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EXISTENCE OF POSITIVE AND SIGN-CHANGING SOLUTIONS FOR p-LAPLACE EQUATIONS WITH POTENTIALS IN $\mathbb{R}^N$

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ABSTRACT. We study the perturbed equation

$$-\varepsilon^p \operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = h(x,u) + K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N$$
$$u(x) \to 0 \quad \text{as } |x| \to \infty .$$

where $2 \le p < N$, $p^* = \frac{pN}{N-p}$, $p < q < p^*$. Under proper conditions on $V(x)$ and $h(x,u)$, we obtain the existence and multiplicity of solutions. We also study the existence of solutions which change sign.

1. Introduction

In this article, we study the equation

$$\begin{gathered}-\varepsilon^p \operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = h(x,u) + K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty\end{gathered} \tag{1.1}$$

where $2 \le p < N$, $p^* = \frac{pN}{N-p}$, $p < q < p^*$, $K(x)$ is a bounded positive functions, and $h(x,u)$ is a superlinear but subcritical function.

When $p = 2$ and $\varepsilon = 1$, this problem is a Schrodinger equation which has been extensively studied; see for example [1, 2, 3, 4, 6, 7, 10, 13, 15]. Authors have used different methods to study this equation. In [18], the authors established many embedding results of weighted Sobolev spaces of radially symmetric functions which be used to obtain ground state solutions. In [6], the authors studied the dependence upon the local behavior of V near its global minimum. In [3], the authors used spectral properties of the Schrodinger operator to study nonlinear Schrodinger equations with steep potential well. In [13], Ding and Szulkin used Rabinowitz's linking theorem to study the equation. In [15], Ding and Szulkin used index theory obtain many solutions of the equation. In [10], the author imposed on functions k and K conditions ensuring that this problem can be written in a variational form. We know that $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space for $1 < p < N$, except for $p = 2$. The space $W^{1,p}(\mathbb{R}^N)$ with $p \neq 2$ does not satisfy the Lieb lemma (see for example [19]). Using $\mathbb{R}^N$ results in the loss of compactness. So there are

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many difficulties to overcome when we study (1.1) of $p \neq 2$ by the usual methods. There seems to be very little work on the case $p \neq 2$, to the best of our knowledge. In this article, we overcome these difficulties and study (1.1) of $p \geq 2$.

When $V(x)$ is a constant and $\varepsilon = 1$, (1.1) becomes the quasilinear elliptic equation

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = f(x,u), \quad \text{in } \Omega \\ u \in W_0^{1,p}(\Omega), \quad u \neq 0 \end{gathered} \tag{1.2}$$

where $1 < p < N$, $N \geq 3$, λ is a parameter, Ω is an unbounded domain in $\mathbb{R}^N$. There are many results about it we can see [5, 8, 9, 11, 12, 24]. Because of the unboundedness of the domain, the Sobolev compact embedding does not hold. There are many methods to overcome this difficulty. In [24], the author used that the projection $u \mapsto f(x,u)$ is weak continuous in $W_0^{1,p}(\Omega)$ to consider the problem. In [8, 9], the authors studied the problem in symmetric Sobolev spaces which possess Sobolev compact embedding. By the result and a min-max procedure formulated by Bahri and Li [5], they considered the existence of positive solutions of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = q(x)u^{\alpha} \quad \text{in } \mathbb{R}^N,$$

where $q(x)$ satisfies many conditions. We can see if $V(x)$ is not constant, then it can not be easily proved by the above methods. In [23], the authors used the concentration-compactness principle posed by Lions and the mountain pass lemma to solve problem with this situation.

Tarantello [21] studied the equation

$$\begin{gathered} -\Delta u = |u|^{2^*-2} + f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is open bounded set. She showed that for f satisfying a suitable condition and $f \neq 0$, the equation (1.3) admits two solutions u_0 and u_1 in $H_0^1(\Omega)$. She used suitable minimization and minimax principles of mountain pass type. The author got the results when f satisfies the following condition

$$\int_\Omega fu \leq C_N(\|\nabla u\|_2)^{(N+2)/2}$$

where $C_N = \frac{4}{N-2}(\frac{N-2}{N+2})^{(N+2)/4}$.

Radulescu and Smets [18] proved existence results for the non autonomous perturbations of critical singular elliptic boundary value problem

$$\begin{gathered} -\operatorname{div}(|x|^{\alpha}\nabla u) = |u|^{2^*-2} + f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where f satisfies suitable conditions. They proved a corresponding multiplicity result for the degenerate problem (1.4). In their case, Ω can be unbounded.

Silva and Xavier [20] used the symmetric Mountain Pass Theorem and the concentration-compactness principle to prove the multiplicity of solutions for the following equation under the presence of symmetry

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu |u|^{p^*-2}u + f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

where $f(x,s)$ is odd and also subcritical in s, and Ω is a bounded smooth domain in $\mathbb{R}^N$. They used the concentration-compactness principle to prove that the Palais-Smale condition is satisfied below a certain level.

In this paper, we inspired by [14, 18, 20, 21, 22] use critical point theory to study the equation (1.2). We extend the equation in [18, 20, 21] where function $V(x) \neq 0$, $\varepsilon \neq 1$, $K(x) \neq 1$ and $p \geq 2$. We will obtain the similar multiplicity results with [18, 20, 21]. However, our method has essential differences with the methods used in [18, 20, 21]. Also we obtain the existence of sign-changing solutions. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all. Since we have to overcome two main difficulties; one is that $\mathbb{R}^N$ results in the loss of compactness; the other is that $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space for $1 < p < N$ and it does not satisfy the Lieb lemma, except for $p = 2$. So we need more delicate estimates and careful analysis. We obtain the existence and the multiplicity of solutions in Theorems 2.1 and 2.2. By the Theorem 2.3 we can obtain the existence of sign-changing solutions.

This paper is organized as follows. In Section 2, we state some condition and main results. Section 3 we obtain many lemmas which will be used in the next section. Section 4 we give the proof of the main result of the paper.

