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# OPTIMIZATION IN PROBLEMS INVOLVING THE P-LAPLACIAN

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Abstract. We minimize the energy integral $\int_\Omega |\nabla u|^p \, dx$, where $g$ is a bounded positive function that varies in a class of rearrangements, $p > 1$, and $u$ is a solution of

$$\begin{gathered} -\Delta_p u = g \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \,. \end{gathered}$$

Also we maximize the first eigenvalue $\lambda = \lambda_g$, where

$$-\Delta_p u = \lambda g u^{p-1} \quad \text{in } \Omega \,.$$

For both problems, we prove existence, uniqueness, and representation of the optimizers.

## 1. Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, and let $g_0$ be a measurable function satisfying $0 < g_0 \le H$ in $\Omega$ for a positive constant $H$. Define $\mathcal{G}$ as the family of measurable functions which are rearrangements of $g_0$. In Section 2 of this article, we consider the problem

$$\begin{gathered} -\Delta_p u = g \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 1$, $g \in \mathcal{G}$. The operator $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, stands for the usual $p$-Laplacian defined as

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

It is well known that (1.1) has a unique solution $u \in W_0^{1,p}(\Omega)$. Corresponding to $g$, we consider the so called energy integral

$$I(g) = \int_\Omega |\nabla u|^p dx.$$

It is useful to investigate the maximum or the minimum of $I(g)$ when $g$ varies in $\mathcal{G}$. Actually, the maximum of $I(g)$ has been discussed in the paper [7]. In the present paper we investigate the minimum of $I(g)$ for $g \in \mathcal{G}$, proving results of existence

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and uniqueness of the minimizer $\check{g}$. We also find a formula of representation for $\check{g}$ in terms of the corresponding solution $u_{\check{g}}$.

In Section 3, we consider the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda g u^{p-1}, \quad u(x) > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\lambda$ is the first eigenvalue. It is well known that problem (1.2) has a first positive eigenvalue $\lambda = \lambda_g$ and a corresponding positive eigenfunction $u = u_g$. It is interesting to investigate the maximum and the minimum of $\lambda_g$ for $g \in \mathcal{G}$. Actually, the minimum of $\lambda_g$ has been discussed in the paper [8]. In the present paper we investigate the maximum of $\lambda_g$ for $g \in \mathcal{G}$, proving results of existence and uniqueness. We also find a formula of representation for the maximizer $\hat{g}$ in terms of the corresponding eigenfunction $u_{\hat{g}}$. We emphasize that the methods developed here are different from those used in the papers [7] and [8].

Since we use the notion of rearrangements, let us recall the definition. Denote with $|E|$ the Lebesgue measure of the (measurable) set $E$. Given a function $g_0(x)$ defined in $\Omega$ and satisfying $0 < g_0(x) \le H$ for a constant $H$. We say that $g(x)$ belongs to the class of rearrangements $\mathcal{G} = \mathcal{G}(g_0)$ if

$$|\{g(x) \ge \beta\}| = |\{g_0(x) \ge \beta\}| \quad \forall \beta \in (0, H).$$

Here we write $\{g(x) \ge \beta\}$ instead of $\{x \in \Omega : g(x) \ge \beta\}$. In what follows, we shall use the following results

**Lemma 1.1.** *Let $g \in L^r(\Omega)$, $r > 1$, and let $u \in L^s(\Omega)$, $s = r/(r-1)$. Suppose that every level set of $u$ has measure zero. Then there exists an increasing function $\phi$ such that $\phi(u)$ is a rearrangement of $g$. Furthermore, there exists a decreasing function $\psi$ such that $\psi(u)$ is a rearrangement of $g$.*

*Proof.* The first assertion follows from [4, Lemma 2.9]. The second assertion follows applying the first one to $-u$. □

Denote with $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ with respect to the weak* topology in $L^\infty(\Omega)$.

