k)]\frac{dk}{d\omega}\frac{1}{\eta}.$$

Since the mapping $k \in (0,1) \mapsto K(k)E(k)$ is a strictly increasing function, and because from Theorem 3.1 we have $\frac{dk}{d\omega} > 0$, the theorem follows. $\square$

Remark. Note that our stability result includes the case of two waves of different polarizations and the case for two waves of different frequencies ω_1 and ω_2 with $\omega_2 \ll \omega_1$ (see the Introduction).

4.3. **Orbital Instability.** In Subsection 4.2, we established a orbital stability result for the periodic waves $\Phi_\gamma(x,t) = (e^{i\gamma t}\phi_\gamma(x), 0)$, where ϕ_γ is given by Theorem 3.1, by periodic perturbations which have the same fundamental period of ϕ_γ. In this subsection, we ask ourselves if such waves are stable/unstable when we consider periodic perturbations which have twice the fundamental period of ϕ_γ. As we will see below, in this case the waves $\Phi_\gamma(x,t)$ are orbitally unstable but in a weaker sense than in Definition 4.5. Actually, here our notion of orbital stability is slightly different from that in Subsection 4.2 and does not include space translations.

Definition 4.7. Let $X_2 = H^1_{\rm per}([0,2L]) \times H^1_{\rm per}([0,2L])$. The orbit generated modulo phase, $\{T_p(\gamma s)(\phi_\gamma,\psi_\gamma); s \in \mathbb{R}\}$, is said to be orbitally stable in X_2 (or X_2-stable) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z_0 \in X_2$ and $\|z_0 - (\phi_\gamma,\psi_\gamma)\|_{X_2} < \delta$, then the solution $z(t) = (u(t), w(t))$ of (1.1) with $z(0) = z_0$ exists for all t and satisfies

$$\sup_{t\in\mathbb{R}}\inf_{s\in\mathbb{R}} \|z(t) - T_p(s)(\phi_\gamma,\psi_\gamma)\|_{X_2} < \varepsilon.$$

Otherwise, the orbit is said to be orbitally unstable in X_2 (or X_2-unstable).

Here, we follow the approach introduced by Grillakis in [11], [12] (see also [4], [5]), which get orbital instability from the linear instability of the zero solution for the linearization of (1.4) around the orbit $\{T_p(\gamma s)(\phi_\gamma,\psi_\gamma); s \in \mathbb{R}\}$.

To start, we define the orbit $\mathcal{O}$ to be

$$\mathcal{O} = \{T_p(\gamma s)(\phi_\gamma, 0, 0, 0); s \in \mathbb{R}\},$$

where $\phi_\gamma = \phi$ is the dnoidal wave given by Theorem 3.1. Next, for $\Phi = (\phi_\gamma, 0, 0, 0)$ and $U = (P, R, Q, S)$ (recall that we have written $u = P + iQ$, $w = R + iS$), we define $V = V(t)$ to be

$$V = T_p(-\gamma t)U - \Phi.$$

By using the group properties of $T_p(s)$ together with the fact that Φ is a critical point of the functional $\mathcal{H} + \gamma\mathcal{F}$, it is easy to see, from the Taylor expansion and (4.1), that $V(t)$ satisfies

$$\frac{dV}{dt} = J\mathcal{L}_\gamma V + O(\|V\|^2), \tag{4.13}$$

where J and $\mathcal{L}_\gamma$ are the operators defined in (4.2) and (4.4), respectively.

To prove that the linearized equation (4.13) has zero as an unstable solution, it is well known that it suffices to show that $J\mathcal{L}_\gamma$ has one and finitely many eigenvalues with strictly positive real part. Moreover, this implies that the orbit $\mathcal{O}$ is orbitally unstable (see [11], [13], [29]). Keeping this in mind, we prove the following result.

