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A PARABOLIC-HYPERBOLIC FREE BOUNDARY PROBLEM MODELING TUMOR GROWTH WITH DRUG APPLICATION

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ABSTRACT. In this article, we study a free boundary problem modeling the growth of tumors with drug application. The model consists of two nonlinear second-order parabolic equations describing the diffusion of nutrient and drug concentration, and three nonlinear first-order hyperbolic equations describing the evolution of proliferative cells, quiescent cells and dead cells. We deal with the radially symmetric case of this free boundary problem, and prove that it has a unique global solution. The proof is based on the L^p theory of parabolic equations, the characteristic theory of hyperbolic equations and the Banach fixed point theorem.

1. INTRODUCTION

In this article we study a free boundary problem modeling the tumor growth with drug application, the mathematical model which neglect the drug application was proposed by A. Friedman (cf. [13]) in 2004. This model consists of three types of cells: proliferative cells; quiescent cells and dead cells, which we denote the corresponding cell densities by P, Q and D, respectively. C and W represent the concentration of nutrient and drug, respectively. We assume $K_B(C)$ is the mitosis rate of proliferative cells when the nutrient supply is at the level C, $K_A(C)$ and $K_D(C)$ are death rates of proliferative cells (apoptosis) and quiescent cells (necrosis), respectively, $K_P(C)$ and $K_Q(C)$ are the transferring rate of quiescent cells to proliferative cells and the transferring rate of proliferative cells to quiescent cells, respectively, K_R denotes the constant rate of dead cells are removed from the tumor.

Fick's law is assumed to describe the diffusion of the nutrient, for reasons stated in [26, 27], the nutrient is consumed at a rate proportional to the rate of cell mitosis, i.e., $(\kappa_1 K_P(C)P + \kappa_2 K_Q(C)Q)C$. Hence, C satisfies the following equation:

$$\frac{\partial C}{\partial t} = D_1 \Delta C - \Big(\kappa_1 K_P(C)P + \kappa_2 K_Q(C)Q\Big)C \quad \text{in } \Omega(t), \tag{1.1}$$

$$C(x,t) = \bar{C} \quad \text{on} \quad \partial\Omega(t), \quad C(x,0) = C_0(x) \quad \text{in } \Omega(0), \tag{1.2}$$

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where $\Omega(t)$ represents the tumor domain at time t, D_1 is the diffusion coefficient of nutrient which is supposed to be a positive constant, κ_1 and κ_2 are two positive constants, $\bar{C}$ is a positive constant reflecting the constant nutrient supply that the tumor receives from its host tissue or the solution in which it is cultivated.

Fick's law is also assumed to describe the diffusion of the drug, we assume $(\mu_1 G_1(W)P + \mu_2 G_2(W)Q)W$ is the drug consumption rate function, μ_1, μ_2 are two positive constants can be viewed as a measure of the drug effectiveness. Hence, W satisfies

$$\frac{\partial W}{\partial t} = D_2 \Delta W - \Big(\mu_1 G_1(W)P + \mu_2 G_2(W)Q\Big)W \quad \text{in } \Omega(t), \tag{1.3}$$

$$W(x,t) = \bar{W} \quad \text{on } \partial\Omega(t), \quad W(x,0) = W_0(x) \quad \text{in } \Omega(0), \tag{1.4}$$

where D_2 is the diffusion coefficient of drug which is supposed to be a positive constant, $\bar{W}$ is a positive constant reflecting the constant drug supply that the tumor receives from its boundary.

Due to the proliferation and removal of cells, there is a continuous motion of cells within the tumor, we denote this movement by the velocity fields $\vec{v}$. We shall assume that the tumor tissue is a porous medium so that by Darcy's law, we have

$$\vec{v} = -\nabla\sigma \quad \text{in } \Omega(t),\ t > 0, \tag{1.5}$$

where σ is the pressure in the tumor.

We also assume that all cells to be mixed together in the tumor which have the same size, and the tumor is uniformly packed with cells, so that

$$P + Q + D = N \quad \text{in } \Omega(t),\ t > 0, \tag{1.6}$$

where N is a total number of cells per unit volume.

The mass conservation law for the densities of the proliferative cells, quiescent cells and dead cells in $\Omega(t)$ take the following form:

$$\begin{gathered}\frac{\partial P}{\partial t} + \operatorname{div}(P\vec{v}) = [K_B(C) - K_Q(C) - K_A(C)]P + K_P(C)Q - \iota_1 G_1(W)P \\ \text{in } \Omega(t),\ t > 0,\end{gathered} \tag{1.7}$$

$$\begin{gathered}\frac{\partial Q}{\partial t} + \operatorname{div}(Q\vec{v}) = K_Q(C)P - [K_P(C) + K_D(C)]Q - \iota_2 G_2(W)Q \\ \text{in } \Omega(t),\ t > 0,\end{gathered} \tag{1.8}$$

$$\begin{gathered}\frac{\partial D}{\partial t} + \operatorname{div}(D\vec{v}) = K_A(C)P + K_D(C)Q - K_R D + \iota_1 G_1(W)P + \iota_2 G_2(W)Q \\ \text{in } \Omega(t),\ t > 0,\end{gathered} \tag{1.9}$$

where $\iota_1 G_1(W)$ is a rate of the proliferative cells become dead cells due to the drug, $\iota_2 G_2(W)$ is a rate of the quiescent cells become dead cells due to the drug, the positive constant ι_1 and ι_2 are the maximum possible rate of drug induced proliferative cells and quiescent cells dead, respectively.

We take the boundary conditions for σ to be

$$\sigma = \theta\kappa \quad \text{on } \partial\Omega(t),\ t > 0, \tag{1.10}$$

$$\frac{\partial\sigma}{\partial n} = -V_n \quad \text{on } \partial\Omega(t),\ t > 0, \tag{1.11}$$

and the initial data

$$P(x,0) = P_0(x), \quad Q(x,0) = Q_0(x), \quad D(x,0) = D_0(x) \quad \text{for } x \in \Omega(0), \tag{1.12}$$

where $\Omega(0)$ is given, θ is the surface tension, κ is the mean curvature of the tumor surface, $\frac{\partial}{\partial n}$ is the derivatives in the direction n of the outward normal, and V_n is the velocity of the free boundary $\partial\Omega(t)$ in the direction n. Equation (1.10) is based on the assumption that the pressure σ on the surface of the tumor is proportional to the surface tension (cf. Greenspan [18, 19]), and (1.11) is a standard kinetic condition.

The model (1.1)–(1.12) without drug application was proposed by A. Friedman (cf. [13]) in 2004. In [12], the authors studied this model, under the case of where the initial data and the solution are spherically symmetric, they proved that there exists a unique global solution. However, for three dimensional model (1.1)–(1.12), as to our knowledge, the global existence is still an open problem. In [9], based on the well-known theory of Hele-Shaw problem, the authors proved the local existence and uniqueness of solution to the system (1.1)–(1.12) without drug application.

There are many mathematical tumor models involving drug therapies (cf. [20, 21, 23, 25, 28]). We know the drug can penetrate tumor tissue mainly by the mechanism of diffusion. In [28], the authors advanced the model which only considered living cells and dead cells with drug application, under the condition of spherical symmetry of the solution. In [23, 25], the authors proved that there exists a unique global solution of the model in [28]. Since the living cells include the proliferative cells and quiescent cells, the model in this article is more reasonable than that of [28]. Some other tumor models and rigorous mathematical analysis of these models we refer the reader to see [1]–[11], [14]–[17], [24] and [26]–[28], and the references there in.

In this article, we consider the drug application, also spherically symmetric solution for the system (1.1)–(1.12). It is clear that, under the condition of spherical symmetry, for given $\vec{v}$ and $R(t)$, σ can easily solved from (1.5) and (1.10). The major difficulty lies in that there is a clear coupling between the evolution of the cells and the nutrient (drug) diffusion-consumption process. By applying the L^p theory of parabolic equations, the characteristic theory of hyperbolic equations and the Banach fixed point theorem, we prove that there exists a unique global solution of (1.1)–(1.12).

It is obvious that if we make an addition to (1.7)–(1.9), then we can get the following equation for $\vec{v}$,

$$\operatorname{div}(\vec{v}) = \frac{1}{N}[K_B(C)P - K_R D] \quad \text{for } x \in \Omega(t),\ t > 0. \tag{1.13}$$

Conversely, from (1.13) and Eqs. (1.7)–(1.9) we have

$$\frac{\partial(P+Q+D)}{\partial t} + \vec{v}\nabla(P+Q+D) = \frac{1}{N}[K_B(C)P - K_R D][N-(P+Q+D)] \tag{1.14}$$

for $x \in \Omega(t)$, $t > 0$. By uniqueness, we can deduce that (1.6) is equivalent to (1.13), later on we shall use (1.13) instead of (1.6).