**Lemma 1.2.** *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^r(\Omega)$, $r > 1$, and let $u \in L^s(\Omega)$, $s = r/(r-1)$. If there is an increasing function $\phi$ such that $\phi(u) \in \mathcal{G}$ then*

$$\int_\Omega gu\,dx \le \int_\Omega \phi(u)u\,dx \quad \forall g \in \overline{\mathcal{G}},$$

*and the function $\phi(u)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function $\psi$ such that $\psi(u) \in \mathcal{G}$ then*

$$\int_\Omega gu\,dx \ge \int_\Omega \psi(u)u\,dx \quad \forall g \in \overline{\mathcal{G}},$$

*and the function $\psi(u)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.*

*Proof.* The first assertion follows from [4, Lemma 2.4]. To prove the second assertion we put $\phi(t) = \psi(-t)$. Since $\phi$ is increasing, applying the previous result we have

$$\int_\Omega g\,(-u)\,dx \le \int_\Omega \phi(-u)\,(-u)\,dx \quad \forall g \in \overline{\mathcal{G}},$$

and $\phi(-u) = \psi(u)$ is the unique function satisfying the inequality. Equivalently, we have

$$\int_\Omega gu\,dx \geq \int_\Omega \psi(u)u\,dx \quad \forall g \in \overline{\mathcal{G}}.$$

The proof is complete. □

## 2. Energy integral

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $p > 1$, $H > 0$ be two real numbers. Let $\mathcal{G}$ be the family of all rearrangements of a given function $g_0$ with $0 < g_0(x) \leq H$. It is convenient to introduce $\overline{\mathcal{G}}$, the closure of $\mathcal{G}$ with respect to the weak* topology in $L^\infty(\Omega)$. By [3] or [4], we know that $\overline{\mathcal{G}}$ is weakly compact and convex. Moreover, each $g \in \overline{\mathcal{G}}$ satisfies $0 < g(x) \leq H$ a.e. in $\Omega$.

For $g \in \overline{\mathcal{G}}$, we consider problem (1.1). It is a classical result that such a problem has a unique positive solution $u \in W_0^{1,p}(\Omega)$ which satisfies

$$\sup_{v\in W_0^{1,p}(\Omega)} \int_\Omega (pgv - |\nabla v|^p)dx = \int_\Omega (pgu - |\nabla u|^p)dx = (p-1)\int_\Omega |\nabla u|^p dx. \tag{2.1}$$

It is also known that the functional $\int_\Omega (pgv - |\nabla v|^p)dx$ has a unique maximizer $u$ in $W_0^{1,p}(\Omega)$, and this maximizer is a solution to problem (1.1). By regularity results (see for example [17]), the solution $u$ belongs to $W^{2,1}(\Omega)$, and equation (1.1) holds a.e. in $\Omega$.

**Lemma 2.1.** *For $g \in \overline{\mathcal{G}}$, let $I(g) = \int_\Omega |\nabla u|^p dx$, where $u$ is the solution to* (1.1).

- (a) *The functional $g \mapsto I(g)$ is continuous with respect to the weak* topology in $L^\infty(\Omega)$.*
- (b) *The functional $g \mapsto I(g)$ is strictly convex in $\overline{\mathcal{G}}$.*
- (c) *The functional $g \mapsto I(g)$ is Gâteaux differentiable with derivative $\frac{p}{p-1}u_g$.*

*Proof.* Part (a). Let $g_n \rightharpoonup g$, and let $u_g$, $u_{g_n}$ be the corresponding solutions to (1.1) with $g$, $g_n$ respectively. Using (2.1) we have

$$\begin{aligned}(p-1)I(g) + \int_\Omega p(g_n - g)u_g\,dx &= \int_\Omega (pg_n u_g - |\nabla u_g|^p)\,dx \leq (p-1)I(g_n) \\ &= \int_\Omega (pgu_{g_n} - |\nabla u_{g_n}|^p)\,dx + \int_\Omega p(g_n - g)u_{g_n} dx \\ &\leq (p-1)I(g) + \int_\Omega p(g_n - g)u_{g_n} dx.\end{aligned} \tag{2.2}$$

By assumption, we have

$$\lim_{n\to\infty} \int_\Omega (g_n - g)u_g\,dx = 0. \tag{2.3}$$

Let us prove that

$$\lim_{n\to\infty} \int_\Omega (g_n - g)u_{g_n}\,dx = 0. \tag{2.4}$$

Using (1.1) with $g = g_n$, Poincaré Theorem and Hölder inequality we have

$$\int_\Omega |\nabla u_{g_n}|^p dx = \int_\Omega g_n\, u_{g_n} dx \leq H\int_\Omega u_{g_n} dx \leq C\Big(\int_\Omega |\nabla u_{g_n}|^p dx\Big)^{1/p} |\Omega|^{(p-1)/p}. \tag{2.5}$$