Theorem 4.8. *Let $\gamma \in \left(\frac{2\pi^2}{L^2}, \infty\right)$ and $\sigma, \eta > 0$. Then, for $\phi = \phi_\gamma$ given by Theorem 3.1, the orbit*

$$\mathcal{O} = \{T_p(\gamma t)(\phi, 0, 0, 0); t \in \mathbb{R}\}$$

is X_2-unstable by the periodic flow of the system (1.4)*.*

Proof. As we have already pointed out, we only have to prove that the operator $J\mathcal{L}_\gamma$ has one and finitely many eigenvalues with strictly positive real part. But, from Lemma 5.6 and Theorem 5.8 in [13], we know that $J\mathcal{L}_\gamma$ has finitely many eigenvalues with strictly positive real part. Now, to prove that $J\mathcal{L}_\gamma$ has at least one eigenvalue with strictly positive real part, in light of Theorem 2.6 in [12], we define

$$\begin{gathered} Y = [\ker(\mathcal{L}_R) \cup \ker(\mathcal{L}_I)]^\perp, \\ \widehat{\mathcal{L}}_R = \text{restriction of } \mathcal{L}_R \text{ on } Y \cap H^2_{\rm per}([0, 2L]), \\ \widehat{\mathcal{L}}_I^{-1} = \text{restriction of } \mathcal{L}_I^{-1} \text{ on } Y \cap H^2_{\rm per}([0, 2L]). \end{gathered}$$

With these notation, Theorem 2.6 in [12] states that $J\mathcal{L}_\gamma$ has exactly

$$\max\{n(\widehat{\mathcal{L}}_R), n(\widehat{\mathcal{L}}_I^{-1})\} - d(C(\widehat{\mathcal{L}}_R) \cap C(\widehat{\mathcal{L}}_I^{-1})) \tag{4.14}$$

$\pm$ pairs of real eigenvalues, where $C(\mathcal{L}) = \{y \in Y; \langle \mathcal{L}y, y\rangle < 0\}$ denotes the negative cone of the operator $\mathcal{L}$ and $d(C(\mathcal{L}))$ denotes the dimension of a maximal linear subspace that is contained in $C(\mathcal{L})$.

Therefore, we have proved the theorem if we show that the number in (4.14) is strictly positive. Since $\mathcal{L}_R$ is a self-adjoint operator on $L^2_{\rm per}([0, 2L])$, its negative eigenspace is orthogonal to its kernel (the same conclusion holds for the operator $\mathcal{L}_I$). Thus, from Theorem 4.4, we have $n(\widehat{\mathcal{L}}_R) = k_0 + 2$ and $n(\widehat{\mathcal{L}}_I) = k_0$. Moreover, from the structure of the operators $\mathcal{L}_R$ and $\mathcal{L}_I$, we see that the negative cone $C(\widehat{\mathcal{L}}_R) \cap C(\widehat{\mathcal{L}}_I^{-1})$ is k_0-dimensional; that is,

$$d(C(\widehat{\mathcal{L}}_R) \cap C(\widehat{\mathcal{L}}_I^{-1})) = k_0.$$

Hence,

$$\max\{n(\widehat{\mathcal{L}}_R), n(\widehat{\mathcal{L}}_I^{-1})\} - d(C(\widehat{\mathcal{L}}_R) \cap C(\widehat{\mathcal{L}}_I^{-1})) = k_0 + 2 - k_0 = 2.$$

This completes the proof of the theorem. □

5. Orbital Instability of the non-semitrivial solutions

This section is mainly devoted to prove the instability results concerning the dnoidal-wave solutions given in Theorem 3.3. For the sake of simplicity, throughout this section we take $\eta = 1$. So, the coupled system (1.4) admits the solutions in Theorem 3.3 for $\sigma \neq 1$ and $b^2 = 1$. We assume throughout that $b = 1$, but a similar analysis can be performed if $b = -1$.

5.1. **Spectral Analysis and Stability/Instability.** Fix $L > 0$ and let $\widetilde{\Phi} = (\phi_\gamma, \phi_\gamma, 0, 0)$, where $\phi = \phi_\gamma$ is the dnoidal wave given by Theorem 3.3 (with $\eta = b = 1$). Consider the linearized operator

$$\mathcal{T}_\gamma = \mathcal{H}''(\widetilde{\Phi}) + \gamma\mathcal{F}''(\widetilde{\Phi}) = \begin{pmatrix} \mathcal{T}_R & 0 \\ 0 & \mathcal{T}_I \end{pmatrix}, \tag{5.1}$$

where

$$\mathcal{T}_R = \begin{pmatrix} -\frac{d^2}{dx^2} + \gamma - (3+\sigma)\,\phi_\gamma^2 & -2\sigma\phi_\gamma^2 \\ -2\sigma\phi_\gamma^2 & -\frac{d^2}{dx^2} + \gamma - (3+\sigma)\,\phi_\gamma^2 \end{pmatrix}, \tag{5.2}$$

$$\mathcal{T}_I = \begin{pmatrix} -\frac{d^2}{dx^2} + \gamma - (1+\sigma)\,\phi_\gamma^2 & 0 \\ 0 & -\frac{d^2}{dx^2} + \gamma - (1+\sigma)\,\phi_\gamma^2 \end{pmatrix}. \tag{5.3}$$

Our first result is about the study of the spectra of the operators $\mathcal{T}_R$ and $\mathcal{T}_I$.