The model (1.1)–(1.12) is a three-dimensional tumor model, in this article we consider the well-posedness of this problem under the case where the initial data and the solution are spherically symmetric. Hence, we assume that C, W, P, Q and D are spherical symmetric in the space variable, let $r = |x|$ we denote

$$C = C(r,t), \quad W = W(r,t), \quad P = P(r,t), \quad Q = Q(r,t), \quad D = D(r,t)$$

for $0 \le r \le R(t)$, $t \ge 0$, and

$$C_0 = C_0(r), \quad W_0 = W_0(r), \quad P_0 = P_0(r), \quad Q_0 = Q_0(r), \quad D_0 = D_0(r)$$

for $0 \le r \le R_0 = R(0)$. We also assume that there is a scalar function $v = v(r,t)$ such that $\vec{v} = v(r,t)\frac{x}{r}$. Since σ is spherically symmetric in the space variable, as we mentioned before, we could eliminate the pressure σ and derive the model (1.1)–(1.12) as follows:

$$\frac{\partial C}{\partial t} = D_1 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial C}{\partial r}) - F(C,P,Q)C \quad \text{for } \ 0 < r < R(t),\ t > 0, \tag{1.15}$$

$$\frac{\partial C}{\partial r}(r,t) = 0 \quad \text{at } r = 0, \quad C(r,t) = \bar{C} \quad \text{at } \ r = R(t) \text{ for } t > 0, \tag{1.16}$$

$$C(r,0) = C_0(r) \quad \text{for } 0 \le r \le R_0, \tag{1.17}$$

$$\frac{\partial W}{\partial t} = D_2 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial W}{\partial r}) - G(W,P,Q)W \quad \text{for } 0 < r < R(t),\ t > 0, \tag{1.18}$$

$$\frac{\partial W}{\partial r}(r,t) = 0 \quad \text{at } r = 0, \quad W(r,t) = \bar{W} \quad \text{at } r = R(t) \text{ for } t > 0, \tag{1.19}$$

$$W(r,0) = W_0(r) \quad \text{for } 0 \le r \le R_0, \tag{1.20}$$

$$\begin{aligned} \frac{\partial P}{\partial t} + v\frac{\partial P}{\partial r} &= g_{11}(C,W,P,Q,D)P + g_{12}(C,W,P,Q,D)Q \\ &\quad + g_{13}(C,W,P,Q,D)D \quad \text{for } 0 \le r \le R(t),\ t > 0, \end{aligned} \tag{1.21}$$

$$\begin{aligned} \frac{\partial Q}{\partial t} + v\frac{\partial Q}{\partial r} &= g_{21}(C,W,P,Q,D)P + g_{22}(C,W,P,Q,D)Q \\ &\quad + g_{23}(C,W,P,Q,D)D \quad \text{for } 0 \le r \le R(t),\ t > 0, \end{aligned} \tag{1.22}$$

$$\begin{aligned} \frac{\partial D}{\partial t} + v\frac{\partial D}{\partial r} &= g_{31}(C,W,P,Q,D)P + g_{32}(C,W,P,Q,D)Q \\ &\quad + g_{33}(C,W,P,Q,D)D \quad \text{for } 0 \le r \le R(t),\ t > 0, \end{aligned} \tag{1.23}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) = h(C,W,P,Q,D) \quad \text{for } 0 < r \le R(t),\ t > 0, \tag{1.24}$$

$$v(0,t) = 0 \quad \text{for } t > 0, \tag{1.25}$$

$$\frac{dR(t)}{dt} = v(R(t),t) \quad \text{for } t > 0, \tag{1.26}$$

$$\begin{gathered} P(r,0) = P_0(r), \quad Q(r,0) = Q_0(r), \quad D(r,0) = D_0(r) \quad \text{for } 0 \le r \le R_0, \\ R(0) = R_0 \quad \text{is prescribed,} \end{gathered} \tag{1.27}$$

where

$$\begin{gathered} F(C,P,Q) = \kappa_1 K_P(C)P + \kappa_2 K_Q(C)Q, \\ G(W,P,Q) = \mu_1 G_1(W)P + \mu_2 G_2(W)Q, \end{gathered}$$

$$\begin{aligned} &g_{11}(C,W,P,Q,D) \\ &= [K_B(C) - K_Q(C) - K_A(C) - \iota_1 G_1(W)] - \frac{1}{N}[K_B(C)P - K_R D], \end{aligned}$$

$$\begin{gathered} g_{12}(C,W,P,Q,D) = K_P(C), \\ g_{13}(C,W,P,Q,D) = 0, \\ g_{21}(C,W,P,Q,D) = K_Q(C), \end{gathered}$$

$$g_{22}(C,W,P,Q,D) = -[K_P(C) + K_D(C) + \iota_2 G_2(W)] - \frac{1}{N}[K_B(C)P - K_R D],$$

$$
\begin{aligned}
g_{23}(C,W,P,Q,D) &= 0,\\
g_{31}(C,W,P,Q,D) &= K_A(C)+\iota_1 G_1(W),\\
g_{32}(C,W,P,Q,D) &= K_D(C)+\iota_2 G_2(W),\\
g_{33}(C,W,P,Q,D) &= -K_R-\frac{1}{N}[K_B(C)P-K_RD],\\
h(C,W,P,Q,D) &= \frac{1}{N}[K_B(C)P-K_RD].
\end{aligned}
$$

Remark 1.1. The following facts will play an important role in our subsequent analysis:

$$
g_{ij}(C,W,P,Q,D)\ge 0 \quad \text{for } i\ne j, \tag{1.28}
$$

$$
\begin{aligned}
&\sum_{i=1}^{3} g_{i1}(C,W,P,Q,D)P+g_{i2}(C,W,P,Q,D)Q+g_{i3}(C,W,P,Q,D)D\\
&=h(C,W,P,Q,D)[N-(P+Q+D)].
\end{aligned} \tag{1.29}
$$

Throughout the whole article we make use of the following notations:
(i) Given an open set $\Omega\subset\mathbb{R}^3$, we denote $W^{2,p}(\Omega):=\{u: D^\alpha u\in L^p(\Omega), 0\le|\alpha|\le 2\}$ is the usual Sobolev space with norm $\|u\|_{W^{2,p}(\Omega)}=\sum_{0\le|\alpha|\le 2}\|D^\alpha u\|_{L^p(\Omega)}$, where $1\le p\le\infty$.
(ii) For $T>0$, given a positive continuous function $R=R(t)$ $(0\le t\le T)$, we denote $Q_T^R=\{(x,t)\in\mathbb{R}^3\times\mathbb{R}: |x|<R(t),\ 0<t<T\}$ and $\overline{Q_T^R}$ denotes the closure of Q_T^R.
(iii) For $R=R(t)$ as in (ii) and $1\le p<\infty$, we denote by $W_p^{2,1}(Q_T^R)$ the usual non-isotropic Sobolev spaces on the parabolic domain Q_T^R, i.e.,

$$
W_p^{2,1}(Q_T^R)=\{u\in L^p(Q_T^R):\ \partial_x^\alpha\partial_t^k u\in L^p(Q_T^R)\quad \text{for } |\alpha|+2k\le 2\},
$$

with the norm

$$
\|u\|_{W_p^{2,1}(Q_T^R)}=\sum_{|\alpha|+2k\le 2}\|\partial_x^\alpha\partial_t^k u\|_{L^p(Q_T^R)}.
$$

(iv) Given an open set $\Omega\subset\mathbb{R}^3$ and for some number $p>5/2$, we denote by $D_p(\Omega)$ the trace space of $W_p^{2,1}(\Omega\times(0,T))$ at $t=0$, i.e., $\varphi\in D_p(\Omega)$ if and only if there exists $u\in W_p^{2,1}(\Omega\times(0,T))$ such that $u(\cdot,0)=\varphi$. The norm equipped in $D_p(\Omega)$ is defined as follows:

$$
\|\varphi\|_{D_p(\Omega)}=\inf\{T^{-\frac{1}{p}}\|u\|_{W_p^{2,1}(\Omega\times(0,T))}: u\in W_p^{2,1}(\Omega\times(0,T)),\ u(\cdot,0)=\varphi\}.
$$

It is well known that if $p>5/2$, then $W_p^{2,1}(\Omega\times(0,T))\subset C(\overline{\Omega}\times[0,T])$ is continuous by the embedding theorem (see [22]). Furthermore, if $\varphi\in W^{2,p}(\Omega)$, then $\varphi\in D_p(\Omega)$ and $\|\varphi\|_{D_p(\Omega)}\le\|\varphi\|_{W^{2,p}(\Omega)}$ since we can take $u(x,t)\equiv\varphi(x)$ for all $0\le t\le T$.

Since the functional dependence of $K_A(C)$, $K_B(C)$, $K_D(C)$, $K_P(C)$ and $K_Q(C)$ with respect to C and $G_1(W)$, $G_2(W)$ with respect to W are not critical to our results, we only need a very simple assumption as follows:

- (A1) $K_A(C)$, $K_B(C)$, $K_D(C)$, $K_P(C)$ and $K_Q(C)$ are non-negative C^1-smooth functions;
- (A2) $G_1(W)$ and $G_2(W)$ are non-negative C^1-smooth functions;
- (A3) P_0, Q_0 and D_0 are non-negative C^1-smooth functions on $[0,R_0]$;
- (A4) $C_0(|x|)$, $W_0(|x|)\in D_p(B_{R_0})$ for some $p>5$, where $B_{R_0}=\{x\in\mathbb{R}^3:\ |x|\le R_0\}$.