By (2.5) we infer that the norm $\|\nabla u_{g_n}\|_{L^p(\Omega)}$ is bounded by a constant independent of $n$. Therefore, a sub-sequence of $u_{g_n}$ (denoted again $u_{g_n}$) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to some function $z \in W_0^{1,p}(\Omega)$. Since

$$\int_\Omega (g_n - g)\, u_{g_n} dx = \int_\Omega (g_n - g) z\, dx + \int_\Omega (g_n - g)(u_{g_n} - z) dx,$$

and since

$$\Big| \int_\Omega (g_n - g)(u_{g_n} - z) dx \Big| \le 2H \|u_{g_n} - z\|_{L^1(\Omega)},$$

Equality (2.4) follows. By (2.2), (2.3) and (2.4) we infer

$$\lim_{n\to\infty} I(g_n) = I(g). \tag{2.6}$$

This yields the weak* continuity. We claim that the function $z$ is actually the solution of (1.1) corresponding to our function $g$. Indeed, from

$$(p-1) I(g_n) = \int_\Omega (p g_n u_{g_n} - |\nabla u_{g_n}|^p)\, dx,$$

$$\lim_{n\to\infty} \int_\Omega g_n u_{g_n}\, dx = \int_\Omega g z\, dx,$$

and the classical result

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{g_n}|^p\, dx \ge \int_\Omega |\nabla z|^p\, dx,$$

using (2.6) and (2.1) we get

$$(p-1) I(g) \le \int_\Omega (p g z - |\nabla z|^p)\, dx \le (p-1) I(g).$$

By the uniqueness of the maximizer of $\int_\Omega (pgv - |\nabla v|^p)\, dx$ we must have $z = u_g$, as claimed.

Proof of (b). Let $f, g \in \overline{\mathcal{G}}$, let $0 < t < 1$ and let $v \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} &\int_\Omega \big(p(tf + (1-t)g)v - |\nabla v|^p\big)\, dx \\ &= t \int_\Omega (pfv - |\nabla v|^p)\, dx + (1-t) \int_\Omega (pgv - |\nabla v|^p)\, dx. \end{aligned}$$

By taking the superior of both sides relative to $v \in W_0^{1,p}(\Omega)$, we get

$$I(tf + (1-t)g)) \le t I(f) + (1-t) I(g),$$

that is, the convexity. Now, suppose equality holds in the above inequality for some $t \in (0,1)$. Then, if $u_t$ is the solution corresponding to $tf + (1-t)g$ we have

$$\begin{aligned} &t \int_\Omega \big(pfu_t - |\nabla u_t|^p\big)\, dx + (1-t) \int_\Omega \big(pgu_t - |\nabla u_t|^p\big)\, dx \\ &= t \int_\Omega (pfu_f - |\nabla u_f|^p)\, dx + (1-t) \int_\Omega (pgu_g - |\nabla u_g|^p)\, dx. \end{aligned}$$

It follows that

$$\begin{aligned} \int_\Omega \big(pfu_t - |\nabla u_t|^p\big)\, dx &= \int_\Omega \big(pfu_f - |\nabla u_f|^p\big)\, dx \\ \int_\Omega \big(pgu_t - |\nabla u_t|^p\big)\, dx &= \int_\Omega \big(pgu_g - |\nabla u_g|^p\big)\, dx. \end{aligned}$$

By the uniqueness of the maximizer, we must have $u_t = u_f = u_g$. Moreover, since

$$-\Delta_p u_f = f, \quad \text{a.e. in } \Omega,$$
$$-\Delta_p u_g = g, \quad \text{a.e. in } \Omega,$$

if $u_f = u_g$, we must have $f(x) = g(x)$ a.e. in $\Omega$, and the strict convexity is proved.