2. Main Results

We make the following assumptions

- (V0) $V \in C(\mathbb{R}^N)$; $\min V = 0$; and there is $b > 0$ such that the set $\upsilon^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure.
- (K0) $K \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$.
- (H0)
  - $h \in C(\mathbb{R}^N \times \mathbb{R})$ and $h(x,u) = o(|u|^{p-1})$ uniformly in x as $u \to 0$;
  - there are $C_0 > 0$ and $q < p^*$ such that $|h(x,u)| \leq C_0(1 + |u|^{q-1})$ for all (x,u);
  - there are $a_0 > 0$, $s > p$ and $\mu > p$ such that $H(x,u) \geq a_0|u|^s$ and $\mu H(x,u) \leq h(x,u)u$, where $H(x,u) = \int_0^u h(x,s)ds$.
- (S) V,K and h are Holder continuous, and there is an orthogonal involution τ such that $V(\tau x) = V(x)$, $K(\tau x) = K(x)$ and $H(\tau x,.) = H(x,.)$ for all $x \in \mathbb{R}^N$.

An example satisfying (H0) is the function $h(x,u) = P(x)|u|^{s-2}u$ with $p < s < p^*$ and $P(x)$ being positive and bounded. Let $\lambda = \varepsilon^{-p}$. (1.1) reads then as

$$
\begin{gathered}
-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda V(x)|u|^{p-2}u = \lambda h(x,u) + \lambda K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N \\
u(x) \to 0 \quad \text{as } |x| \to \infty
\end{gathered}
\tag{2.1}
$$

We introduce the space

$$
E = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\}.
$$

It follows from (V0) and Poincare inequality that E continuously in $W^{1,p}(\mathbb{R}^N)$. It is thus clear that, for each $s \in [p,p^*]$, there is $\upsilon_s > 0$ independent of λ such that if $\lambda \geq 1$,

$$
|u|_s \leq \upsilon_s\|u\| \leq \upsilon_s\|u\|_\lambda \quad \text{for all } u \in E \tag{2.2}
$$

Set

$$
\begin{aligned}
g(x,u) &= K(x)|u|^{p^*-2}u + h(x,u), \\
G(x,u) &= \int_0^u g(x,s)ds = \frac{1}{p^*}K(x)|u|^{p^*} + H(x,u)
\end{aligned} \tag{2.3}
$$

Consider the functional

$$
\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda V(x)|u|^p) - \lambda\int_{\mathbb{R}^N} G(x,u) = \frac{1}{p}\|u\|_\lambda^p - \lambda\int_{\mathbb{R}^N} G(x,u).
$$

Under the assumptions, $\Phi_\lambda \in C^1(E,R)$ and its critical points are solutions of $(NS)_\lambda$. Set $g^+(x,u) = g(x,u^+)$, $G^+(x,u) = G(x,u^+)$ and define, on E,

$$
\Psi_\lambda(u) = \frac{1}{p}\|u\|_\lambda^p - \lambda\int_{\mathbb{R}^N} G^+(x,u)
$$

where as usual $u^\pm = \max\{\pm u, 0\}$. Then $\Psi_\lambda \in C^1(E,R)$ and critical points of Ψ_λ are positive solutions of $(NS)_\lambda$.

Let $\{u_n\}$ denote a $(PS)_c$-sequence. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\widetilde{u_j}(x) = \eta(2|x|/j)u(x)$. Then

$$
\|u - \widetilde{u_j}\| \to 0 \quad \text{as } j \to \infty \tag{2.4}
$$

Set

$$
u_n^1 = u_n - \widetilde{u_n}
$$

Then $u_n - u = u_n^1 + (\widetilde{u_n} - u)$ and by (2.4), $u_n \to u$ if and only if $u_n^1 \to 0$. If we can shows that $\lim_{n\to\infty}\Phi_\lambda(u_n^1) \le c - \Phi_\lambda(u)$ and $\Phi'_\lambda(u_n^1) \to 0$. Note that

$$
\Phi_\lambda(u_n^1) - \frac{1}{p}\Phi'_\lambda(u_n^1)u_n^1 \ge \frac{\lambda}{N}\int_{\mathbb{R}^N} K(x)|u_n^1|^{p^*} \ge \frac{\lambda K_{\min}}{N}\int_{\mathbb{R}^N}|u_n^1|^{p^*}
$$

where $K_{\min} = \inf_{x\in R^N} K(x) > 0$, hence

$$
|u_n^1|_{p^*}^{p^*} \le \frac{N(c - \Phi_\lambda(u))}{\lambda K_{\min}} + o(1) \tag{2.5}
$$

Let

$$
V_b(x) = \max\{V(x), b\} \tag{2.6}
$$

where b is the positive constant from the assumption (V0).

Since the set v^b has finite measure and $u_n^1 \to 0$ in L^p_{loc}, we see that

$$
\int_{\mathbb{R}^N} V(x)|u_n^1|^p = \int_{R^N} V_b(x)|u_n^1|^p + o(1).
$$

It follows from the definition (2.3) of $g(x,u)$ and the assumptions (K0) and (H0) that there exists a constant $\gamma_b > 0$ such that

$$
g(x,u)u \le b|u|^p + \gamma_b|u|^{p^*} \quad \text{for all } (x,u) \tag{2.7}
$$

Let S be the best Sobolev constant:

$$
S|u|_{p^*}^p \le \int_{R^N}|\nabla u|^p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N)
$$

In the following we will find special finite-dimensional subspaces by which we construct sufficiently small minimax levels. Recall that the assumption (V0) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$. Observe that, by (H0),

$$
G(x,u) \ge H(x,u) \ge a_0|u|^s
$$

Define the functional $J_\lambda \in C^1(E,R)$ by setting

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda V(x)|u|^p) - a_0\lambda\int_{\mathbb{R}^N}|u|^s$$

Then

$$\Phi_\lambda(u) \le J_\lambda(u) \quad \text{for all } u \in E$$

and it suffices to construct small minimax levels for J_λ.