The first three conditions are clearly very natural from biological point of view.

We also assume that the initial data satisfy the following compatible conditions:

$$
\begin{gathered}
0 \le C_0(r) \le \bar{C}, \quad 0 \le W_0(r) \le \bar{W} \quad \text{for } 0 \le r \le R_0, \\
C_0'(0) = 0, \quad C_0(R_0) = \bar{C}, \quad W_0'(0) = 0, \quad W_0(R_0) = \bar{W}, \\
P_0(r) \ge 0, \quad Q_0(r) \ge 0, \quad D_0(r) \ge 0 \quad \text{for } 0 \le r \le R_0, \\
P_0(r) + Q_0(r) + D_0(r) = N \quad \text{for } 0 \le r \le R_0.
\end{gathered} \tag{1.30}
$$

Remark 1.2. Under the assumptions (A1)–(A4), we can easily deduce the following facts:

- $F \ge 0$, $G \ge 0$.
- F, G and h are C^1-functions.
- g_{ij} $(i, j = 1, 2, 3)$ are C^1-functions.

Now we give our main results.

Theorem 1.3. *Under the assumptions* (A1)–(A4) *and initial condition* (1.30)*, the free boundary problem* (1.15)–(1.27) *has a unique solution* (R, C, W, P, Q, D) *for all* $t \ge 0$*. In addition, for any* $T > 0$*,* $R(t) \in C^1[0, T]$*,* C*,* $W \in W_p^{2,1}(Q_T^R)$ *and* P*,* Q*,* $D \in C^1(Q_T^R)$*. Furthermore, the following estimates hold:*

$$
\begin{gathered}
R(t) > 0 \quad \textit{for } t > 0, \\
0 < C(r, t) \le \bar{C}, \quad 0 < W(r, t) \le \bar{W} \quad \textit{for } 0 \le r \le R(t),\ t \ge 0, \\
P(r, t) \ge 0, \quad Q(r, t) \ge 0, \quad D(r, t) \ge 0 \quad \textit{for } 0 \le r \le R(t),\ t \ge 0, \\
P(r, t) + Q(r, t) + D(r, t) = N \quad \textit{for } 0 \le r \le R(t),\ t \ge 0.
\end{gathered} \tag{1.31}
$$

This article is organized as follows. In Section 2, we transform the problem (1.15)–(1.27) for a moving domain into an equivalent one which defined on a fixed domain. Section 3 is devoted to presenting some preliminary lemmas that will be used in the later analysis. In section 4 we prove local existence and uniqueness of the transformed problem by applying Banach fixed point theorem. We prove Theorem 1.3 in Section 5.

2. Reformulation of the problem

To transform the varying domain $\{(x, t) : |x| = r < R(t),\ t \ge 0\}$ into a fixed domain, let we assume (R, C, W, P, Q, D) is a solution of (1.15)–(1.27) and $R(t) > 0$ $(t \ge 0)$, and make the following change of variables,

$$
\begin{gathered}
\rho = \frac{r}{R(t)}, \quad \tau = \int_0^t \frac{ds}{R^2(s)}, \quad \eta(\tau) = R(t), \quad c(\rho, \tau) = C(r, t), \quad w(\rho, \tau) = W(r, t), \\
p(\rho, \tau) = P(r, t), \quad q(\rho, \tau) = Q(r, t), \quad d(\rho, \tau) = D(r, t), \quad u(\rho, \tau) = R(t) v(r, t),
\end{gathered} \tag{2.1}
$$

then the free boundary problem (1.15)–(1.27) is transformed into the following initial-boundary value problem on the fixed domain $\{(\rho, \tau) : 0 \le \rho \le 1,\ \tau \ge 0\}$:

$$
\frac{\partial c}{\partial \tau} = D_1 \frac{1}{\rho^2} \frac{\partial}{\partial \rho}(\rho^2 \frac{\partial c}{\partial \rho}) + u(1, \tau) \rho \frac{\partial c}{\partial \rho} - \eta^2 f(c, p, q) c \quad \text{for } 0 < \rho < 1,\ \tau > 0, \tag{2.2}
$$

$$
\frac{\partial c}{\partial \rho}(0, \tau) = 0, \quad c(1, \tau) = \bar{c} \quad \text{for } \tau > 0, \tag{2.3}
$$

$$
c(\rho, 0) = c_0(\rho) \quad \text{for } 0 \le \rho \le 1, \tag{2.4}
$$

$$\frac{\partial w}{\partial \tau} = D_2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial w}{\partial \rho}) + u(1,\tau)\rho\frac{\partial w}{\partial \rho} - \eta^2 g(w,p,q)w \quad \text{for } 0<\rho<1,\ \tau>0, \tag{2.5}$$

$$\frac{\partial w}{\partial \rho}(0,\tau) = 0, \quad w(1,\tau) = \bar{w} \quad \text{for } \tau > 0, \tag{2.6}$$

$$w(\rho,0) = w_0(\rho) \quad \text{for } 0 \le \rho \le 1, \tag{2.7}$$

$$\frac{\partial p}{\partial \tau} + \nu\frac{\partial p}{\partial \rho} = \eta^2[g_{11}(c,w,p,q,d)p + g_{12}(c,w,p,q,d)q + g_{13}(c,w,p,q,d)d] \quad \text{for } 0\le\rho\le 1, \tau>0, \tag{2.8}$$

$$\frac{\partial q}{\partial \tau} + \nu\frac{\partial q}{\partial \rho} = \eta^2[g_{21}(c,w,p,q,d)p + g_{22}(c,w,p,q,d)q + g_{23}(c,w,p,q,d)d] \quad \text{for } 0\le\rho\le 1, \tau>0, \tag{2.9}$$

$$\frac{\partial d}{\partial \tau} + \nu\frac{\partial d}{\partial \rho} = \eta^2[g_{31}(c,w,p,q,d)p + g_{32}(c,w,p,q,d)q + g_{33}(c,w,p,q,d)d] \quad \text{for } 0\le\rho\le 1, \tau>0, \tag{2.10}$$

$$\nu(\rho,\tau) = u(\rho,\tau) - \rho u(1,\tau) \quad \text{for } 0\le\rho\le 1,\ \tau>0, \tag{2.11}$$

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 u) = \eta^2(\tau)h(c,w,p,q,d) \quad \text{for } 0<\rho\le 1,\ \tau>0, \tag{2.12}$$

$$u(0,\tau) = 0 \quad \text{for } \tau>0, \tag{2.13}$$

$$\frac{d\eta(\tau)}{d\tau} = \eta(\tau)u(1,\tau) \quad \text{for } \tau>0, \tag{2.14}$$

$$p(\rho,0) = p_0(\rho), \quad q(\rho,0) = q_0(\rho), \quad d(\rho,0) = d_0(\rho) \quad \text{for } 0\le\rho\le 1, \tag{2.15}$$

$$\eta(0) = \eta_0, \tag{2.16}$$

where

$$\begin{gathered} f(c,p,q) = F(c,p,q), \quad g(w,p,q) = G(w,p,q), \\ \bar{c} = \bar{C}, \quad \bar{w} = \bar{W}, \quad c_0(\rho) = C_0(\rho R_0), \quad w_0(\rho) = W_0(\rho R_0), \\ p_0(\rho) = P_0(\rho R_0), \quad q_0(\rho) = Q_0(\rho R_0), \quad d_0(\rho) = D_0(\rho R_0), \\ \eta_0 = R_0. \end{gathered}$$

Conversely, if (η, c, w, p, q, d, u) is a solution of (2.2)–(2.16) such that $\eta(\tau) > 0$ for $\tau \ge 0$, then by making the change of variables

$$\begin{gathered} r = \rho\eta(\tau), \quad t = \int_0^\tau \eta^2(s)ds, \quad R(t) = \eta(\tau), \quad C(r,t) = c(\rho,\tau), \\ W(r,t) = w(\rho,\tau), \quad P(r,t) = p(\rho,\tau), \quad Q(r,t) = q(\rho,\tau), \\ D(r,t) = d(\rho,\tau), \quad v(r,t) = \frac{u(\rho,\tau)}{\eta(\tau)}. \end{gathered} \tag{2.17}$$

One can easily verify that (R, C, W, P, Q, D, v) is a solution to (1.15)–(1.27). Hence, we summarize the above result in the following lemma.

Lemma 2.1. *Under the change of variables* (2.1) *or its inverse* (2.17)*, the free boundary problem* (1.15)–(1.27) *is equivalent to the initial-boundary value problem* (2.2)–(2.16).