Proof of (c). Let $t_n > 0$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Let $f, g \in \overline{\mathcal{R}}$, and let $g_n = g + t_n(f-g)$. Then, by (2.2) we find

$$\begin{aligned} I(g) + t_n \int_\Omega (f-g)\frac{p}{p-1} u_g\,dx &\le I\big(g + t_n(f-g)\big) \\ &\le I(g) + t_n \int_\Omega (f-g)\frac{p}{p-1} u_{g_n}\,dx, \end{aligned}$$

and

$$\int_\Omega (f-g)\frac{p}{p-1} u_g\,dx \le \frac{I(g+t_n(f-g)) - I(g)}{t_n} \le \int_\Omega (f-g)\frac{p}{p-1} u_{g_n}\,dx.$$

As already observed, the sequence $u_{g_n}$ converges to $u_g$ in the norm of $L^p(\Omega)$. Therefore,

$$\lim_{n\to\infty} \int_\Omega (f-g)u_{g_n}\,dx = \int_\Omega (f-g)u_g\,dx.$$

Hence, since the sequence $t_n$ is arbitrary, we have

$$\lim_{t\to 0} \frac{I(g+t(f-g)) - I(g)}{t} = \int_\Omega (f-g)\frac{p}{p-1} u_g\,dx.$$

It follows that $I(g)$ is Gâteaux differentiable with derivative $\frac{p}{p-1}u_g$. The proof is complete. □

**Theorem 2.2.** *Let $0 < g_0(x) \le H$, and let $\mathcal{G}$ be the class of all rearrangements of $g_0$. There exists a unique $\breve{g} \in \mathcal{G}$ such that*

$$I(\breve{g}) = \inf_{g\in\mathcal{G}} I(g).$$

*Furthermore, we have $\breve{g} = \psi(u_{\breve{g}})$ for some decreasing function $\psi$.*

*Proof.* By the compactness of $\overline{\mathcal{G}}$ and the weak continuity of $I(g)$ (proved in Lemma 2.1), we know that a minimizer $\breve{g}$ exists in $\overline{\mathcal{G}}$. Since by Lemma 2.1, $I(g)$ is strictly convex, the minimizer $\breve{g}$ is unique. We have to show that $\breve{g} \in \mathcal{G}$.

With $0 < t < 1$ and $g \in \overline{\mathcal{G}}$, let $g_t = \breve{g} + t(g - \breve{g})$. Since $I(g)$ is Gâteaux differentiable at $\breve{g}$, we have

$$I(g_t) = I(\breve{g}) + t\int_\Omega (g-\breve{g})\frac{p}{p-1} u_{\breve{g}}\,dx + o(t).$$

Since $I(g_t) \ge I(\breve{g})$, we find

$$I(\breve{g}) \le I(\breve{g}) + t\int_\Omega (g-\breve{g})\frac{p}{p-1} u_{\breve{g}}\,dx + o(t).$$

It follows that

$$0 \le \int_\Omega (g-\breve{g})\frac{p}{p-1} u_{\breve{g}}\,dx + \frac{o(t)}{t}.$$

As $t \to 0$ we find that

$$0 \le \int_\Omega (g-\breve{g}) u_{\breve{g}}\,dx,$$

and

$$\int_\Omega g u_{\breve g}\,dx \ge \int_\Omega \breve g u_{\breve g}\,dx, \quad \forall g \in \overline{\mathcal{G}}. \tag{2.7}$$

The function $u = u_{\breve g}$ satisfies the equation $-\Delta_p u = \breve g > 0$ a.e. in $\Omega$, therefore $u_{\breve g}$ cannot have flat zones in $\Omega$ (see [12, Lemma 7.7]). By Lemma 1.1 and Lemma 1.2 we can find a decreasing function $\psi$ such that $\psi(u_{\breve g})$ is a rearrangement of $g_0$ and

$$\int_\Omega g u_{\breve g}\,dx \ge \int_\Omega \psi(u_{\breve g}) u_{\breve g}\,dx, \quad \forall g \in \overline{\mathcal{G}}.$$

Comparing the latter inequality with inequality (2.7) and using Lemma 1.2 again, we must have $\breve g = \psi(u_{\breve g}) \in \mathcal{G}$. The proof is complete. □

We remark that Theorem 2.2 gives some information on the shape of the minimizer $\breve g$. Indeed, since the associate solution $u_{\breve g}$ is positive in $\Omega$, vanishes on the boundary $\partial\Omega$, and $\breve g = \psi(u_{\breve g})$ with $\psi$ decreasing, $\breve g$ has to be large where $u_{\breve g}$ is small, that is close to $\partial\Omega$.