Theorem 5.1. *Let $\phi = \phi_\gamma$ be the dnoidal wave given by Theorem 3.3 and assume that $\sigma > 0$. Then the following spectral properties hold.*

(i) *If $\sigma > 1$ then the operator $\mathcal{T}_R$ in (5.2) defined in $L^2_{\rm per}([0,L])$ with domain $H^2_{\rm per}([0,L])$ has exactly one negative eigenvalue which is simple; zero is a simple eigenvalue. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

(ii) *If $0 < \sigma < 1$ then the operator $\mathcal{T}_R$ in (5.2) defined in $L^2_{\rm per}([0,L])$ with domain $H^2_{\rm per}([0,L])$ has exactly two negative eigenvalue which are simple; zero is a simple eigenvalue. Moreover the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

(iii) *The operator $\mathcal{T}_I$ in (5.3) defined in $L^2_{\rm per}([0,L])$ with domain $H^2_{\rm per}([0,L])$ has no negative eigenvalues; its kernel is two-dimensional and spanned by $(0,\phi)$ and $(\phi,0)$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. Let

$$\mathcal{T}_1 = -\frac{d^2}{dx^2} + \gamma - 3(1+\sigma)\phi_\gamma^2, \quad \mathcal{T}_2 = -\frac{d^2}{dx^2} + \gamma - (1+\sigma)\,\phi_\gamma^2. \tag{5.4}$$

It is not difficult to check that the same results as in Lemma 4.2 hold for the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ replacing $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. So, since $\mathcal{T}_2$ has no negative eigenvalues and zero is a simple eigenvalue we have proved part (iii).

To show parts (i) and (ii), we note that the operator $\mathcal{T}_R$ can be diagonalized under a similarity transformation. Indeed, let

$$\mathcal{A}_R = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{5.5}$$

Then

$$\mathcal{T}_{DR} := \mathcal{A}_R \mathcal{T}_R \mathcal{A}_R^{-1} = \begin{pmatrix} \mathcal{T}_1 & 0 \\ 0 & \mathcal{T}_3 \end{pmatrix} \tag{5.6}$$

where $\mathcal{T}_1$ is defined in (5.4) and

$$\mathcal{T}_3 = \mathcal{T}_1 + 4\sigma\phi_\gamma^2. \tag{5.7}$$

Since $\mathcal{T}_1$ has exactly one negative eigenvalue, zero is the second eigenvalue and $\sigma > 0$, the Comparison Theorem implies that $\mathcal{T}_3$ has at most one negative eigenvalue and the second eigenvalue is strictly positive.

Now, note that from (5.7), we can also write

$$\mathcal{T}_3 = \mathcal{T}_2 + 2(\sigma - 1)\phi_\gamma^2. \tag{5.8}$$

Because $\mathcal{T}_2$ has no negative eigenvalues, it follows from (5.8), the above observation, and the Comparison Theorem that $\mathcal{T}_3$ is a strictly positive operator if $\sigma > 1$, and it has a unique negative eigenvalue if $0 < \sigma < 1$. This proves the theorem. $\square$

Remark 5.2. It follows from Theorem 5.1 that the kernel of the operator $\mathcal{T}_\gamma$ is 3-dimensional. Hence, the Grillakis, Shatah, and Strauss theory cannot be applied to get a stability result as in Theorem 4.6.

In view of Remark 5.2, we propose in proving an instability result (modulo phase) in the same spirit of Subsection 4.3. First, we linearize (1.4) around the orbit $\widetilde{\mathcal{O}} = \{T_p(\gamma s)(\phi_\gamma, \phi_\gamma, 0, 0); s \in \mathbb{R}\}$. Defining $W = W(t)$ to be

$$W = T_p(-\gamma t)U - (\phi_\gamma, \phi_\gamma, 0, 0),$$

we get the linearized equation

$$\frac{dW}{dt} = J\mathcal{T}_\gamma W, \tag{5.9}$$

where J is the matrix in (4.2) and $\mathcal{T}_\gamma$ is defined in (5.1).