Remark 2.2. Note that from the (2.12), we obtain

$$u(\rho,\tau)=\frac{\eta^2(\tau)}{\rho^2}\int_0^\rho h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau))s^2ds. \tag{2.18}$$

Then, using (2.14) and (2.18) we have

$$\frac{d\eta(\tau)}{d\tau}=\eta^3(\tau)\int_0^1 h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau))s^2ds. \tag{2.19}$$

At first glance, we can not expect the solution of (2.2)–(2.16) exists for all $\tau\geq 0$, but since we make the change of variables $t=\int_0^\tau \eta^2(s)ds$ and $\tau=\int_0^t \frac{ds}{R^2(s)}$, we can prove the solution of (2.2)–(2.16) exists actually for all $\tau\geq 0$.

3. Preliminary Lemmas

In this section we present some preliminary lemmas which can be found in [12]. Let $Q_T=\{(x,\tau)\in\mathbb{R}^3\times\mathbb{R}:\ |x|<1,\ 0<\tau<T\}$ and $\bar{Q}_T$ denotes the closure of Q_T. For a vector-valued function (p,q,d) we denote

$$\|(p,q,d)\|_{L^\infty}=(\|p\|^2_{L^\infty}+\|q\|^2_{L^\infty}+\|d\|^2_{L^\infty})^{\frac{1}{2}}.$$

Without confusion we do not point out the explicit domain in the L^∞-norm in the whole article.

Lemma 3.1. *Let $\phi(\tau)$, $\varphi(\rho,\tau)$ and $\psi(\rho,\tau)$ be bounded continuous functions on $[0,T]$ and $[0,1]\times[0,T]$ $(T>0)$, respectively. Let $\bar{\sigma}$ be a constant, and σ_0 be a function on $[0,1]$ such that $\sigma_0(|x|)\in D_p(B_1)$ for some $p>\frac{5}{2}$, where B_1 denotes the unit ball in $\mathbb{R}^3$. Then the initial value problem*

$$\begin{gathered}\frac{\partial\sigma}{\partial\tau}=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\sigma}{\partial\rho})+\phi(\tau)\rho\frac{\partial\sigma}{\partial\rho}+\varphi(\rho,\tau)\sigma+\psi(\sigma,\rho)\\ \text{for } 0<\rho<1,\ 0<\tau\leq T,\end{gathered} \tag{3.1}$$

$$\frac{\partial\sigma}{\partial\rho}(0,\tau)=0,\quad \sigma(1,\tau)=\bar{\sigma}\quad \text{for } 0<\tau\leq T, \tag{3.2}$$

$$\sigma(\rho,0)=\sigma_0(\rho)\quad \text{for } 0\leq\rho\leq 1, \tag{3.3}$$

has a unique solution $\sigma(\rho,\tau)$ such that $\sigma(|x|,\tau)\in W^{2,1}_p(Q_T)$. Moreover, there exists a positive constant A depending only on p, T, $\|\phi\|_{L^\infty}$ and $\|\varphi\|_{L^\infty}$, such that

$$\|\sigma(|x|,\tau)\|_{W^{2,1}_p(Q_T)}\leq A(|\bar{\sigma}|+\|\sigma_0(|x|)\|_{D_p(B_1)}+\|\psi\|_{L^p}), \tag{3.4}$$

where A is bounded for T in any bounded set. Furthermore, the following estimate holds:

$$\|\sigma\|_{L^\infty}\leq\max\{|\bar{\sigma}|,\|\sigma_0\|_{L^\infty}\}+Te^{A_0T}\|\psi\|_{L^\infty}, \tag{3.5}$$

where $A_0=0$ if $\varphi\leq 0$ and $A_0=\max_{\overline{Q}_T}\varphi$ otherwise.

Proof. The proof of Lemma 3.1 relies on the standard L^p theory for parabolic equations and maximum principle. See [12] for more details. □

Lemma 3.2. *Assume that $\nu(\rho,\tau)$, $g_{ij}(\rho,\tau)$ $(i,j=1,2,3)$ and $f_i(\rho,\tau)$ $(i=1,2,3)$ are bounded functions defined on $[0,1]\times[0,T]$, $\nu(\rho,\tau)$ is continuously differentiable* *with respect to* ρ *and* $\nu(0,\tau)=\nu(1,\tau)=0$. *Then for any* α_0, β_0, $\gamma_0 \in C[0,1]$, *the initial value problem*

$$\frac{\partial\alpha}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\alpha}{\partial\rho}=g_{11}(\rho,\tau)\alpha+g_{12}(\rho,\tau)\beta+g_{13}(\rho,\tau)\gamma+f_1(\rho,\tau) \quad \text{for } 0\le\rho\le1,\quad 0<\tau\le T, \tag{3.6}$$

$$\frac{\partial\beta}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\beta}{\partial\rho}=g_{21}(\rho,\tau)\alpha+g_{22}(\rho,\tau)\beta+g_{23}(\rho,\tau)\gamma+f_2(\rho,\tau) \quad \text{for } 0\le\rho\le1,\quad 0<\tau\le T, \tag{3.7}$$

$$\frac{\partial\gamma}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\gamma}{\partial\rho}=g_{31}(\rho,\tau)\alpha+g_{32}(\rho,\tau)\beta+g_{33}(\rho,\tau)\gamma+f_3(\rho,\tau) \quad \text{for } 0\le\rho\le1,\quad 0<\tau\le T, \tag{3.8}$$

$$\alpha(\rho,0)=\alpha_0(\rho),\quad \beta(\rho,0)=\beta_0(\rho),\quad \gamma(\rho,0)=\gamma_0(\rho)\quad \text{for } 0\le\rho\le1, \tag{3.9}$$

has a unique weak solution α, β, $\gamma\in C([0,1]\times[0,T])$ *and the following estimate holds:*

$$\|(\alpha,\beta,\gamma)\|_{L^\infty}\le e^{TA_0(T)}\Big(\|(\alpha_0,\beta_0,\gamma_0)\|_{L^\infty}+T\|(f_1,f_2,f_3)\|_{L^\infty}\Big), \tag{3.10}$$

where $A_0(T)=2\max\{\|g_{ij}\|_{L^\infty}: i,j=1,2,3\}$. *If we assume further that* $g_{ij}(\rho,\tau)$ $(i,j=1,2,3)$ *and* $f_i(\rho,\tau)$ $(i=1,2,3)$ *are also continuously differentiable with respect to* ρ, *and* α_0, β_0, $\gamma_0\in C^1[0,1]$, *then the weak solution of* (3.6)–(3.9) *is a classical solution, and the following estimate holds:*

$$\begin{aligned}&\|(\frac{\partial\alpha}{\partial\rho},\frac{\partial\beta}{\partial\rho},\frac{\partial\gamma}{\partial\rho})\|_{L^\infty}\\&\le e^{T(A(T)+A_0(T))}\Big(\|(\alpha_0',\beta_0',\gamma_0')\|_{L^\infty}+TA_1(T)e^{TA(T)}\|(\alpha_0,\beta_0,\gamma_0)\|_{L^\infty}\\&\quad+Te^{TA(T)}\|(\frac{\partial f_1}{\partial\rho},\frac{\partial f_2}{\partial\rho},\frac{\partial f_3}{\partial\rho})\|_{L^\infty}\Big),\end{aligned} \tag{3.11}$$

where $A_0(T)$ *is as before, and*

$$A(T)=\|\frac{\partial\nu}{\partial\rho}\|_{L^\infty},\quad A_1(T)=\max\{\|\frac{\partial g_{ij}}{\partial\rho}\|_{L^\infty}:\ i,j=1,2,3\}.$$

If in addition, $g_{ij}\ge0$ *for* $i\ne j$, *and*

$$\alpha_0(\rho)\ge0,\quad \beta_0(\rho)\ge0,\quad \gamma_0(\rho)\ge0,\quad f_i(\rho,\tau)\ge0\ (i=1,2,3),$$

then we have

$$\alpha(\rho,\tau)\ge0,\quad \beta(\rho,\tau)\ge0,\quad \gamma(\rho,\tau)\ge0\quad \text{for } 0\le\rho\le1,\quad 0\le t\le T.$$

Proof. Using the characteristic theory of hyperbolic equations, we can transform (3.6)–(3.9) into an ordinary differential equations, the desired results readily follow from a simple analysis of this transformed equations. See [12] for more details. □

Lemma 3.3. *Let* $f_i(\rho,\tau,\alpha,\beta,\gamma)$ $(i=1,2,3)$ *be functions defined in* $[0,1]\times[0,T]\times\mathbb{R}^3$ *are continuous in all arguments and continuously differentiable in* $(\rho,\alpha,\beta,\gamma)$. *Let* $\nu(\rho,\tau)$ *be as in Lemma 3.2, consider the following initial value problem:*

$$\frac{\partial\alpha}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\alpha}{\partial\rho}=f_1(\rho,\tau,\alpha,\beta,\gamma)\quad \text{for } 0\le\rho\le1,\ 0<\tau\le T, \tag{3.12}$$

$$\frac{\partial\beta}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\beta}{\partial\rho}=f_2(\rho,\tau,\alpha,\beta,\gamma)\quad \text{for } 0\le\rho\le1,\ 0<\tau\le T, \tag{3.13}$$

$$\frac{\partial\gamma}{\partial\tau}+\nu(\rho,\tau)\frac{\partial\gamma}{\partial\rho}=f_3(\rho,\tau,\alpha,\beta,\gamma)\quad\text{for } 0\le\rho\le 1,\ 0<\tau\le T, \tag{3.14}$$

$$\alpha(\rho,0)=\alpha_0(\rho),\quad \beta(\rho,0)=\beta_0(\rho),\quad \gamma(\rho,0)=\gamma_0(\rho)\quad\text{for } 0\le\rho\le 1. \tag{3.15}$$