## 3. Principal eigenvalue

We use the same assumptions and notation as in the previous section. For $g \in \overline{\mathcal{G}}$, we consider problem (1.2). It is known that such a problem has a principal positive eigenvalue $\lambda_g$ to which corresponds a unique (up to a normalization) positive eigenfunction $u_g$ [1]. We have [14]

$$\frac{1}{\lambda_g} = \sup_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega p g v^p\,dx}{\int_\Omega |\nabla v|^p\,dx}. \tag{3.1}$$

Following Auchmuty [2], we can prove that

$$\frac{p^2}{4}\frac{1}{\lambda_g^2} = \sup_{v \in W_0^{1,p}(\Omega)} \Big[\int_\Omega p g |v|^p\,dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2\Big] = \int_\Omega p g |u|^p\,dx - \Big(\int_\Omega |\nabla u|^p dx\Big)^2. \tag{3.2}$$

Since we know that the principal eigenfunction is positive, we can take $v > 0$ in (3.2). Therefore, for $v > 0$, define

$$A(v) = \int_\Omega p g v^p\,dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2.$$

With $t > 0$, we have

$$A(tv) = t^p \int_\Omega p g v^p\,dx - t^{2p}\Big(\int_\Omega |\nabla v|^p dx\Big)^2.$$

It is easy to see that, for $v$ fixed, $A(tv) \le A(t_0 v)$ with

$$t_0^p = \frac{\int_\Omega p g v^p\,dx}{2\big(\int_\Omega |\nabla v|^p\,dx\big)^2} \tag{3.3}$$

Therefore,

$$A(tv) \le \frac{p^2}{4}\Big(\frac{\int_\Omega p g v^p\,dx}{\int_\Omega |\nabla v|^p\,dx}\Big)^2.$$

It follows that

$$\sup_{v \in W_0^{1,p}(\Omega)} A(v) = \frac{p^2}{4} \sup_{v \in W_0^{1,p}(\Omega)} \Big(\frac{\int_\Omega p g v^p\,dx}{\int_\Omega |\nabla v|^p\,dx}\Big)^2.$$

Equation (3.2) follows from the latter equation and (3.1).

Note that if $v$ is a maximizer in (3.1) then also $\nu v$ with $\nu \neq 0$ is a maximizer. A maximizer $u$ in (3.1) is also a maximizer in (3.2) when $u$ is normalized so that $t_0 = 1$ in (3.3), that is

$$\int_\Omega pgu^p\,dx = 2\Big(\int_\Omega |\nabla u|^p\,dx\Big)^2. \tag{3.4}$$

Therefore, the (positive) maximizer $u = u_g$ in (3.2) satisfies (3.4) and is unique.

**Lemma 3.1.** *For $g \in \overline{\mathcal{G}}$, let $J(g) = \frac{p^2}{4}\frac{1}{\lambda_g^2}$, where $\lambda_g$ is the principal eigenvalue of problem* (1.2).

(a) *The functional $g \mapsto J(g)$ is continuous with respect to the weak* topology in $L^\infty(\Omega)$.*
(b) *The functional $g \mapsto J(g)$ is strictly convex in $\overline{\mathcal{G}}$.*
(c) *The functional $g \mapsto J(g)$ is Gâteaux differentiable with derivative $pu_g^p$.*

*Proof.* Parts (a) and (b) of this lemma are essentially proved in [8]; however we give here a slightly different proof.

Proof of (a). Let $g_n \rightharpoonup g$, and let $u_g$, $u_{g_n}$ be the corresponding maximizers of (3.2) (eigenfunctions) with $g$, $g_n$ respectively. Using (3.2) we have

$$\begin{aligned} J(g) + \int_\Omega p(g_n - g)u_g^p\,dx &= \int_\Omega pg_n u_g^p\,dx - \Big(\int_\Omega |\nabla u_g|^p\Big)^2 dx \le J(g_n) \\ &= \int_\Omega pgu_{g_n}^p\,dx - \Big(|\nabla u_{g_n}|^p\Big)^2 dx + \int_\Omega p(g_n - g)u_{g_n}^p\,dx \\ &\le J(g) + \int_\Omega p(g_n - g)u_{g_n}^p\,dx. \end{aligned} \tag{3.5}$$

By assumption, we have

$$\lim_{n\to\infty}\int_\Omega (g_n - g)u_g^p\,dx = 0. \tag{3.6}$$

Let us prove that

$$\lim_{n\to\infty}\int_\Omega (g_n - g)u_{g_n}^p\,dx = 0. \tag{3.7}$$

Using (3.4) with $g = g_n$ and Poincaré Theorem we have

$$2\Big(\int_\Omega |\nabla u_{g_n}|^p\,dx\Big)^2 = \int_\Omega pg_n\,u_{g_n}^p\,dx \le pH\int_\Omega u_{g_n}^p\,dx \le C\int_\Omega |\nabla u_{g_n}|^p dx. \tag{3.8}$$