The first natural question is the following: is the orbit $\widetilde{\mathcal{O}}$ unstable with regard to periodic perturbations which have the same fundamental period as ϕ_γ? In this case, the method of proof employed in Theorem 4.8 does not give a result concerning the orbital instability as in Subsection 4.3, and we can only prove a spectral stability result. More precisely, we prove the following.

Theorem 5.3. *Let $\phi = \phi_\gamma$ be the dnoidal wave given by Theorem 3.3. Then the orbit*

$$\widetilde{\mathcal{O}} = \{T_p(\gamma s)(\phi_\gamma, \phi_\gamma, 0, 0); s \in \mathbb{R}\}$$

is spectrally stable, which is to say, the spectrum of the linearized operator $J\mathcal{T}_\gamma$, in the space $L^2_{\rm per}([0, L])$, entirely lies on the imaginary axis.

Proof. As in the proof of Theorem 4.8, we already know that the operator $J\mathcal{T}_\gamma$ has finitely many eigenvalues with strictly positive real part. Moreover, since $n(\mathcal{T}_I) = 0$ (see Theorem 5.1), we deduce that the unstable eigenvalues of $J\mathcal{T}_\gamma$, that is, those with a strictly positive real part, may occur only as real positive eigenvalues (see e.g. [11] or [26]). Thus, since $J\mathcal{T}_\gamma$ has exactly

$$\max\{n(\widehat{\mathcal{T}}_R), n(\widehat{\mathcal{T}}_I^{-1})\} - d(C(\widehat{\mathcal{T}}_R) \cap C(\widehat{\mathcal{T}}_I^{-1})) \tag{5.10}$$

$\pm$ pairs of real eigenvalues (see Subsection 4.3 for the definitions), we only have to prove that such a number is zero. But, since $n(\mathcal{T}_I) = 0$, the number in (5.10) reduces to $n(\widehat{\mathcal{T}}_R)$. So, our task is to show that no eigenfunctions of $\mathcal{T}_R$, associated to negative eigenvalues, are orthogonal to $\ker(\mathcal{T}_I)$. To do this, let $\overrightarrow{u} = (u_1, u_2) \neq 0$ and $\lambda < 0$ such that $\mathcal{T}_R\overrightarrow{u} = \lambda\overrightarrow{u}$. From (5.6), we see that

$$\mathcal{A}_R\overrightarrow{u} = \begin{pmatrix} u_1 + u_2 \\ -\frac{1}{2}u_1 + \frac{1}{2}u_2 \end{pmatrix}$$

is an eigenfunction of $\mathcal{T}_{DR}$ associated to the negative eigenvalue λ. As a consequence, either $\mathcal{T}_1 u = \lambda u$ or $\mathcal{T}_3 v = \lambda v$, where $u = u_1 + u_2$ and $v = \frac{1}{2}(u_2 - u_1)$. Now, from Theorem 5.1, it follows that $u \equiv 0$ or $v \equiv 0$. This means that either

(i) $\overrightarrow{u} = (-u_2, u_2)$ and $\mathcal{T}_3 u_2 = \lambda u_2$; or
(ii) $\overrightarrow{u} = (u_1, u_1)$ and $\mathcal{T}_1 u_1 = \lambda u_1$.

Because $\mathcal{T}_1$ and $\mathcal{T}_3$ have at most one negative eigenvalue, it follows from the Floquet theory that if (i) occurs then $u_2 > 0$ and if (ii) occurs then $u_1 > 0$. Hence, for any $\overrightarrow{\phi} = (\alpha_1\phi_\gamma, \alpha_2\phi_\gamma) \in \ker(\mathcal{T}_I)$ with $\alpha_1 \neq \alpha_2$, we have $\langle\overrightarrow{u}, \overrightarrow{\phi}\rangle_{L^2_{\rm per}} \neq 0$. In consequence, $\overrightarrow{u} \notin [\ker(\mathcal{T}_I)]^\perp$. This completes the proof of the theorem. $\square$

In analogy to Theorem 4.8, we can also prove that the orbit $\widetilde{\mathcal{O}}$ is orbitally unstable with regard to periodic perturbations which have twice the fundamental period of ϕ_γ.

