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EXISTENCE AND EXPONENTIAL STABILITY FOR ANTI-PERIODIC SOLUTIONS FOR SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH CONTINUOUSLY DISTRIBUTED DELAYS

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Abstract. This article concerns anti-periodic solutions for shunting inhibitory cellular neural networks (SICNNs), with continuously distributed delays, arising from the description of the neurons state in delayed neural networks. Without assuming global Lipschitz conditions of activation functions, we obtain existence and local exponential stability of anti-periodic solutions.

1. Introduction

We consider shunting inhibitory cellular neural networks (SICNNs) with continuously distributed delays described by

$$\begin{aligned} x'_{ij}(t) &= -a_{ij}(t)x_{ij}(t) \\ &\quad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)f(x_{kl}(t-u))du\,x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{1.1}$$

where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, C_{ij} denotes the cell at the (i, j) position of the lattice, the r-neighborhood $N_r(i, j)$ of C_{ij} is given by

$$N_r(i,j) = \{C_{kl} : \max(|k-i|, |l-j|) \le r, 1 \le k \le m, 1 \le l \le n\}.$$

Here x_{ij} represents the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, the constant $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity, $C_{ij}^{kl}(t) \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f(\cdot)$ is a continuous function representing the output or firing rate of the cell C_{kl}.

It is well known that studies on SICNNs not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior and almost periodic oscillatory properties. Therefore, considerable effort has been devoted to study dynamic behaviors, in particular, the existence and stability of periodic and almost periodic solutions of SICNNs in the literature (see, e.g., [3, 4, 5, 8, 9, 10] and the references therein). Recently, in [11, 12] the authors

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studied the existence and stability of anti-periodic solutions for SICNNs with the following assumption:

(T0) The function f is global Lipschitz continuous; that is, there exists a constant $\mu > 0$ such that for all $x, y \in \mathbb{R}$,
$$|f(x) - f(y)| \le \mu |x - y|.$$

To the best of our knowledge, very few authors have considered problems of anti-periodic solutions of SICNNs (1.1) without Assumption (T0). Moreover, it is well known that the existence and stability of anti-periodic solutions play a key role in characterizing the behavior of nonlinear differential equations; see [1, 2, 6, 13]. Since SICNNs can be analog voltage transmission, and voltage transmission process often a anti-periodic process. Thus, it is worth while to continue to investigate the existence and stability of anti-periodic solutions of SICNNs (1.1) without Assumption (T0).

The purpose of this article is to present sufficient conditions for the existence and local exponential stability of anti-periodic solutions of (1.1). Moreover, we do not assume (T0). An example is provided to illustrate our results.

Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous in t. $u(t)$ is said to be T-anti-periodic on $\mathbb{R}$ if,
$$u(t + T) = -u(t) \quad \text{for all } t \in \mathbb{R}.$$
In this article, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, it is assumed that $a_{ij}, C_{ij}^{kl} : \mathbb{R} \to [0, +\infty)$, $K_{ij} : [0, +\infty) \to \mathbb{R}$ and $L_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous functions, and
$$\begin{gathered} a_{ij}(t + T) = a_{ij}(t), \quad C_{ij}^{kl}(t + T) = C_{ij}^{kl}(t), \\ f(-u) = f(u), \quad L_{ij}(t + T) = -L_{ij}(t), \end{gathered} \tag{1.2}$$
for all $t, u \in \mathbb{R}$. Let
$$\{x_{ij}(t)\} = (x_{11}(t), \dots, x_{1n}(t), \dots, x_{i1}(t), \dots, x_{in}(t), \dots, x_{m1}(t), \dots, x_{mn}(t))$$
be an element in $\mathbb{R}^{m \times n}$. For $x(t) = \{x_{ij}(t)\}$ in $\mathbb{R}^{m \times n}$, define the norm $\|x(t)\| = \max_{(i,j)}\{|x_{ij}(t)|\}$.

We shall use the following conditions

(H1) For $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$, the delay kernels $K_{ij} : [0, \infty) \to \mathbb{R}$ are continuous and integrable;

(H2) there exists a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $r > 0$,
$$|f(u) - f(v)| \le L(r)|u - v|, \quad |u|, |v| \le r. \tag{1.3}$$

(H3) there exists a constant $r_0 > 0$ such that
$$D[|f(0)|r_0 + L(r_0)r_0^2] + L^+ \le r_0,$$
where
$$D = \max_{(i,j)} \big\{ \frac{\sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \int_0^\infty |K_{ij}(u)| du}{\underline{a}_{ij}} \big\} > 0, \quad L^+ = \max_{(i,j)} \frac{\overline{L}_{ij}}{\underline{a}_{ij}},$$
$$\overline{L}_{ij} = \sup_{t \in \mathbb{R}} |L_{ij}(t)|, \quad \bar{C}_{ij}^{kl} = \sup_{t \in \mathbb{R}} C_{ij}^{kl}(t), \quad \underline{a}_{ij} = \inf_{t \in \mathbb{R}} a_{ij}(t) > 0.$$

The initial conditions associated with system (1.1) are
$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0], \ i = 1, 2, \dots, m, \ j = 1, 2, \dots, n, \tag{1.4}$$
where $\varphi_{ij}(\cdot)$ denotes real-valued bounded continuous function defined on $(-\infty, 0]$.