By (3.8) we infer that the norm $\|\nabla u_{g_n}\|_{L^p(\Omega)}$ is bounded by a constant independent of $n$. Therefore, a sub-sequence of $u_{g_n}$ (denoted again $u_{g_n}$) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to some function $z \in W_0^{1,p}(\Omega)$. Since

$$\int_\Omega (g_n - g)\,u_{g_n}^p\,dx = \int_\Omega (g_n - g)\,z^p\,dx + \int_\Omega (g_n - g)(u_{g_n}^p - z^p)dx,$$

and since

$$\begin{aligned} \Big|\int_\Omega (g_n - g)(u_{g_n}^p - z^p)dx\Big| &\le 2HC_p\int_\Omega |u_{g_n} - z|(u_{g_n} + z)^{p-1}dx \\ &\le 2HC_p\|u_{g_n} - z\|_{L^p(\Omega)}\Big(\int_\Omega (u_{g_n} + z)^p\,dx\Big)^{(p-1)/p}, \end{aligned}$$

Equality (3.7) follows. By (3.5), (3.6) and (3.7) we infer

$$\lim_{n\to\infty} J(g_n) = J(g). \tag{3.9}$$

This yields the weak* continuity. We claim that the function $z$ is actually the eigenfunction corresponding to $g$. Indeed, from

$$J(g_n) = \int_\Omega pg_n u_{g_n}^p dx - \Big(\int_\Omega |\nabla u_{g_n}|^p dx)\Big)^2,$$
$$\lim_{n\to\infty} \int_\Omega g_n u_{g_n}^p\, dx = \int_\Omega gz^p\, dx,$$
$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{g_n}|^p\, dx \ge \int_\Omega |\nabla z|^p\, dx,$$

using (3.9) and (3.2), we obtain

$$J(g) \le \int_\Omega pgz^p dx - \Big(\int_\Omega |\nabla z|^p dx\Big)^2 \le J(g).$$

By the uniqueness of the maximizer of $\int_\Omega pgv^p dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2$ we must have $z = u_g$, as claimed.

Proof of (b). Let $f, g \in \overline{\mathcal{G}}$, let $0 < t < 1$ and let $v \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\int_\Omega p(tf + (1-t)g)v^p\, dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2 \\
&= t\int_\Omega pfv^p\, dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2 + (1-t)\int_\Omega pgv^p\, dx - \Big(\int_\Omega |\nabla v|^p dx\Big)^2.
\end{aligned}$$

By taking the superior of both sides relative to $v \in W_0^{1,p}(\Omega)$, we get

$$J(tf + (1-t)g)) \le tJ(f) + (1-t)J(g),$$

that is, the convexity. To prove strict convexity, suppose equality holds in the above inequality for some $t \in (0,1)$. Then, if $u_t$ is the eigenfunction corresponding to $tf + (1-t)g$ we have

$$\begin{aligned}
&t\Big[\int_\Omega pfu_t^p dx - \Big(\int_\Omega |\nabla u_t|^p dx\Big)^2\Big] + (1-t)\Big[\int_\Omega pgu_t^p dx - \Big(\int_\Omega |\nabla u_t|^p dx\Big)^2\Big] \\
&= t\Big[\int_\Omega pfu_f^p dx - \Big(\int_\Omega |\nabla u_f|^p dx\Big)^2\Big] + (1-t)\Big[\int_\Omega pgu_g^p dx - \Big(\int_\Omega |\nabla u_g|^p dx\Big)^2\Big].
\end{aligned}$$

It follows that

$$\int_\Omega pfu_t^p dx - \Big(\int_\Omega |\nabla u_t|^p dx\Big)^2 = \int_\Omega pfu_f^p dx - \Big(\int_\Omega |\nabla u_f|^p dx\Big)^2,$$
$$\int_\Omega pfu_t^p dx - \Big(\int_\Omega |\nabla u_t|^p dx\Big)^2 = \int_\Omega pgu_g^p dx - \Big(\int_\Omega |\nabla u_g|^p dx\Big)^2.$$

By the uniqueness of the maximizer, we must have $u_t = u_f = u_g$ and $\lambda_f = \lambda_g$. Moreover, since

$$-\Delta_p u_f = \lambda_f f u_f^{p-1}, \quad \text{a.e. in}\Omega,$$
$$-\Delta_p u_g = \lambda_g g u_g^{p-1}, \quad \text{a.e. in}\Omega,$$

if $u_f = u_g$ and $\lambda_f = \lambda_g$, we must have $f(x) = g(x)$ a.e. in $\Omega$, and the strict convexity is proved.