In $W^{1,p}$ for $p > 1$ the Sobolev constant is never achieved on any domain Ω different from $\mathbb{R}^N$. Moreover, that for $u \in C_0^\infty(\mathbb{R}^N)$ the support of u lies in a fixed compact set Ω different from $\mathbb{R}^N$. And combined with Lions [16, 17]. It implies that

$$\inf\{\int_{\mathbb{R}^N}|\nabla\varphi|^p : \varphi \in C_0^\infty(\mathbb{R}^N), \quad |\varphi|_s = 1\} = 0.$$

For any $\delta > 0$ one can choose $\varphi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\varphi_\delta|_s = 1$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla\varphi_\delta|_p^p < \delta$. Set

$$e_\lambda(x) = \varphi_\delta(\lambda^{1/p}x) \tag{2.8}$$

Then $\operatorname{supp} e_\lambda \subset B_{\lambda^{1/p}r_\delta}(0)$. For $t \ge 0$,

$$\begin{aligned} J_\lambda(te_\lambda) &= \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla e_\lambda|^p + \lambda V(x)|e_\lambda|^p - a_0\lambda t^s\int_{\mathbb{R}^N}|e_\lambda|^s \\ &= \lambda^{1-\frac{N}{p}}(\frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^p + V(\lambda^{-1/p}x)|\varphi_\delta|^p - a_0t^s\int_{\mathbb{R}^N}|\varphi_\delta|^s) \\ &= \lambda^{1-\frac{N}{p}}I_\lambda(t\varphi_\delta), \end{aligned}$$

where $I_\lambda \in C^1(E,R)$ defined by

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p + V(\lambda^{-1/p}x)|u|^p - a_0\int_{\mathbb{R}^N}|u|^s$$

and

$$\max_{t\ge 0} I_\lambda(t\varphi_\delta) = \frac{s-p}{sp(sa_0)^{p/(s-p)}}(\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^p + V(\lambda^{-1/p}x)|\varphi_\delta|^p)^{s/(s-p)}.$$

Since $V(0) = 0$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$, there is $\widehat{\Lambda}_\delta > 0$ such that

$$V(\lambda^{-1/p}x) \le \frac{\delta}{|\varphi_\delta|_p^p} \quad \text{for all } |x| \le r_\delta \quad \text{and} \quad \lambda \ge \widehat{\Lambda}_\delta.$$

This implies that

$$\max_{t\ge 0} I_\lambda(t\varphi_\delta) \le \frac{s-p}{sp(sa_0)^{p/(s-p)}}(2\delta)^{s/(s-p)}. \tag{2.9}$$

Therefore, for all $\lambda \ge \widehat{\Lambda}_\delta$,

$$\max_{t\ge 0}\Phi_\lambda(te_\lambda) \le \frac{s-p}{sp(sa_0)^{p/(s-p)}}(2\delta)^{s/(s-p)}\lambda^{1-\frac{N}{p}}. \tag{2.10}$$

In general, for any $m \in N$, one can choose m functions $\varphi_\delta^j \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\ \varphi_\delta^i \cap \operatorname{supp}\ \varphi_\delta^k = \emptyset$ if $i \ne k$, $|\varphi_\delta^i|_s = 1$ and $|\nabla\varphi_\delta^i|_p^p < \delta$.

Let $r_\delta^m > 0$ be such that $\operatorname{supp}\varphi_\delta^j \subset B_{r_\delta^m}(0)$ for $j = 1,\dots,m$. Set

$$e_\lambda^j(x) = \varphi_\delta^j(\lambda^{1/p}x) \quad \text{for} \quad j = 1,\dots,m$$

and $H^m_{\lambda\delta} = \operatorname{span}\{e^1_\lambda, \dots, e^m_\lambda\}$. Observe that for each $u = \Sigma^m_{j=1} C_j e^j_\lambda \in H^m_{\lambda\delta}$,

$$\int_{\mathbb{R}^N} |\nabla u|^p = \Sigma^m_{j=1} |C_j|^p \int_{\mathbb{R}^N} |\nabla e^j_\lambda|^p,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p = \Sigma^m_{j=1} |C_j|^p \int_{\mathbb{R}^N} V(x)|e^j_\lambda|^p,$$

$$\int_{\mathbb{R}^N} G(x,u) = \Sigma^m_{j=1} \int_{\mathbb{R}^N} G(x, C_j e^j_\lambda)$$

Hence

$$\Phi_\lambda(u) = \Sigma^m_{j=1} \Phi_\lambda(C_j e^j_\lambda)$$

and as before

$$\Phi_\lambda(C_j e^j_\lambda) \le \lambda^{1-\frac{N}{p}} I_\lambda(|C_j| e^j_\lambda)$$

Set

$$\beta_\delta = \max\{|\varphi^j_\delta|^p_p : j = 1, \dots, m\}$$

and choose $\widehat{\Lambda}_{m\delta}$ so that

$$V(\lambda^{-1/p} x) \le \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \le r^m_\delta$$

and $\lambda \ge \widehat{\Lambda}_{m\delta}$. As before, one obtains easily that

$$\sup_{u \in H^m_{\lambda\delta}} \Phi_\lambda(u) \le \frac{s-p}{sp(sa_0)^{p/(s-p)}} (2\delta)^{s/(s-p)} \lambda^{1-\frac{N}{p}} \tag{2.11}$$

for all $\lambda \ge \widehat{\Lambda}_{m\delta}$.

Remark. Let $h(x,u)$ is odd in u and $\tau : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal involution. Then τ induces an involution on E which we denote again by $\tau : E \to E$ as follows $(\tau u)(x) = -u(\tau x)$. If (S) is satisfied, then $\int_{\mathbb{R}^N} G(x, \tau u) = \int_{\mathbb{R}^N} G(x,u)$. This implies that Φ_λ is τ-invariant: $\Phi_\lambda(\tau u) = \Phi_\lambda(u)$ and Φ'_λ is τ-equivalent: $\Phi'_\lambda(\tau u) = \tau \Phi'_\lambda(u)$. In particular, if $\tau u = u$ then $\tau \Phi'_\lambda(u) = \Phi'_\lambda(u)$. Let $E^\tau = \{u \in E : \tau u = u\}$. It is known that critical points of the restriction of Φ_λ on E^τ are solutions of (2.1) satisfying $u(\tau x) = -u(x)$.

We modify the method developed in [14, 18, 20, 21, 22], and obtain the following Theorems.

Theorem 2.1. *Let* (V0), (K0), (H0) *be satisfied. Then for any $\sigma > 0$ there is $\omega_\sigma > 0$ such that if $\varepsilon \le \omega_\sigma$,* (1.1) *has at least one positive solution u_ε of least energy satisfying*

$$\frac{\mu - p}{p} \int_{\mathbb{R}^N} H(x, u_\varepsilon) + \frac{1}{N} \int_{\mathbb{R}^N} K(x)|u_\varepsilon|^{p^*} dx \le \sigma \varepsilon^N, \tag{2.12}$$

$$\frac{\mu - p}{p\mu} \int_{\mathbb{R}^N} (\varepsilon^p |\nabla u_\varepsilon|^p + V(x)|u_\varepsilon|^p) dx \le \sigma \varepsilon^N \tag{2.13}$$

Theorem 2.2. *Let* (V0), (K0), (H0) *be satisfied. If moreover $h(x,u)$ is odd in u, then for any $m \in N$ and $\sigma > 0$ there is $\omega_{m\sigma} > 0$ such that if $\varepsilon \le \omega_{m\sigma}$,* (1.1) *has at least m pairs of solutions u_ε which satisfy the estimates* (2.12) *and* (2.13).