Theorem 5.4. *Let $\phi = \phi_\gamma$ be the dnoidal wave given by Theorem* 3.3*. Then the orbit*

$$\widetilde{\mathcal{O}} = \{T_p(\gamma s)(\phi_\gamma, \phi_\gamma, 0, 0); s \in \mathbb{R}\}$$

is X_2-unstable, in the sense of Definition 4.7, by the periodic flow of system (1.4)*.*

Proof. The proof follows the same arguments as in Theorem 4.8. Actually, similarly to the proof of Theorem 4.4, we can prove that the operator $\mathcal{T}_1$, with domain $H^2_{\rm per}([0, 2L])$ has exactly three negative eigenvalues, for which, the eigenfunctions corresponding to the second and third eigenvalues are orthogonal to ϕ_γ. Therefore,

$$\max\{n(\widehat{\mathcal{T}}_R), n(\widehat{\mathcal{T}}_I^{-1})\} - d(C(\widehat{\mathcal{T}}_R) \cap C(\widehat{\mathcal{T}}_I^{-1})) = n(\widehat{\mathcal{T}}_R) \geq 2,$$

and the theorem is proved. □

Finally, we observe that the solutions in Theorem 3.3 also make sense when $-1 < \sigma < 0$. In this case, Theorem 5.1, (5.7), and the Comparison Theorem imply that the operator $\mathcal{T}_3$, defined in $L^2_{\rm per}([0, L])$ with domain $H^2_{\rm per}([0, L])$, has at least two negative eigenvalues and so $\mathcal{T}_R$ has at least three negative eigenvalues. Thus, we can establish the following.

Theorem 5.5. *Let $\phi = \phi_\gamma$ be the dnoidal wave given by Theorem* 3.3 *and $-1 < \sigma < 0$. Then the orbit*

$$\widetilde{\mathcal{O}} = \{T_p(\gamma s)(\phi_\gamma, \phi_\gamma, 0, 0); s \in \mathbb{R}\}$$

is X_1-unstable, in the sense of Definition 4.7, by the periodic flow of system (1.4)*.*

Proof. As we have already pointed out, the advantage in using the Grillakis approach is that we do not need to know the exact number of negative eigenvalues of the operator $\mathcal{T}_\gamma$, but only to have an estimate of the number in (5.10). In the present case, since

$$\max\{n(\widehat{\mathcal{T}}_R), n(\widehat{\mathcal{T}}_I^{-1})\} - d(C(\widehat{\mathcal{T}}_R) \cap C(\widehat{\mathcal{T}}_I^{-1})) = n(\widehat{\mathcal{T}}_R)$$

we can use the estimate (see e.g. [5] or [26])

$$n(\widehat{\mathcal{T}}_R) \geq n(\mathcal{T}_R) - \dim(\ker(\mathcal{T}_I)) \tag{5.11}$$

to conclude that

$$\max\{n(\widehat{\mathcal{T}}_R), n(\widehat{\mathcal{T}}_I^{-1})\} - d(C(\widehat{\mathcal{T}}_R) \cap C(\widehat{\mathcal{T}}_I^{-1})) \geq 3 - 2 = 1.$$

This proves the theorem. □

Remark 5.6. The estimate (5.11) may be proved following similar arguments as in the proof of [11, Theorem 3.2] (see also [31]).

Acknowledgments. The author was supported by grant no. 152234/2007-1 from CNPq-Brazil. The author thanks Professor Felipe Linares for reading the first draft of the paper. The author also thanks the referee for the suggestions, which improved the presentation of the paper.