Proof of (c). Let $t_n > 0$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Let $f, g \in \overline{\mathcal{R}}$, and let $g_n = g + t_n(f - g)$. Then, by (3.5) we find

$$J(g) + t_n \int_\Omega (f-g) p u_g^p \, dx \le J\big(g + t_n(f-g)\big) \le J(g) + t_n \int_\Omega (f-g) p u_{g_n}^p \, dx,$$
$$\int_\Omega (f-g) p u_g^p \, dx \le \frac{J(g + t_n(f-g)) - J(g)}{t_n} \le \int_\Omega (f-g) p u_{g_n}^p \, dx.$$

As already observed, the sequence $u_{g_n}$ converges to $u_g$ in the norm of $L^p(\Omega)$. Therefore,

$$\lim_{n\to\infty} \int_\Omega (f-g) u_{g_n}^p \, dx = \int_\Omega (f-g) u_g^p \, dx.$$

Hence, since the sequence $t_n$ is arbitrary, we have

$$\lim_{t\to 0} \frac{J(g + t(f-g)) - J(g)}{t} = \int_\Omega (f-g) p u_g^p \, dx.$$

It follows that $J(g)$ is Gâteaux differentiable with derivative $p u_g^p$. The proof is complete. □

**Theorem 3.2.** *Let $0 < g_0(x) \le H$, and let $\mathcal{G}$ be the class of all rearrangements of $g_0$. There exists a unique $\hat{g} \in \mathcal{G}$ such that*

$$J(\hat{g}) = \inf_{g\in\mathcal{G}} J(g).$$

*Furthermore, $\hat{g} = \psi(u_{\hat{g}})$ for some decreasing function $\psi$.*

*Proof.* By the compactness of $\overline{\mathcal{G}}$ and the weak continuity of $J(g)$ (proved in Lemma 3.1), we know that a minimizer $\hat{g}$ exists in $\overline{\mathcal{G}}$. Since by Lemma 3.1 $J(g)$ is strictly convex, the minimizer $\hat{g}$ is unique. We have to show that $\hat{g} \in \mathcal{G}$.

With $0 < t < 1$ and $g \in \overline{\mathcal{G}}$, let $g_t = \hat{g} + t(g - \hat{g})$. Since $J(g)$ is Gâteaux differentiable at $\hat{g}$, we have

$$J(g_t) = J(\hat{g}) + t \int_\Omega (g - \hat{g}) p u_{\hat{g}}^p dx + o(t).$$

Since $J(g_t) \ge J(\hat{g})$, we find

$$J(\hat{g}) \le J(\hat{g}) + t \int_\Omega (g - \hat{g}) p u_{\hat{g}} \, dx + o(t).$$

It follows that

$$0 \le \int_\Omega (g - \hat{g}) p u_{\hat{g}} \, dx + \frac{o(t)}{t}.$$

As $t \to 0$ we find

$$0 \le \int_\Omega (g - \hat{g}) u_{\hat{g}} \, dx,$$

and

$$\int_\Omega g u_{\hat{g}}^p \, dx \ge \int_\Omega \hat{g} u_{\hat{g}}^p \, dx, \quad \forall g \in \overline{\mathcal{G}}. \tag{3.10}$$

The function $u = u_{\hat{g}}$ satisfies the equation $-\Delta_p u = \lambda_{\hat{g}} \hat{g} u_{\hat{g}}^{p-1} > 0$ a.e. in $\Omega$; therefore, $u_{\hat{g}}^p$ cannot have flat zones in $\Omega$. By Lemmas 1.1 and 1.2 we can find a decreasing function $\psi$ such that $\psi(u_{\hat{g}}^p)$ is a rearrangement of $g_0$ and

$$\int_\Omega g u_{\hat{g}}^p \, dx \ge \int_\Omega \psi(u_{\hat{g}}) u_{\hat{g}} \, dx, \quad \forall g \in \overline{\mathcal{G}}.$$