If $\alpha_0,\ \beta_0,\ \gamma_0\in C^1[0,1]$, then there exists $0<T_1\le T$ depending only on $M_0=\|(\alpha_0,\beta_0,\gamma_0\|_\infty$ and supremum norms of f_i, $\frac{\partial f_i}{\partial\alpha}$, $\frac{\partial f_i}{\partial\beta}$, $\frac{\partial f_i}{\partial\gamma}$ $(i=1,2,3)$ on the set $[0,1]\times[0,T]\times[-2M_0,2M_0]\times[-2M_0,2M_0]\times[-2M_0,2M_0]$ such that the above problem has a unique weak solution in $[0,1]\times[0,T_1]$ satisfying

$$\|(\alpha,\beta,\gamma)\|_{L^\infty}\le 2M_0. \tag{3.16}$$

Furthermore, if $\alpha_0,\ \beta_0,\ \gamma_0\in C^1[0,1]$, then this weak solution is actually a classical solution, and the following estimate holds

$$\begin{aligned}&\|(\frac{\partial\alpha}{\partial\rho},\frac{\partial\beta}{\partial\rho},\frac{\partial\gamma}{\partial\rho})\|_{L^\infty}\\&\le e^{T(A(T)+B_0(T))}\Big(\|(\alpha_0',\beta_0',\gamma_0')\|_{L^\infty}+Te^{TA(T)}\|(\frac{\partial f_1}{\partial\rho},\frac{\partial f_2}{\partial\rho},\frac{\partial f_3}{\partial\rho})\|_{L^\infty}\Big),\end{aligned} \tag{3.17}$$

where $A(T)=\|\frac{\partial\nu}{\partial\rho}\|_{L^\infty}$, and $B_0(T)=\max_{1\le i\le 3}\max\{\|\frac{\partial f_i}{\partial\alpha}\|_{L^\infty},\|\frac{\partial f_i}{\partial\beta}\|_{L^\infty},\|\frac{\partial f_i}{\partial\gamma}\|_{L^\infty}\}$.

Proof. Using the same argument as that of Lemma 3.2, and via a standard contraction argument, we can prove Lemma 3.3. See [12] for more details. □

4. Existence of a local solution

From the assumptions (A1)–(A4) in Section 1 and transformation (2.1) in Section 2, we can readily verify the following conditions hold:

- (B1) f, g and h are C^1-smooth functions;
- (B2) g_{ij} $(i,j=1,2,3)$ are C^1-smooth functions;
- (B3) p_0, q_0 and d_0 are C^1-smooth functions;
- (B4) $c_0(|x|),\ w_0(|x|)\in D_p(B_1)$ for some $p>5$.

We shall prove the local existence and uniqueness of solution to (2.2)–(2.16) by using Banach fixed point theorem and then prove it is actually a global one in Section 5. To this purpose, let

$$\begin{gathered}M_0=\|(p_0,q_0,d_0)\|_{L^\infty};\\ A_0=2\max\{|g_{ij}(c,w,p,q,d)|:0\le c\le\bar c,\ 0\le w\le\bar w,\\ |p|\le 2M_0,\ |q|\le 2M_0,\ |d|\le 2M_0,\ i,j=1,2,3\};\\ B_0=\max\{|h(c,w,p,q,d)|:0\le c\le\bar c,\ 0\le w\le\bar w,\\ |p|\le 2M_0,\ |q|\le 2M_0,\ |d|\le 2M_0\}.\end{gathered}$$

Now, given $T>0$, we introduce a metric space (X_T,d) as

$$\begin{aligned}X_T=\Big\{&(\eta(\tau),c(\rho,\tau),w(\rho,\tau),p(\rho,\tau),q(\rho,\tau),d(\rho,\tau))\ (0\le\rho\le 1,\ 0\le\tau\le T):\\&(\eta,c,w,p,q,d)\text{ satisfying the following conditions (C1)–(C4)}\Big\},\end{aligned}$$

- (C1) $\eta\in C[0,1]$, $\eta(0)=\eta_0$ and $\frac12\eta_0\le\eta(\tau)\le 2\eta_0$ $(0\le\tau\le T)$;
- (C2) $c\in C([0,1]\times[0,T])$, $c(\rho,0)=c_0(\rho)$, $c(1,\tau)=\bar c$ and $0\le c(\rho,\tau)\le\bar c$ for $0\le\rho\le 1$, $0\le\tau\le T$;
- (C3) $w\in C([0,1]\times[0,T])$, $w(\rho,0)=w_0(\rho)$, $w(1,\tau)=\bar w$ and $0\le w(\rho,\tau)\le\bar w$ for $0\le\rho\le 1$, $0\le\tau\le T$;

(C4) $p(\rho,\tau)$, $q(\rho,\tau)$, $d(\rho,\tau) \in C([0,1]\times[0,T])$, $p(\rho,0)=p_0(\rho)$, $q(\rho,0)=q_0(\rho)$, $d(\rho,0)=d_0(\rho)$ and $|p(\rho,\tau)|\le 2M_0$, $|q(\rho,\tau)|\le 2M_0$, $|d(\rho,\tau)|\le 2M_0$ for $0\le\rho\le1$, $0\le\tau\le T$.

The metric d in X_T is defined by

$$\begin{aligned}
&d\Big((\eta_1,c_1,w_1,p_1,q_1,d_1),(\eta_2,c_2,w_2,p_2,q_2,d_2)\Big)\\
&=\|\eta_1-\eta_2\|_{L^\infty}+\|c_1-c_2\|_{L^\infty}+\|w_1-w_2\|_{L^\infty}\\
&\quad+\|p_1-p_2\|_{L^\infty}+\|q_1-q_2\|_{L^\infty}+\|d_1-d_2\|_{L^\infty}.
\end{aligned}$$

It is easy to see (X_T,d) is a complete metric space.

Given any $(\eta,c,w,p,q,d)\in X_T$, set

$$\begin{gathered}
u(\rho,\tau)=\frac{\eta^2(\tau)}{\rho^2}\int_0^\rho h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau))s^2ds,\\
\nu(\rho,\tau)=u(\rho,\tau)-\rho u(1,\tau),\\
\phi(\rho,\tau)=\eta^2(\tau)f(c(s,\tau),p(s,\tau),q(s,\tau)),\\
\varphi(\rho,\tau)=\eta^2(\tau)g(w(s,\tau),p(s,\tau),q(s,\tau)).
\end{gathered}$$

Consider the following problem for $(\tilde\eta,\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)$:

$$\frac{d\tilde\eta}{d\tau}=\tilde\eta(\tau)u(1,\tau)\quad\text{for } 0<\tau\le T, \tag{4.1}$$
$$\tilde\eta(0)=\eta_0, \tag{4.2}$$
$$\frac{\partial\tilde c}{\partial\tau}=\frac{D_1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\tilde c}{\partial\rho})+u(1,\tau)\rho\frac{\partial\tilde c}{\partial\rho}-\phi(\rho,\tau)\tilde c\quad\text{for } 0<\rho<1,\ 0<\tau\le T, \tag{4.3}$$
$$\frac{\partial\tilde c}{\partial\rho}(0,\tau)=0,\quad \tilde c(1,\tau)=\bar c\quad\text{for } 0<\tau\le T, \tag{4.4}$$
$$\tilde c(\rho,0)=c_0(\rho)\quad\text{for } 0\le\rho\le1, \tag{4.5}$$
$$\frac{\partial\tilde w}{\partial\tau}=\frac{D_2}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\tilde w}{\partial\rho})+u(1,\tau)\rho\frac{\partial\tilde w}{\partial\rho}-\varphi(\rho,\tau)\tilde w\quad\text{for } 0<\rho<1,\ 0<\tau\le T, \tag{4.6}$$
$$\frac{\partial\tilde w}{\partial\rho}(0,\tau)=0,\quad \tilde w(1,\tau)=\bar w\quad\text{for } 0<\tau\le T, \tag{4.7}$$
$$\tilde w(\rho,0)=w_0(\rho)\quad\text{for } 0\le\rho\le1, \tag{4.8}$$
$$\begin{gathered}\frac{\partial\tilde p}{\partial\tau}+\nu\frac{\partial\tilde p}{\partial\rho}=\eta^2[g_{11}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde p+g_{12}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde q+g_{13}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde d]\\ \text{for } 0\le\rho\le1,\ 0<\tau\le T,\end{gathered} \tag{4.9}$$
$$\begin{gathered}\frac{\partial\tilde q}{\partial\tau}+\nu\frac{\partial\tilde q}{\partial\rho}=\eta^2[g_{21}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde p+g_{22}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde q+g_{23}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde d]\\ \text{for } 0\le\rho\le1,\ 0<\tau\le T,\end{gathered} \tag{4.10}$$
$$\begin{gathered}\frac{\partial\tilde d}{\partial\tau}+\nu\frac{\partial\tilde d}{\partial\rho}=\eta^2[g_{31}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde p+g_{32}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde q+g_{33}(\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)\tilde d]\\ \text{for } 0\le\rho\le1,\ 0<\tau\le T,\end{gathered} \tag{4.11}$$
$$\tilde p(\rho,0)=p_0(\rho),\quad \tilde q(\rho,0)=q_0(\rho),\quad \tilde d(\rho,0)=d_0(\rho)\quad\text{for } 0\le\rho\le1. \tag{4.12}$$

With this problem solved, we define a mapping $F:(\eta,c,w,p,q,d)\mapsto(\tilde\eta,\tilde c,\tilde w,\tilde p,\tilde q,\tilde d)$. Next we shall prove that F is a contraction mapping from X_T to X_T provided T is sufficiently small.