References

[1] G. Agrawal; *Nonlinear Fiber Optics*, Academic Press, Boston, (1989).
[2] A. Ambrosetti and E. Colorado; Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Acad. Sci. Paris, Ser. I* **342** (2006), 453–458.
[3] J. Angulo; Stability of periodic travelling waves solutions to the Schrödinger and modified Korteweg-de Vries equations, *J. Differential Equations* **235** (2007), 1-30.
[4] J. Angulo and F. Linares; Periodic pulses of coupled nonlinear Schrödinger equations in optics, *Indiana Univ. Math. J.* **56** (2007), 847–878.
[5] J. Angulo and A. Pastor; Stability of periodic optical solitons for a nonlinear Schrödinger system, *Proc. Roy. Soc. Edinburgh Sect. A* **139** (2009), 927–959.
[6] J. Bourgain; Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II, *Geom. Funct. Anal.* **3** (1993), 107–156, 209–262.
[7] J. Bourgain; *Global Solutions of Nonlinear Schrödinger Equations*, Amer. Math. Soc. Coll. Publ., **46**, American Mathematical Society, Providence (1999).
[8] P. F. Byrd and M. D. Friedman; *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer Verlag, Nova York (1971).
[9] D. G. de Figueiredo and O. Lopes; Solitary waves for some nonlinear Schrödinger systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), 149–161.
[10] M.S.P. Eastham; *The Spectral Theory of Periodic Differential Equtions*, Scottish Academic Press, London (1973).
[11] M. Grillakis; Linearized instability for nonlinear Schrödinger and Klein-Gordon equations, *Comm. Pure Appl. Math.* **41** (1988), 747–774.
[12] M. Grillakis; Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system, *Comm. Pure Appl. Math.* **43** (1990), 299–333.
[13] M. Grillakis, J. Shatah, and W. Strauss; Stability theory of solitary waves in the presence of symmetry II, *J. Funct. Anal.* **94** (1990), 308–348.
[14] E. L. Ince; The periodic Lamé functions, *Proc. Roy. Soc. Edinburgh* **60** (1940), 47–63.
[15] C. E. Kenig, G. Ponce, and L. Vega; Quadratic forms for the 1-D semilinear Schrödinger equation, *Trans. Amer. Math. Soc.* **348** (1996), 3323–3353.
[16] C. E. Kenig, G. Ponce, and L. Vega; The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, *Duke Math. J.* **71** (1993), 1–21.
[17] Y. S. Kivshar; Bright and dark spatial solitons in non-Kerr media, *J. Opt. Quantum Electron.* **30** (1998), 571–614.
[18] Y. S. Kivshar and B. Luther-Davies; Dark optical solitons: physics and applications, *Physics Reports* **298** (1998), 81–197.
[19] T. C. Lin and J. Wei; Ground states of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Commun. Math. Phys.* **255** (2005), 629–653.
[20] T. C. Lin and J. Wei, Erratum; Ground states of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Commun. Math. Phys.* **277** (2008), 573–576.
[21] W. Magnus and Winkler; *Hill's Equation*, vol **20**, Interscience, Tracts in Pure and Appl. Math., 1976.
[22] C. G. Menyuk; Pulse propagation in an elliptically birefringent Kerr medium, *IEEE. J. Quantum Electron* **25** (1989) 2674–2682.
[23] M. Ohta; Stability of solitary waves for coupled nonlinear Schrödinger equations, *Nonlinear Anal.* **26** (1996), 933–939.
[24] A. Pastor; Nonlinear and spectral stability of periodic travelling wave solutions for a nonlinear Schrödinger system, *Differential Integral Equations* **23** (2010), 125–154.
[25] D. E. Pelinovsky and Y. S. Kivshar; Stability criterion for multicomponent solitary waves, *Phys. Rev. E* **62** (2000), 8668–8676.
[26] D. E. Pelinovsky; Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations, *Proc. Roy. Soc. Edinburgh Sect. A* **461** (2005), 783–812.
[27] A. R. Sammut, A. V. Buryak, and Y. S. Kivshar; Modification of solitary waves by third-harmonic generation, *Opt. Lett.* **22** (1997), 1385–1387.
[28] A. R. Sammut, A. V. Buryak, and Y. S. Kivshar; Bright and dark solitary waves in the presence of the third-harmonic generation, *J. Opt. Soc. Am. B* **15** (1998), 1488–1496.
[29] J. Shatah and W. Strauss; Spectral condition for instability, *Contemp. Math.* **255** (2000), 189–198.

[30] B. Sirakov; Least energy solitary waves for a system of nonlinear schrödinger equations in $\mathbb{R}^n$, *Commun. Math. Phys.* **271** (2007), 199–221.

[31] A. C. Yew; Stability analysis of multipulses in nonlinearly-coupled Schrödinger equations, *Indiana Univ. Math. J.* **49** (2000), 1079–1124.

Ademir Pastor Ferreira
IMPA, Estrada Dona Castorina 110, CEP 22460-320, Rio de Janeiro, RJ, Brazil
E-mail address: apastor@impa.br