Comparing the latter inequality with inequality (3.10) and using Lemma 1.2 again, we must have $\hat{g} = \psi(u_{\hat{g}}^p) \in \mathcal{G}$, and the statement of the theorem follows. □

**Remarks.** Since $J(g) = \frac{p^2}{4}\frac{1}{\lambda_g^2}$, the minimization of $J(g)$ corresponds to the maximization of $\lambda_g$. Theorem 3.2 gives some information on the shape of the maximizer of $\lambda_g$, $\hat{g}$. Indeed, since the associate eigenfunction $u_{\hat{g}}$ is positive in $\Omega$, vanishes on the boundary $\partial\Omega$, and $\hat{g} = \psi(u_{\hat{g}}^p)$ with $\psi$ decreasing, $\hat{g}$ has to be large where $u_{\hat{g}}$ is small, that is close to $\partial\Omega$.

We underline that the maximization and the minimization of $\lambda_g$ for $g \in \mathcal{G}$ in case of $p = 2$ are discussed in [9]. However, the (interesting) method developed in [9] for the investigation of the maximum of $\lambda_g$ seems to not work in the nonlinear case $p \neq 2$. Related problems are discussed in [6, 10, 11, 13, 15, 16].

## References

[1] A. Anane; Simplicité et isolation de la première valeur propre du $p$-Laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305 (1987), 725–728.

[2] G. Auchmuty; Dual variational principles for eigenvalue problems, in Nonlinear Analysis and its applications (ed. F.E. Browder), *Proc. Symposia in Pure Mathematics*, Vol. 45, Part. 1, AMS 1986, 55–72.

[3] G. R. Burton; Rearrangements of functions, maximization of convex functionals and vortex rings. *Math. Ann.*, 276 (1987), 225–253.

[4] G. R. Burton; Variational problems on classes of rearrangements and multiple configurations for steady vortices. *Ann. Inst. Henri Poincaré*, 6(4) (1989), 295–319.

[5] G. R. Burton and J. B. McLeod; Maximisation and minimisation on classes of rearrangements. *Proc. Roy. Soc. Edinburgh Sect. A*, 119 (3-4) (1991), 287–300.

[6] S. Chanillo, D. Grieser, M. Imai, K. Kurata, I. Ohnishi; Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes. *Commun. Math. Phys.*, 214 (2000), 315–337.

[7] F. Cuccu, B. Emamizadeh, G. Porru; Nonlinear elastic membrane involving the p-Laplacian operator, *Electronic Journal of Differential Equations.* Vol 2006, no. 49 (2006), 1–10.

[8] F. Cuccu, B. Emamizadeh, G. Porru, G.; Optimization of the first eigenvalue in problems involving the p-Laplacian, *Proc. Amer. Math. Soc.* 137 (2009), 1677–1687.

[9] S. J. Cox, J. R. McLaughlin; Extremal eigenvalue problems for composite membranes, I, II. *Appl. Math. Optim.*, 22 (1990),153–167; 169–187.

[10] F. Cuccu, G. Porcu; Existence of solutions in two optimization problems. *Comp. Rend. de l'Acad. Bulg. des Sciences*, 54(9) (2001), 33–38.

[11] J. García-Melián, J. Sabina de Lis; Maximum and comparison principles involving the $p$-Laplacian. *Journal Math. Anal. Appl.*, 218 (1998), 49–65.

[12] D. Gilbarg, N. S. Trudinger; Elliptic Partial Differential Equations of Second Order, Second edition, *Springer*, Berlin, 1998.

[13] A. Henrot; Extremum problems for eigenvalues of elliptic operators. *Frontiers in Mathematics*, Birkhuser Verlag, Basel, 2006.

[14] B. Kawohl, M. Lucia, S. Prashanth; Simplicity of the principal eigenvalue for indefinite quasilinear problems. *Adv. Differential Equations*, 12 (2007), 407–434.

[15] J. Nycander, B. Emamizadeh; Variational problems for vortices attached to seamounts. *Nonlinear Analysis*, 55 (2003), 15–24.

[16] W. Pielichowski; The optimization of eigenvalue problems involving the $p$-Laplacian. *Univ. Iag. Acta Math.*, 42 (2004), 109–122.

[17] P. Tolksdorf; Regularity for a more general class of quasilinear elliptic equations. *Journal of Differential Equations*, 51 (1984), 126–150.

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