Step 1 First we prove F maps X_T into itself. It is obvious that (4.1)–(4.2) has a unique solution $\tilde\eta\in C^1[0,T]$ and

$$\tilde\eta(\tau)=\eta_0\exp\{\int_0^\tau u(1,s)ds\}\quad\text{for } 0\le\tau\le T. \tag{4.13}$$

From the fact that $\|h(c(\rho,\tau),w(\rho,\tau),p(\rho,\tau),q(\rho,\tau),d(\rho,\tau))\|_{L^\infty}\le B_0$ and $\frac12\eta_0<\eta(\tau)\le 2\eta_0$, we know $\|u(1,\tau)\|_{L^\infty}\le\frac43B_0\eta_0^2$, then we have

$$\eta_0\exp\{-\frac43B_0\eta_0^2T\}\le\tilde\eta(\tau)\le\eta_0\exp\{\frac43B_0\eta_0^2T\}\quad\text{for } 0\le\tau\le T. \tag{4.14}$$

So if we choose T sufficiently small such that $\exp\{\frac43B_0\eta_0^2T\}\le 2$, we have $\frac12\eta_0\le\tilde\eta\le 2\eta_0$, that implies $\tilde\eta$ satisfies the condition (C1).

Next we consider (4.3)–(4.5) and (4.6)–(4.8). Since $c_0(|x|)$, $w_0(|x|)\in D_p(B_1)$ for some $p>5$, then from Lemma 3.1 we know (4.3)–(4.5) and (4.6)–(4.8) has a unique solution $\tilde c(|x|,\tau)\in W_p^{2,1}(Q_T)$ and $\tilde w(|x|,\tau)\in W_p^{2,1}(Q_T)$, respectively. According to the embedding theorem, $W_p^{2,1}(Q_T)\hookrightarrow C^{\lambda,\frac{\lambda}{2}}(\overline{Q}_T)$, where $\lambda=2-\frac5p$ (see [22]), then we know $\tilde c(|x|,\tau)$, $\tilde w(|x|,\tau)\in C([0,1]\times[0,T])$. By applying the maximum principle we have $0\le\tilde c\le\bar c$ and $0\le\tilde w\le\bar w$. Furthermore, by (3.4) and the embedding $W_p^{2,1}(Q_T)\hookrightarrow C^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q}_T)$ with $\lambda=1-\frac5p$ (see [22]), we have

$$\|\frac{\partial\tilde c}{\partial\rho}\|_{L^\infty}\le A(T),\quad \|\frac{\partial\tilde w}{\partial\rho}\|_{L^\infty}\le A(T).$$

From above results, we know $\tilde c$ satisfies the condition (C2) and $\tilde w$ satisfies the condition (C3).

Finally we consider (4.9)–(4.12). Since $\nu(\rho,\tau)$, $\tilde c(\rho,\tau)$ and $\tilde w(\rho,\tau)$ are continuously differentiable, then from Lemma 3.3 we obtain that if we take T small enough, (4.9)–(4.12) has a unique classical solution $(\tilde p,\tilde q,\tilde d)\in C^1([0,1]\times[0,T])$ satisfying

$$|\tilde p|\le 2M_0,\quad |\tilde q|\le 2M_0,\quad |\tilde d|\le 2M_0\quad\text{for } 0\le\rho\le1,\ 0\le\tau\le T. \tag{4.15}$$

Furthermore, by (3.17) in Lemma 3.3, if T is small enough, then we have

$$\|\Big(\frac{\partial\tilde p}{\partial\rho},\frac{\partial\tilde q}{\partial\rho},\frac{\partial\tilde d}{\partial\rho}\Big)\|_{L^\infty}\le 2M_1\quad\text{for } 0\le\rho\le1,\ 0\le\tau\le T, \tag{4.16}$$

where $M_1=\|(p_0',q_0',d_0')\|_{L^\infty}$. This implies $\tilde p$, $\tilde q$ and $\tilde d$ satisfy the condition (C4).

Now we can see that for a sufficiently small T, $F:X_T\mapsto X_T$ is well-defined. To obtain the desired result we only need to prove $F:X_T\mapsto X_T$ is a contraction mapping if T is further small enough.

Step 2 Let $(\eta_i,c_i,w_i,p_i,q_i,d_i)\in X_T$ $(i=1,2)$, set

$$u_i(\rho,\tau)=\frac{\eta_i^2(\tau)}{\rho^2}\int_0^\rho h(c_i(s,\tau),w_i(s,\tau),p_i(s,\tau),q_i(s,\tau),d_i(s,\tau))s^2ds,$$
$$\nu_i(\rho,\tau)=u_i(\rho,\tau)-\rho u_i(1,\tau),$$
$$(\tilde\eta_i,\tilde c_i,\tilde w_i,\tilde p_i,\tilde q_i,\tilde d_i)=F(\eta_i,c_i,w_i,p_i,q_i,d_i),$$
$$d=d\Big((\eta_1,c_1w_1,p_1,q_1,d_1),(\eta_2,c_2,w_2,p_2,q_2,d_2)\Big).$$

Firstly from $\|h(c_i(\rho,\tau),w_i(\rho,\tau),p_i(\rho,\tau),q_i(\rho,\tau),d_i(\rho,\tau))\|_{L^\infty} \le B_0$ and $\frac{1}{2}\eta_0 < \eta_i(\tau) \le 2\eta_0$, we can easily calculate that

$$|u_1(\rho,\tau)-u_2(\rho,\tau)| \le A(T)d. \tag{4.17}$$

Then by (4.13) we get

$$\|\tilde\eta_1-\tilde\eta_2\|_{L^\infty} \le \max_{0\le\tau\le T}|\tilde\eta_1(\tau)-\tilde\eta_2(\tau)| \le TA(T)d. \tag{4.18}$$

Next, let $\tilde c_* = \tilde c_1-\tilde c_2$ and $\tilde w_* = \tilde w_1 - \tilde w_2$, we have

$$\begin{gathered}\frac{\partial \tilde c_*}{\partial\tau} = \frac{D_1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\tilde c_*}{\partial\rho}) + u_1(1,\tau)\rho\frac{\partial\tilde c_*}{\partial\rho} - \phi(\rho,\tau)\tilde c_* + F(\rho,\tau)\\ \text{for } 0<\rho<1,\ 0<\tau\le T,\end{gathered} \tag{4.19}$$

$$\frac{\partial\tilde c_*}{\partial\rho}(0,\tau)=0, \quad \tilde c_*(1,\tau)=0 \quad \text{for } 0\le\tau\le T, \tag{4.20}$$

$$\tilde c_*(\rho,0)=0 \quad \text{for } 0\le\rho\le 1, \tag{4.21}$$

$$\begin{gathered}\frac{\partial \tilde w_*}{\partial\tau} = \frac{D_2}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial\tilde w_*}{\partial\rho}) + u_1(1,\tau)\rho\frac{\partial\tilde w_*}{\partial\rho} - \varphi(\rho,\tau)\tilde w_* + G(\rho,\tau)\\ \text{for } 0<\rho<1,\ 0<\tau\le T,\end{gathered} \tag{4.22}$$

$$\frac{\partial\tilde w_*}{\partial\rho}(0,\tau)=0, \quad \tilde w_*(1,\tau)=0 \quad \text{for } 0\le\tau\le T, \tag{4.23}$$

$$\tilde w_*(\rho,0)=0 \quad \text{for } 0\le\rho\le 1, \tag{4.24}$$

where

$$\phi(\rho,\tau) = \eta_1^2(\tau)f(c_1(s,\tau),p_1(s,\tau),q_1(s,\tau)),$$
$$\varphi(\rho,\tau) = \eta_1^2(\tau)g(w_1(s,\tau),p_1(s,\tau),q_1(s,\tau)),$$
$$F(\rho,\tau) = [u_1(1,\tau)-u_2(1,\tau)]\rho\frac{\partial\tilde c_2}{\partial\rho} + [\eta_2^2(\tau)f(c_2,p_2,q_2) - \eta_1^2(\tau)f(c_1,p_1,q_1)]\tilde c_2,$$
$$G(\rho,\tau) = [u_1(1,\tau)-u_2(1,\tau)]\rho\frac{\partial\tilde w_2}{\partial\rho} + [\eta_2^2(\tau)g(w_2,p_2,q_2) - \eta_1^2(\tau)g(w_1,p_1,q_1)]\tilde w_2.$$