Theorem 2.3. *Let* (V0), (K0), (H0), (S) *be satisfied. If moreover $h(x,u)$ is odd in u, then for any $\sigma > 0$ there exists $\omega_\sigma > 0$ such that if $\varepsilon \le \omega_\sigma$,* (1.1) *has at least one pair of solutions which change sign exactly once and satisfy the estimates* (2.12) *and* (2.13).

3. Preliminaries

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $p \geq 2$. Then*

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p dx + \lim_{n\to\infty} \int_\Omega |\nabla u|^p dx$$

Proof. When $p = 2$, from Brezis-Lieb Lemma (see [11, lemma 1.32]) we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx = \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^2 dx + \lim_{n\to\infty} \int_\Omega |\nabla u|^2 dx$$

when $3 \geq p > 2$, using the lower semi-continuity of the L^p-norm with respect to the weak convergence and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, we deduce

$$\langle |\nabla u_n|^{p-2} \nabla u_n, \nabla u_n \rangle \geq \langle |\nabla u|^{p-2} \nabla u, \nabla u \rangle + o(1)$$

and

$$\begin{aligned} &\lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} (\nabla u_n - \nabla u), \nabla u_n - \nabla u \rangle \\ &\geq 0 = \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} (\nabla u - \nabla u), \nabla u - \nabla u \rangle \end{aligned}$$

So

$$\begin{aligned} \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u_n, \nabla u_n \rangle &\geq \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u_n, \nabla u \rangle \\ &= \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u, \nabla u_n \rangle \\ &= \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u, \nabla u \rangle \end{aligned}$$

Then

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega (|\nabla u_n|^p - |\nabla u|^p) dx \\ &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} (|\nabla u_n|^2 - |\nabla u|^2) dx + \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2} - |\nabla u|^{p-2}) |\nabla u|^2 dx \\ &= \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2} + |\nabla u|^{p-2})(|\nabla u_n|^2 - |\nabla u|^2) dx \\ &\quad + \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2} |\nabla u|^2 - |\nabla u|^{p-2} |\nabla u_n|^2) dx. \end{aligned}$$

From $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, it follows that

$$\lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2} |\nabla u|^2 - |\nabla u|^{p-2} |\nabla u_n|^2) dx = 0\,.$$

So that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega (|\nabla u_n|^p - |\nabla u|^p) dx \\ &= \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p-2} + |\nabla u|^{p-2})(|\nabla u_n|^2 - |\nabla u|^2) dx \\ &\geq \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^{p-2} (|\nabla u_n|^2 - |\nabla u|^2). \end{aligned}$$

So we have

$$\begin{aligned}&\langle|\nabla u_n|^{p-2}\nabla u_n,\nabla u_n\rangle+\langle|\nabla u_n-\nabla u|^{p-2}\nabla u,\nabla u_n\rangle+\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n,\nabla u\rangle\\&\geq\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n,\nabla u_n\rangle+\langle|\nabla u_n-\nabla u|^{p-2}\nabla u,\nabla u\rangle\\&\quad+\langle|\nabla u|^{p-2}\nabla u,\nabla u\rangle+o(1).\end{aligned}$$

Then

$$\begin{aligned}&\langle|\nabla u_n|^{p-2}\nabla u_n,\nabla u_n\rangle\\&\geq\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n-\nabla u,\nabla u_n-\nabla u\rangle+\langle|\nabla u|^{p-2}\nabla u,\nabla u\rangle+o(1)\end{aligned}$$

and

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^pdx\geq\lim_{n\to\infty}\int_\Omega|\nabla u_n-\nabla u|^pdx+\lim_{n\to\infty}\int_\Omega|\nabla u|^pdx$$

when $p>3$, there exist a $k\in N$ that $0<p-k\leq 1$. Then, we only need to prove the following inequality

$$\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^p-|\nabla u|^p)dx\geq\lim_{n\to\infty}\int_\Omega|\nabla u_n-\nabla u|^{p-k}(|\nabla u_n|^k-|\nabla u|^k).$$

The proof is similar to the proof above, so we omit it. The lemma is proved. □

Recall that a sequence $\{u_n\}\subset E$ is a (PS) sequence at level c if $\Phi_\lambda(u_n)\to c$ and $\Phi_\lambda{}'(u_n)\to 0$. Φ_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$-sequence contains a convergent subsequence.

Lemma 3.2. *Assume that* (V0), (K0), (H0) *be satisfied. Let $\{u_n\}$ be a $(PS)_c$-sequence for Φ_λ. Then $c\geq 0$ and $\{u_n\}$ is bounded in E.*

Proof. Let $\{u_n\}$ be a $(PS)_c$-sequence

$$\Phi_\lambda(u_n)\to c\quad\text{and}\quad\Phi_\lambda{}'(u_n)\to 0.$$

By (H0) we have

$$\begin{aligned}&d+\|u_n\|_\lambda+o(1)\\&\geq\Phi_\lambda(u_n)-\frac{1}{\mu}\Phi_\lambda{}'(u_n)u_n\\&=(\frac{1}{p}-\frac{1}{\mu})\int_{\mathbb{R}^N}(|\nabla u_n|^p+\lambda V(x)|u_n|^p)\\&\quad+\lambda\int_{\mathbb{R}^N}(\frac{1}{\mu}h(x,u_n)u_n-H(x,u_n))+(\frac{1}{\mu}-\frac{1}{p^*})\lambda\int_{\mathbb{R}^N}K(x)|u_n|^{p^*}\\&\geq(\frac{1}{p}-\frac{1}{\mu})\int_{\mathbb{R}^N}(|\nabla u_n|^p+\lambda V(x)|u_n|^p);\end{aligned}\tag{3.1}$$

hence for n large, $d+\|u_n\|_\lambda\geq\|u_n\|_\lambda^p$, where d is a positive constant. This implies that $\{u_n\}$ is bounded. Taking the limit in (3.1) shows that $c\geq 0$. □

Let $\{u_n\}$ denote a $(PS)_c$-sequence. By the above lemma, it is bounded, hence, without loss of generality, we may assume $u_n\rightharpoonup u$ in E, L^s and L^{p^*}, $u_n\to u$ in L^t_{loc} for $1\leq t<p^*$, and $u_n\to u$ a.e.for $x\in\mathbb{R}^N$.

Lemma 3.3. *Let $s\in[2,p^*)$. There is a subsequence (u_{n_j}) such that for each $\varepsilon>0$, there exists $r_\varepsilon>0$ with*

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|u_{n_j}|^sdx\leq\varepsilon$$

for all $r \geq r_\varepsilon$*, where* $B_k = \{x \in \mathbf{R}^N : |x| \leq k\}$*.*

Proof. Note that for each $j \in N$, $\int_{B_j} |u_n|^s \to \int_{B_j} |u|^s$ as $n \to \infty$. There exists $\widehat{n_j} \in N$ such that $\int_{B_j}(|u_n|^s - |u|^s) < \frac{1}{j}$ for all $n = \widehat{n_j} + i$, $i = 1, 2, 3, \dots$. Without loss of generality we can assume $\widehat{n_{j+1}} \geq \widehat{n_j}$. In particular, for $n_j = \widehat{n_j} + j$ we have

$$\int_{B_j} (|u_{n_j}|^s - |u|^s) < \frac{1}{j}$$

Observe that there is r_ε satisfying

$$\int_{\mathbb{R}^N \setminus B_r} |u|^s < \varepsilon \tag{3.2}$$

for all $r \geq r_\varepsilon$. Since

$$\begin{aligned}\int_{B_j \setminus B_r} |u_{n_j}|^s &= \int_{B_j} (|u_{n_j}|^s - |u|^s) + \int_{B_j \setminus B_r} |u|^s + \int_{B_r} (|u|^s - |u_{n_j}|^s) \\ &\leq \frac{1}{j} + \int_{R^N \setminus B_r} |u|^s + \int_{B_r} (|u|^s - |u_{n_j}|^s)\end{aligned}$$

the lemma follows. □

Recall that, by (H0), $|h(x, u)| \leq C_1(|u| + |u|^{q-1})$ for all (x, u). Let firstly $\{u_{n_j}\}_{j \in N}$ be a subsequence of $\{u_n\}_{n \in N}$ such that Lemma 3.3 holds for $s = 2$. Repeating the argument we can then find a subsequence $\{u_{n_{ji}}\}_{i \in N}$ of $\{u_{n_j}\}_{j \in N}$ such that Lemma 3.3 holds for $s = q$. Therefore, for notational convenience, we can assume in the following that Lemma 3.3 holds for both $s = 2$ and $s = q$ with the same subsequence.

Lemma 3.4. *We have*

$$\lim_{j \to \infty} |\int_{\mathbb{R}^N} (h(x, u_{n_j}) - h(x, u_{n_j} - \widetilde{u_j}) - h(x, \widetilde{u_j}))\varphi| = 0$$

uniformly in $\varphi \in E$ *with* $\|\varphi\| \leq 1$*.*

Proof. Note that (2.4) and the local compactness of Sobolev embedding imply that, for any $r > 0$.

$$\lim_{j \to \infty} |\int_{B_r} (h(x, u_{n_j}) - h(x, u_{n_j} - \widetilde{u_j}) - h(x, \widetilde{u_j}))\varphi| = 0$$

uniformly in $\|\varphi\| \leq 1$. For any $\varepsilon > 0$ it follows from (3.2) that

$$\limsup_{j \to \infty} \int_{B_j \setminus B_r} |\widetilde{u}_j|^s dx \leq \varepsilon \leq \int_{\mathbb{R}^N \setminus B_r} |u|^s < \varepsilon$$

for all $r \geq r_\varepsilon$. Using Lemma 3.3 for $s = 2, q$ we get

$$\begin{aligned}
&\limsup_{j\to\infty} \Big| \int_{\mathbb{R}^N} (h(x,u_{n_j}) - h(x,u_{n_j} - \widetilde{u}_j) - h(x,\widetilde{u}_j))\varphi \Big| \\
&= \limsup_{j\to\infty} \Big| \int_{B_j\setminus B_r} (h(x,u_{n_j}) - h(x,u_{n_j} - \widetilde{u}_j) - h(x,\widetilde{u}_j))\varphi \Big| \\
&\leq C_2 \limsup_{j\to\infty} \int_{B_j\setminus B_r} (|u_{n_j}| + |\widetilde{u}_j|)|\varphi| + C_3 \limsup_{j\to\infty} \int_{B_j\setminus B_r} (|u_{n_j}|^{q-1} + |\widetilde{u}_j|^{q-1})|\varphi| \\
&\leq C_2 \limsup_{j\to\infty} (|u_{n_j}|_{L^2(B_j\setminus B_r)} + |\widetilde{u}_j|_{L^2(B_j\setminus B_r)})|\varphi|_2 \\
&\quad + C_3 \limsup_{j\to\infty} (|u_{n_j}|^q_{L^q(B_j\setminus B_r)} + |\widetilde{u}_j|^q_{L^q(B_j\setminus B_r)})|\varphi|_q \\
&\leq C_4 \varepsilon^{\frac{1}{2}} + C_5 \varepsilon^{\frac{(q-1)}{q}}
\end{aligned}$$

the conclusion as required. ∎

Lemma 3.5. *One has along a subsequence: (1)* $\lim_{n\to\infty} \Phi_\lambda(u_n - \widetilde{u_n}) \leq c - \Phi_\lambda(u)$, *and (2)* $\Phi'_\lambda(u_n - \widetilde{u_n}) \to 0$.

Proof. From Lemma 3.1 we have

$$\begin{aligned}
\Phi_\lambda(u_n - \widetilde{u_n}) &\leq \Phi_\lambda(u_n) - \Phi_\lambda(\widetilde{u_n}) + \frac{\lambda}{p^*} \int_{R^N} K(x)(|u_n|^{p^*} - |u_n - \widetilde{u_n}|^{p^*} - |\widetilde{u_n}|^{p^*}) \\
&\quad + \lambda \int_{\mathbb{R}^N} (H(x,u_n) - H(x,u_n - \widetilde{u_n}) - H(x,\widetilde{u_n}))
\end{aligned}$$

Using (2.4) and the Lieb Lemma, we have

$$\begin{gathered}
\int_{\mathbb{R}^N} K(x)(|u_n|^{p^*} - |u_n - \widetilde{u_n}|^{p^*} - |\widetilde{u_n}|^{p^*}) \to 0\,, \\
\int_{\mathbb{R}^N} (H(x,u_n) - H(x,u_n - \widetilde{u_n}) - H(x,\widetilde{u_n})) \to 0
\end{gathered}$$

This, together with the facts $\Phi_\lambda(u_n) \to c$ and $\Phi_\lambda(\widetilde{u_n}) \to \Phi_\lambda(u)$, gives (1).