As for $\tilde c$, from the Lemma 3.1 we know $\|\frac{\partial\tilde c_2}{\partial\rho}\|_{L^\infty} \le A(T)$ and $0\le\tilde c_2(\rho,\tau)\le\bar c$ by maximum principle. Note that f is continuously differentiable and η_i, p_i, q_i are bounded, so we can deduce that

$$\|F\|_{L^\infty} \le A(T)\|u_1-u_2\|_{L^\infty} + \|\eta_2^2 f(c_2,p_2,q_2) - \eta_1^2 f(c_1,p_1,q_1)\|_{L^\infty} \le A(T)d. \tag{4.25}$$

Then from Lemma 3.1 again we obtain that

$$\|\tilde c_1-\tilde c_2\|_{L^\infty} = \|\tilde c_*\|_{L^\infty} \le T\|F\|_{L^\infty} \le TA(T)d. \tag{4.26}$$

Similarly, for $\tilde w$, we obtain

$$\|G\|_{L^\infty} \le A(T)\|u_1-u_2\|_{L^\infty} + A(T)d \le A(T)d. \tag{4.27}$$

Then from Lemma 3.1 again we obtain

$$\|\tilde w_1-\tilde w_2\|_{L^\infty} = \|\tilde w_*\|_{L^\infty} \le T\|G\|_{L^\infty} \le TA(T)d. \tag{4.28}$$

Finally, letting $\tilde p_* = \tilde p_1-\tilde p_2$, $\tilde q_* = \tilde q_1 - \tilde q_2$, $\tilde d_* = \tilde d_1 - \tilde d_2$, we have:

$$\begin{gathered}\frac{\partial\tilde p_*}{\partial\tau} + \nu_1\frac{\partial\tilde p_*}{\partial\rho} = \lambda_{11}(\rho,\tau)\tilde p_* + \lambda_{12}(\rho,\tau)\tilde q_* + \lambda_{13}(\rho,\tau)\tilde d_* + F_1(\rho,\tau)\\ \text{for } 0\le\rho\le 1,\ 0<\tau\le T,\end{gathered} \tag{4.29}$$

$$\frac{\partial \tilde{q}_*}{\partial \tau} + \nu_1 \frac{\partial \tilde{q}_*}{\partial \rho} = \lambda_{21}(\rho,\tau)\tilde{p}_* + \lambda_{22}(\rho,\tau)\tilde{q}_* + \lambda_{23}(\rho,\tau)\tilde{d}_* + F_2(\rho,\tau) \quad \text{for } 0 \le \rho \le 1, \quad 0 < \tau \le T, \tag{4.30}$$

$$\frac{\partial \tilde{d}_*}{\partial \tau} + \nu_1 \frac{\partial \tilde{d}_*}{\partial \rho} = \lambda_{31}(\rho,\tau)\tilde{p}_* + \lambda_{32}(\rho,\tau)\tilde{q}_* + \lambda_{33}(\rho,\tau)\tilde{d}_* + F_3(\rho,\tau) \quad \text{for } 0 \le \rho \le 1, \ 0 < \tau \le T, \tag{4.31}$$

$$\tilde{p}_*(\rho,0) = 0, \quad \tilde{q}_*(\rho,0) = 0, \quad \tilde{d}_*(\rho,0) = 0 \quad \text{for } 0 \le \rho \le 1, \tag{4.32}$$

where

$$\lambda_{ij} = \eta_1^2(\tau) g_{ij}(\tilde{c}_1, \tilde{w}_1, \tilde{p}_1, \tilde{q}_1, \tilde{d}_1) \quad (i,j = 1,2,3),$$

$$F_i(\rho,\tau) = (\nu_2 - \nu_1)\frac{\partial \tilde{\xi}_i}{\partial \rho} + \sum_{j=1}^{3} [\eta_1^2 g_{ij}(\tilde{c}_1, \tilde{w}_1, \tilde{p}_1, \tilde{q}_1, \tilde{d}_1) - \eta_2^2 g_{ij}(\tilde{c}_2, \tilde{w}_2, \tilde{p}_2, \tilde{q}_2, \tilde{d}_2)]\tilde{\xi}_j,$$

and $\tilde{\xi}_1 = \tilde{p}_2$, $\tilde{\xi}_2 = \tilde{q}_2$, $\tilde{\xi}_3 = \tilde{d}_2$. Then from (4.15)–(4.16) we know that

$$\|\tilde{p}_i\|_{L^\infty} \le 2M_0, \quad \|\tilde{q}_i\|_{L^\infty} \le 2M_0, \quad \|\tilde{d}_i\|_{L^\infty} \le 2M_0, \quad i = 1,2,$$

$$\|(\frac{\partial \tilde{p}_i}{\partial \rho}, \frac{\partial \tilde{q}_i}{\partial \rho}, \frac{\partial \tilde{d}_i}{\partial \rho})\|_{L^\infty} \le 2M_1, \quad i = 1,2,$$

and since g_{ij} $(i,j = 1,2,3)$ are continuously differentiable, we deduce that

$$\begin{aligned} \|F_i\|_{L^\infty} &\le A(T)\|\nu_1 - \nu_2\|_{L^\infty} + A(T)\sum_{j=1}^{3} \|\eta_1^2 g_{ij}(\tilde{c}_1, \tilde{w}_1, \tilde{p}_1, \tilde{q}_1, \tilde{d}_1) \\ &\quad - \eta_2^2 g_{ij}(\tilde{c}_2, \tilde{w}_2, \tilde{p}_2, \tilde{q}_2, \tilde{d}_2)\|_{L^\infty} \le A(T)d, \quad i = 1,2,3. \end{aligned} \tag{4.33}$$

It is easy to see λ_{ij} $(i,j = 1,2,3)$ are bounded by a constant independent of the choice of $(\eta_i, c_i, p_i, q_i, d_i)$, so from (3.17) in Lemma 3.3 and (4.33) we have

$$\|(\tilde{p}_1 - \tilde{p}_2, \tilde{q}_1 - \tilde{q}_2, \tilde{d}_1 - \tilde{d}_2)\|_{L^\infty} = \|(\tilde{p}_*, \tilde{q}_*, \tilde{d}_*)\|_{L^\infty} \le TA(T)d. \tag{4.34}$$

By (4.16), (4.26), (4.28) and (4.34) we conclude that

$$d\Big((\tilde{\eta}_1, \tilde{c}_1, \tilde{w}_1, \tilde{p}_1, \tilde{q}_1, \tilde{d}_1), (\tilde{\eta}_2, \tilde{c}_2, \tilde{w}_2, \tilde{p}_2, \tilde{q}_2, \tilde{d}_2)\Big) \le TA(T)d.$$

Hence, if we choose T sufficiently small such that $TA(T) < 1$, then F is a contraction mapping from X_T into X_T.

According to the Banach fixed point theorem we know that if T is small enough then F has a unique fixed point (η, c, w, p, q, d) for $0 \le \tau \le T$. By the definition of the mapping F, it is clearly that (η, c, w, p, q, d) is the unique solution of the problem (2.2)–(2.16) for $0 \le \tau \le T$.

Theorem 4.1. *Under the assumptions of Theorem 1.3, there exists $T > 0$ depending only on $\|c_0(|x|)\|_{D_p(B_{R_0})}$, $\|w_0(|x|)\|_{D_p(B_{R_0})}$, $\|(p_0, q_0, d_0)\|_{L^\infty}$, $\|(p_0', q_0', d_0')\|_{L^\infty}$, such that the problem* (2.2)–(2.16) *has a unique solution for $0 \le \tau \le T$.*

5. Existence of global solutions

Note that from Theorem 4.1 we know (2.2)–(2.16) has a unique local solution, then by Lemma 2.1, we know the problem (1.15)–(1.27) has also a unique local solution for $0 \le \tau \le T$, where T is some positive constant which may depend on the bound of R_0, $\|C_0(|x|)\|_{D_p(B_{R_0})}$, $\|W_0(|x|)\|_{D_p(B_{R_0})}$, $\|(P_0, Q_0, D_0)\|_{L^\infty(B_{R_0})}$ and

$\|(P_0', Q_0', D_0')\|_{L^\infty(B_{R_0})}$. To get the global result of Theorem 1.3, we establish the following two preliminary lemmas.