To verify (2), observe that, as $\widetilde{u_n} \to u$ and $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ so $u_n - \widetilde{u_n} \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} (|\nabla(u_n - \widetilde{u_n})|^{p-2}\nabla(u_n - \widetilde{u_n})\nabla\varphi + \lambda V(x)|u_n - \widetilde{u_n}|^{p-2}(u_n - \widetilde{u_n})\varphi) = o(1),$$

for any $\varphi \in E$. So for any $\varphi \in E$,

$$\begin{aligned}
&|\Phi'_\lambda(u_n - \widetilde{u_n})\varphi| \\
&\leq |\Phi'_\lambda(u_n)\varphi| + |\Phi'_\lambda(\widetilde{u_n})\varphi| \\
&\quad + \lambda \int_{\mathbb{R}^N} K(x)(|u_n|^{p^*-2}u_n - |u_n - \widetilde{u_n}|^{p^*-2}(u_n - \widetilde{u_n}) - |\widetilde{u_n}|^{p^*-2}\widetilde{u_n})\varphi \\
&\quad + \lambda \int_{\mathbb{R}^N} (h(x,u_n) - h(x,u_n - \widetilde{u_n}) - h(x,\widetilde{u_n}))\varphi
\end{aligned}$$

It follows, again from a standard argument, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)(|u_n|^{p^*} - |u_n - \widetilde{u_n}|^{p^*} - |\widetilde{u_n}|^{p^*})\varphi = 0$$

uniformly in $\|\varphi\| \le 1$. By Lemma 3.4 we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (h(x,u_n) - h(x,u_n - \widetilde{u_n}) - h(x,\widetilde{u_n}))\varphi = 0$$

uniformly in $\|\varphi\| \le 1$, proving (2). □

Lemma 3.6. *Under the assumptions of Lemma 3.2, there is a constant $\alpha_0 > 0$ independent of λ such that, for any $(PS)_c$-sequence (u_n) for Φ_λ with $u_n \rightharpoonup u$, either $u_n \to u$ or*

$$c - \Phi_\lambda(u) \ge \alpha_0 \lambda^{1-\frac{N}{p}}$$

where $\alpha_0 = S^{N/p} {\gamma_b}^{-N/p} N^{-1} K_{\min}$.

Proof. Assume u_n doesn't tend to u. Then $\liminf_{n\to\infty} \|u_n^1\|_\lambda > 0$ and $c - \Phi_\lambda(u) > 0$. By the Sobolev inequality, (2.6) and (2.7),

$$\begin{aligned}
S|u_n^1|_{p^*}^p &\le \int_{\mathbb{R}^N} |\nabla u_n^1|^p + \lambda V(x)|u_n^1|^p - \lambda \int_{\mathbb{R}^N} V(x)|u_n^1|^p \\
&= \lambda \int_{\mathbb{R}^N} g(x,u_n^1)u_n^1 - \lambda \int_{\mathbb{R}^N} V_b(x)|u_n^1|^p + o(1) \\
&\le \lambda \int_{\mathbb{R}^N} g(x,u_n^1)u_n^1 - \lambda b \int_{\mathbb{R}^N} |u_n^1|^p + o(1) \\
&\le \lambda\gamma_b |u_n^1|_{p^*}^{p^*} + o(1).
\end{aligned}$$

Thus by (2.5)

$$\begin{aligned}
S &\le \lambda\gamma_b |u_n^1|_{p^*}^{p^*-p} + o(1) \\
&\le \lambda\gamma_b \Big(\frac{N(c-\Phi_\lambda(u))}{\lambda K_{\min}}\Big)^{p/N} + o(1) \\
&= \lambda^{1-\frac{p}{N}} \gamma_b \Big(\frac{N}{K_{\min}}\Big)^{p/N} (c-\Phi_\lambda(u))^{p/N} + o(1)
\end{aligned}$$

or

$$\alpha_0 \lambda^{1-\frac{p}{N}} \le c - \Phi_\lambda(u) + o(1)$$

where

$$\alpha_0 = S^{N/p} {\gamma_b}^{-N/p} N^{-1} K_{\min}$$

The proof is complete. □

Lemma 3.7. *Under the assumptions of Lemma 3.2, Ψ_λ satisfies the $(PS)_c$ condition for all $c < \alpha_0 \lambda^{1-\frac{p}{N}}$.*

Proof. Assume (u_n) is a $(PS)_c$ sequence for Ψ_λ. Then $o(1)\|u_n^-\|_\lambda \ge {\Psi_\lambda}'(u_n)u_n^- = \|u_n^-\|_\lambda^p$ which implies $\|u_n^-\|_\lambda \to 0$. In addition,

$$\Psi_\lambda(u_n) - \frac{1}{p}{\Psi_\lambda}'(u_n)u_n \ge \frac{\lambda}{N}\int_{\mathbb{R}^N} K(x)|u_n^+|^{p^*}$$

and

$$o(1)\|u_n^+\|_\lambda \ge {\Psi_\lambda}'(u_n)u_n^+ = \|u_n^+\|_\lambda^p - \int_{\mathbb{R}^N} g(x,u_n^+)u_n^+$$

Using the above argument, it is not difficult to check that under the assumptions of Lemma 3.2, Ψ_λ satisfies the $(PS)_c$ condition for all $c < \alpha_0\lambda^{1-\frac{p}{N}}$. □

We consider $\lambda \ge 1$. The following two Lemmas imply that Φ_λ possesses the mountain-pass structure.

Lemma 3.8. *Assume* (V0), (K0), (H0) *hold. There exist* α_λ, $\rho_\lambda > 0$ *such that* $\Phi_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $\Phi_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$, *where*

$$B_{\rho_\lambda} = \{u \in E : \|u\|_\lambda \leq \rho_\lambda\}.$$

Proof. By (H0), for $\delta \leq (2p\lambda\upsilon_p^p)^{-1}$ there is $C_\delta > 0$ such that $G(x,u) \leq \delta|u|^p + C_\delta|u|^{p^*}$ for all (x,u), where υ_p is the embedding constant of (2.2). Thus

$$\Phi_\lambda(u) \geq \frac{1}{p}\|u\|_\lambda^p - \lambda\delta|u|_p^p - \lambda C_\delta|u|_{p^*}^{p^*} \geq \frac{1}{2p}\|u\|_\lambda^p - \lambda C_\delta \upsilon_{p^*}^{p^*}\|u\|_\lambda^{p^*}.$$

Consequently the conclusion follows because $p^* > p$. $\square$

Lemma 3.9. *Under the assumptions of Lemma 3.8, for any finite dimensional subsequence* $F \subset E$, $\Phi_\lambda(u) \to -\infty$ *as* $u \in F$, $\|u\|_\lambda \to \infty$.

Proof. By (H0),

$$\Phi_\lambda(u) \leq \frac{1}{p}\|u\|_\lambda^p - \lambda_0 a_0 |u|_s^s$$

for all $u \in E$. Since all norms in a finite-dimensional space are equivalent and $s > p$, one obtains easily the desired conclusion. $\square$

Lemma 3.10. *Under the assumptions of Lemma 3.8, for any* $\sigma > 0$ *there exists* $\Lambda_\sigma > 0$, *such that, for each* $\lambda \geq \Lambda_\sigma$, *there is* $\bar{e}_\lambda \in E$ *with* $\|\bar{e}_\lambda\| > \sigma_\lambda$, $\Phi_\lambda(\bar{e}_\lambda) \leq 0$ *and*

$$\max_{t\in[0,1]} \Phi_\lambda(t\bar{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{p}},$$

where ρ_λ *is from Lemma 3.8.*

Proof. Choose $\delta > 0$ so small that

$$\frac{s-p}{sp(sa_0)^{p/(s-p)}}(2\delta)^{s/(s-p)} \leq \sigma$$

and let $e_\lambda \in E$ be the function defined by (2.8). Take $\Lambda_\sigma = \widehat{\Lambda}_\delta$. Let $\bar{t}_\lambda > 0$ be such that $\bar{t}_\lambda\|e_\lambda\|_\lambda > \rho_\lambda$ and $\Phi_\lambda(te_\lambda) \leq 0$ for all $t > \bar{t}_\lambda$. Then by (2.10), $\bar{e}_\lambda = \bar{t}_\lambda e_\lambda$ satisfies the requirements. $\square$

Lemma 3.11. *Under the assumptions of Lemma 3.8, for any* $m \in N$ *and* $\sigma > 0$ *there exist* $\Lambda_{m\sigma} > 0$, *such that, for each* $\lambda \geq \Lambda_{m\sigma}$, *there exists an* m*-dimensional subspace* $F_{\lambda m}$ *satisfying*

$$\sup_{u\in F_{\lambda m}} \Phi_\lambda(u) \leq \sigma\lambda^{1-\frac{N}{p}}.$$

Proof. Choose $\delta > 0$ small so that

$$\frac{s-p}{sp(sa_0)^{p/(s-p)}}(2\delta)^{s/(s-p)} \leq \sigma$$

and take $F_{\lambda m} = H_{\lambda\delta}^m$. Then (2.11) yields the conclusion as required. $\square$

4. Proof of Main Theorems

Theorem 4.1. *Let* (V0), (K0), (H0) *be satisfied. Then for any* $\sigma > 0$ *there is* $\Lambda_\sigma > 0$ *such that if* $\lambda \geq \Lambda_\sigma$*, then* (2.1) *has at least one positive solution* u_λ *of least energy satisfying*

$$\frac{\mu - p}{p} \int_{\mathbb{R}^N} H(x, u_\lambda) + \frac{1}{N} \int_{\mathbb{R}^N} K(x)|u_\lambda|^{p^*} dx \leq \sigma \lambda^{-\frac{N}{p}}, \tag{4.1}$$

$$\frac{\mu - p}{p\mu} \int_{\mathbb{R}^N} (\varepsilon^p |\nabla u_\lambda|^p + V(x)|u_\lambda|^p) dx \leq \sigma \lambda^{1-\frac{N}{p}} \tag{4.2}$$

Proof. Consider the functional Ψ_λ. For any $0 < \sigma < a_0$, we choose Λ_σ and define for $\lambda \geq \Lambda_\sigma$ the minimax value

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} \Psi_\lambda(\gamma(t))$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = \bar{e}_\lambda\}$. By Lemma 3.8,

$$\alpha_\lambda \leq c_\lambda \leq \sigma \lambda^{1-\frac{N}{p}}$$

Since by Lemma 3.7, Ψ_λ satisfies the $(PS)_{c_\lambda}$-condition, the mountain-pass theorem implies that there is $u_\lambda \in E$ such that $\Psi'_\lambda(u_\lambda) = 0$ and $\Psi_\lambda(u_\lambda) = c_\lambda$. Then u_λ is a positive solution of (2.1). Moreover, it is well known that such a Mountain-Pass solution is a least energy solution of (2.1).

Since u_λ is a critical point of Ψ_λ, for $\nu \in [p, p^*]$,

$$\begin{aligned}
&\sigma \lambda^{1-\frac{N}{p}} \\
&\geq \Psi_\lambda(u_\lambda) \\
&= \Psi_\lambda(u_\lambda) - \frac{1}{\nu} \Psi'_\lambda(u_\lambda) u_\lambda \\
&\geq \Big(\frac{1}{p} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} (|\nabla u_\lambda|^p + \lambda V(x)|u_\lambda|^p) \\
&\quad + \lambda\Big(\frac{1}{\nu} - \frac{1}{p^*}\Big) \int_{\mathbb{R}^N} K(x)|u_\lambda|^{p^*} + \lambda\Big(\frac{\mu}{\nu} - 1\Big) \int_{R^N} H(x, u_\lambda),
\end{aligned}$$

where μ is the constant in (H0). Taking $\nu = p$ yields the estimate (4.1), and taking $\nu = \mu$ gives the estimate (4.2). The proof is complete. □

Theorem 4.2. *Let* (V0), (K0), (H0) *be satisfied. If moreover* $h(x, u)$ *is odd in* u*, then for any* $m \in N$ *and* $\sigma > 0$ *there is* $\Lambda_{m\sigma} > 0$ *such that if* $\lambda \geq \Lambda_{m\sigma}$*,* (2.1) *has at least m pairs of solutions* u_λ *which satisfy the estimates* (4.1) *and* (4.2).