Lemma 5.1. *Under the assumptions of Theorem 1.3, if*
$$\Big(R(t), C(r,t), W(r,t), P(r,t), Q(r,t), D(r,t)\Big)$$
is a solution of (1.15)–(1.27) *for* $0 \le t < T$*, then*
$$0 \le C(r,t) \le \bar{C} \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T, \tag{5.1}$$
$$0 \le W(r,t) \le \bar{W} \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T, \tag{5.2}$$
$$P(r,t) \ge 0, \quad Q(r,t) \ge 0, \quad D(r,t) \ge 0 \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T, \tag{5.3}$$
$$P(r,t) + Q(r,t) + D(r,t) = N \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T, \tag{5.4}$$
$$R_0 \exp\{-\frac{1}{3}B_0 t\} \le R(t) \le R_0 \exp\{\frac{1}{3}B_0 t\} \quad \text{for } 0 \le t < T, \tag{5.5}$$
$$-\frac{1}{3}B_0 R(t) \le \frac{dR(t)}{dt} \le \frac{1}{3}B_0 R(t) \quad \text{for } 0 \le t < T, \tag{5.6}$$
where
$$B_0 = \max\{|h(C, W, P, Q, D)| : 0 \le C \le \bar{C},\ 0 \le W \le \bar{W},\ 0 \le P,\ Q,\ D \le N\}.$$

Proof. Note that (5.1) and (5.2) are immediate results by applying the maximum principle. From (1.28) in Remark 1.1 and Lemma 3.3 we know (5.3) holds. To prove (5.4) we represent $M(r,t) = P(r,t) + Q(r,t) + D(r,t)$, then summing up (1.21)–(1.23) and using (1.29) in Remark 1.1, we can get $M(r,t)$ satisfies the following equation:
$$\frac{\partial M}{\partial t} + v\frac{\partial M}{\partial r} = \frac{1}{N}[K_B(C) - K_R D](N - M) \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T, \tag{5.7}$$
$$M(r,0) = P_0(r) + Q_0(r) + D_0(r) = N \quad \text{for } 0 \le r \le R_0. \tag{5.8}$$
It is clear that $M(r,t) = N$ is a solution of (5.7)–(5.8), by uniqueness we obtain that $M(r,t) = N$ for all $0 \le r \le R(t)$, $0 \le t < T$, this completes the proof of (5.4). From (5.3) and (5.4) we get
$$0 \le P(r,t),\ Q(r,t),\ R(r,t) \le N \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T.$$
It is obvious that $|h(C, W, P, Q, D)| \le B_0$, then by (1.24), we have
$$-\frac{1}{3}B_0 r \le v(r,t) \le \frac{1}{3}B_0 r \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T. \tag{5.9}$$
From (1.26) we can see
$$-\frac{1}{3}B_0 R(t) \le \frac{dR(t)}{dt} \le \frac{1}{3}B_0 R(t).$$
Hence, we complete the proof of (5.6). (5.5) is an immediate consequence of (5.6). □

Lemma 5.2. *Under the assumptions of Theorem 1.3, if*
$$\Big(R(t), C(r,t), W(r,t), P(r,t), Q(r,t), D(r,t)\Big)$$
is a solution of (1.15)–(1.27) *for* $0 \le t < T$*, then*
$$\begin{gathered}\|C(r,t)\|_{W_p^{2,1}(Q_T^R)} \le A(T), \quad \|W(r,t)\|_{W_p^{2,1}(Q_T^R)} \le A(T) \\ \text{for } 0 \le r \le R(t),\ 0 \le t < T,\end{gathered} \tag{5.10}$$

$$\|(\frac{\partial P}{\partial r}, \frac{\partial Q}{\partial r}, \frac{\partial D}{\partial r})\|_{L^\infty} \le A(T) \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T. \tag{5.11}$$

An immediate consequence from (5.10) *we obtain that for any* $t_0 \in [0, T)$*,*

$$\|C(r, t_0)\|_{D_p(B(t_0))} \le A(T), \quad \|W(r, t_0)\|_{D_p(B(t_0))} \le A(T). \tag{5.12}$$

Proof. From Lemma 5.1 we know $R(t)$ has a positive lower bound $R_0 \exp\{-\frac{1}{3}B_0 T\}$ and a finite upper bound $R_0 \exp\{\frac{1}{3}B_0 T\}$, $\frac{dR(t)}{dt}$ is also bound for $0 \le t < T$, by (5.3) and (5.4) we know P, Q and D are also bound. Let

$$c(x, t) = C(|x|R(t), t), \quad w(x, t) = W(|x|R(t), t) \quad \text{for } |x| \le 1,\ 0 \le t < T,$$

and we denote $\dot{R}(t) = \frac{dR(t)}{dt}$. Then from (1.15)–(1.17) and (1.18)–(1.20) we can see that c is a solution of the following problem:

$$\frac{\partial c}{\partial t} = \frac{D_1}{R^2(t)}\Delta c + \frac{\dot{R}(t)}{R(t)}(x \cdot \nabla c) - f(x, t)c \quad \text{for } |x| < 1,\ 0 < t < T, \tag{5.13}$$

$$c(x, t) = \bar{C} \quad \text{for } |x| = 1,\ 0 < t < T, \tag{5.14}$$

$$c(x, 0) = C_0(|x|R_0) \quad \text{for } |x| \le 1, \tag{5.15}$$

and w is a solution of the following problem:

$$\frac{\partial w}{\partial t} = \frac{D_2}{R^2(t)}\Delta w + \frac{\dot{R}(t)}{R(t)}(x \cdot \nabla w) - g(x, t)w \quad \text{for } |x| < 1,\ 0 < t < T, \tag{5.16}$$

$$w(x, t) = \bar{W} \quad \text{for } |x| = 1,\ 0 < t < T, \tag{5.17}$$

$$w(x, 0) = W_0(|x|R_0) \quad \text{for } |x| \le 1. \tag{5.18}$$

Here

$$\begin{aligned} f(x, t) &= F(C(|x|R(t), t), P(|x|R(t), t), Q(|x|R(t), t)), \\ g(x, t) &= G(W(|x|R(t), t), P(|x|R(t), t), Q(|x|R(t), t)). \end{aligned}$$

Since all coefficients in (5.13) and (5.16) are bounded continuous functions, then from Lemma 3.1 we get

$$\|c\|_{W_p^{2,1}(Q_T)} \le A(T), \quad \|w\|_{W_p^{2,1}(Q_T)} \le A(T).$$

Now transforming back to the original variables we know

$$\|C(|x|, t)\|_{W_p^{2,1}(Q_T^R)} \le A(T), \quad \|W(|x|, t)\|_{W_p^{2,1}(Q_T^R)} \le A(T).$$

Besides, since all coefficients in (1.21)–(1.23) are bounded continuously differentiable functions, so from Lemma 3.3 we get

$$\|(\frac{\partial P}{\partial r}, \frac{\partial Q}{\partial r}, \frac{\partial D}{\partial r})\|_{L^\infty} \le A(T) \quad \text{for } 0 \le r \le R(t),\ 0 \le t < T.$$

We complete the proof of Lemma 5.2. □

From a priori estimates established in Lemma 5.1 and Lemma 5.2, now we can extend the local solution of (1.15)–(1.27) to the global one.

Theorem 5.3. *Under the assumptions of Theorem 1.3, there exists a unique global solution* $\big(R(t),\ C(r, t), W(r, t), P(r, t), Q(r, t), D(r, t)\big)$ *of* (1.15)–(1.27).

Proof. From Section 4 we know that (1.15)–(1.27) has a unique local (in time) solution, we can extend this local solution step by step to get a solution defined in a maximal time interval $[0, T)$ with either $T = \infty$ or $0 < T < \infty$. In what follows we show, by using the method of reducing into absurdity, that the second case cannot occur. Hence we assume that $T < \infty$, then for any $0 < t, t_0 < T$, from Lemma 5.1 and Lemma 5.2 we have

$$\begin{gathered}
\|C(|x|,t)\|_{W_p^{2,1}(Q_T^R)} \le A(T), \quad \|W(|x|,t)\|_{W_p^{2,1}(Q_T^R)} \le A(T),\\
\|C(|x|,t_0)\|_{D_p(B(t_0))} \le A(T), \quad \|W(|x|,t_0)\|_{D_p(B(t_0))} \le A(T),\\
\|(P,Q,D)\|_{L^\infty} \le A(T),\\
\|(\frac{\partial P}{\partial r}, \frac{\partial Q}{\partial r}, \frac{\partial D}{\partial r})\|_{L^\infty} \le A(T),\\
R_0 \exp\{-\frac{1}{3}B_0 t\} \le R(t) \le R_0 \exp\{\frac{1}{3}B_0 t\},\\
-\frac{1}{3}B_0 R(t) \le \frac{dR(t)}{dt} \le \frac{1}{3}B_0 R(t).
\end{gathered}$$

Hence if we consider the initial value problem (1.15)–(1.27) with initial data given at t_0 for every $t_0 \in [0, T)$, then by Theorem 4.1, there exists a common constant $\delta > 0$ such that the problem (1.15)–(1.27) always has a solution on the time interval $[t_0, t_0+\delta)$. It follows that the solution $\Big(R(t), C(r,t), W(r,t), P(r,t), Q(r,t), D(r,t)\Big)$ is extended to the time interval $[0, T+\delta)$, which contradicts the definition of T. Hence the solution of (1.15)–(1.27) exists for all $t \ge 0$. $\square$

By Lemma 2.1 and Theorem 5.3, we accomplish the proof of Theorem 1.3.

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