Proof. Consider the functional Φ_λ. By virtue of Lemma 3.11, for any $m \in N$ and $\sigma \in (0, a_0)$ there is $\Lambda_{m\sigma}$ such that for each $\lambda \geq \Lambda_{m\sigma}$, we can choose a m-dimensional subspace $F_{\lambda m}$ with $\max \Phi_\lambda(F_{\lambda m}) \leq \sigma \lambda^{1-\frac{N}{p}}$. By Lemma 3.9, there is $R > 0$ which depending on λ and m such that $\Phi_\lambda(u) \leq 0$ for all $u \in F_{\lambda m} \setminus B_R$.

Denote the set of all symmetric (in the sense that -A=A) and closed subsets of E by Σ. For each $A \in \Sigma$ let gen(A) be the Krasnoselski genus and

$$i(A) = \min_{h \in \Gamma_m} \operatorname{gen}(h(A) \cap \partial B_{\rho_\lambda})$$

where Γ_m is the set of all odd homeomorphisms $h \in C(E, E)$ and ρ_λ is the number from Lemma 3.8. Then (4.1) is a version of Benci's pseudoindex. Let

$$c_{\lambda_j} = \inf_{i(A)\geq j} \sup_{u\in A} \Phi_\lambda(u), \quad 1 \leq j \leq m$$

Since $\Phi_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B_{\rho_\lambda}$ and since $i(F_{\lambda m}) = \dim F_{\lambda m} = m$,

$$\alpha_\lambda \leq c_{\lambda_1} \leq \cdots \leq c_{\lambda_m} \leq \sup_{u\in F_{\lambda m}} \Phi_\lambda(u) \leq \sigma\lambda^{1-\frac{N}{p}}.$$

It follows from Lemma 3.6 that Φ_λ satisfies the $(PS)_c$-condition at all levels $c < \lambda^{1-\frac{N}{p}}\alpha_0$. By the critical point theory, all e_{λ_j} are critical levels and Φ_λ has at least m pairs of nontrivial critical points satisfying

$$\alpha_\lambda \leq \Phi_\lambda(u_\lambda) \leq \sigma\lambda^{1-\frac{N}{p}}$$

Therefore, $(NS)_\lambda$ has at least m pairs of solutions. Finally, as in the proof of Theorem 4.1 one sees that these solutions satisfy the estimates (i) and (ii). □

Theorem 4.3. *Let* (V0), (K0), (H0), (S) *be satisfied. If moreover $h(x, u)$ is odd in u, then for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$ such that if $\Lambda \geq \omega_\sigma$,* (2.1) *has at least one pair of solutions which change sign exactly once and satisfy the estimates* (4.1) *and* (4.2).

Proof. We say that a function $u : \mathbb{R}^N \to \mathbb{R}$ changes sign n times if the set $\{x \in \mathbb{R}^N : u(x) \neq 0\}$ has $n + 1$ connected components. If u is a solution of (2.1) then it is of class C^2 and τ induces a bijection between the connected components of $\{x \in \mathbb{R}^N : u(x) > 0\}$ and those of $\{x \in \mathbb{R}^N : u(x) < 0\}$. So u changes sign an odd number of times. Define the τ-Nehari manifold

$$N_\lambda^\tau = \{u \in E^\tau : u \neq 0, \ \ \Phi_\lambda'(u)u = 0\}.$$

Then critical points of the restriction of Φ_λ on N_λ^τ are solutions of (2.1). Set

$$c_\lambda^\tau = \inf\{\Phi_\lambda(u) : u \in N_\lambda^\tau\}.$$

Assume (S) holds. If $u \in E$ then the function $\widetilde{u} = (u + \tau u)/2$ satisfies $\tau\widetilde{u} = \widetilde{u}$; i.e., $\widetilde{u} \in E^\tau$. It is clear that if $(\varphi_j) \subset C_0^\infty(\mathbb{R}^N)$, $|\varphi_j|_s = 1$ and $|\nabla\varphi_j|_p \to 0$, then $\widetilde{\varphi_j} = (\varphi_j + \tau\varphi_j)/2 \in E^\tau$ and $|\nabla\widetilde{\varphi_j}|_p \to 0$. Arguing as before, we see the conclusion: Assume (V0), (K0), (H0) and (S) be satisfied. Then for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_{m\sigma}$ there exists $0 \neq \overline{e}_\lambda \in E^\tau$ such that $\Phi_\lambda'(\overline{e}_\lambda)\overline{e}_\lambda = 0$ and

$$\Phi_\lambda(\overline{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{p}}.$$

So for any $\sigma \in (0, a_0)$, there is $\Lambda_\sigma > 0$ such that

$$0 < C_\lambda^\tau \leq \sigma\lambda^{1-\frac{N}{p}} \quad \text{if } \lambda \geq \Lambda_\sigma.$$

By Lemma 3.6, Φ_λ satisfies the $(PS)_{c_\lambda^\tau}$ condition. Thus c_λ^τ is a critical value of Φ_λ. Let $u_\lambda \in E^\tau$ be the relative critical point which is a solution of (2.1) with $u_\lambda(\tau x) = -u_\lambda(x)$. It remains to show that u_λ changes sign exactly once.

Observe that if $u \in N_\lambda^\tau$ is a solution of (2.1) which changes sign $2m-1$ times, then $\Phi_\lambda(u) \geq mc_\lambda^\tau$. Indeed, the set $\{x \in \mathbb{R}^N : u(x) > 0\}$ has m connected components $X_1, \ldots, X_m$. Let $u_i(x) = u(x)$ if $x \in X_i \cup \tau X_i$ and $u_i(x) = 0$ otherwise. Since u is a critical point of Φ_λ,

$$\Phi_\lambda'(u)u_i = \|u_i\|_\lambda^p - \int_{\mathbb{R}^N} g(x, u_i)u_i = 0.$$

Thus $u_i \in N_\lambda^\tau$ for $i = 1, \dots, m$, and

$$\Phi_\lambda(u) = \Phi_\lambda(u_1) + \cdots + \Phi_\lambda(u_m) \geq mc_\lambda^\tau.$$

Now since $\Phi_\lambda(u_\lambda) = c_\lambda^\tau$, one concludes that u_λ changes sign only $m = 1$ time. Final, as before one sees that u_λ satisfies (i) and (ii). The proof is complete. $\square$

Remark. Clearly we can see that the Theorems 2.1, 2.2 and 2.3 also be proofed. Indeed, (1.1)$\sim$2.1. Our methods and results can also be applicable to subcritical nonlinear problems (1.2).

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