The remaining parts of this paper are organized as follows. In Section 2, sufficient conditions are derived for the boundedness of solution of (1.1). In Section 3, we present sufficient conditions for the existence and local exponential stability of anti-periodic solution of (1.1). In Section 4, an illustrative example is given to show the proposed theory and method.

2. Preliminary Results

The following lemmas will be used to prove our main results in Section 3.

Lemma 2.1. *Assume* (H1)–(H3). *Suppose that* $\widetilde{x}(t) = \{\widetilde{x}_{ij}(t)\}$ *is a solution of* (1.1) *with initial conditions*

$$\widetilde{x}_{ij}(s) = \widetilde{\varphi}_{ij}(s), \quad |\widetilde{\varphi}_{ij}(s)| < r_0, \quad s \in (-\infty, 0],\ ij = 11, 12, \dots, mn \tag{2.1}$$

Then

$$|\widetilde{x}_{ij}(t)| < r_0, \quad \text{where } t \geq 0, ij = 11, 12, \dots, mn. \tag{2.2}$$

Proof. Assume, by way of contradiction, that (2.2) does not hold. Then, there exist $ij \in \{11, 12, \dots, mn\}$ and $\rho > 0$ such that

$$\widetilde{x}_{ij}(\rho) = r_0, \quad \widetilde{x}_{\bar{i}\bar{j}}(t) < r_0 \quad \text{for all } t \in (-\infty, \rho),\ \bar{i}\bar{j} = 11, 12, \dots, mn. \tag{2.3}$$

Calculating the the upper left derivative of $|\widetilde{x}_{ij}(t)|$, together with (H1)-(H3), (1.3) and (2.3), we obtain

$$\begin{aligned}
&0 \leq D^+(|\widetilde{x}_{ij}(\rho)|) \\
&\leq -a_{ij}(\rho)\widetilde{x}_{ij}(\rho) + |\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(\rho)\int_0^\infty K_{ij}(u) f(\widetilde{x}_{kl}(\rho - u)) du \widetilde{x}_{ij}(\rho) + L_{ij}(\rho)| \\
&\leq -\underline{a}_{ij} r_0 + \sum_{C_{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl} \int_0^\infty |K_{ij}(u)|(|f(0)| + L(r_0)|\widetilde{x}_{kl}(\rho - u)|) du r_0 + |L_{ij}(\rho)| \\
&\leq \underline{a}_{ij}(-r_0 + \frac{\sum_{C_{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\int_0^\infty |K_{ij}(u)|}{\underline{a}_{ij}}(|f(0)|r_0 + L(r_0)r_0^2) + \frac{\overline{L}_{ij}}{\underline{a}_{ij}}) < 0.
\end{aligned}$$

This is a contradiction and hence (2.2) holds. This completes the proof. □

Remark 2.2. Under Conditions (H1)–(H3), the solution of (1.1) always exists [7]. In view of the boundedness of this solution, from the theory of functional differential equations in [7], it follows that $\widetilde{x}(t)$ can be defined on $(-\infty, \infty)$.

Lemma 2.3. *Assume* (H1)-(H3)*, and let* $x^*(t) = \{x_{ij}^*(t)\}$ *be the solution of* (1.1) *with initial value* $\varphi^* = \{\varphi_{ij}^*(t)\}$*, where*

$$|\varphi_{ij}^*(s)| < r_0, \quad s \in (-\infty, 0],\ ij = 11, 12, \dots, mn. \tag{2.4}$$

Also assume that

(H4) *there exists a constant* $r_1 \geq r_0$ *such that*

$$|f(0)| + L(r_0)r_0 + L(r_1)r_1 < \frac{1}{D};$$

(H5) *for* $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$*, there exists a constant* $\lambda_0 > 0$ *such that*

$$\int_0^\infty |K_{ij}(s)| e^{\lambda_0 s} ds < +\infty.$$

Then there is a positive constant λ *such that for every solution* $x(t) = \{x_{ij}(t)\}$ *of* (1.1) *with any initial value* $\sup_{t\in(-\infty,0]} \|\varphi(t)\| \le r_1$,

$$\|x(t) - x^*(t)\| \le Me^{-\lambda t}, \quad \forall t > 0,$$

where $M = \sup_{-\infty\le t\le 0} \|\varphi(t) - x^*(t)\|$.

Proof. In view (2.4) and Lemma 2.1, we have

$$|x^*_{ij}(t)| < r_0, \quad \text{for all } t \in \mathbb{R},\ ij = 11, 12, \dots, mn. \tag{2.5}$$

Set

$$\begin{aligned}\Gamma_{ij}(\alpha) = \alpha - \underline{a}_{ij} + \sum_{C_{kl}\in N_r(i,j)} &\bar{C}^{kl}_{ij}(|f(0)| + L(r_0)r_0 \int_0^\infty |K_{ij}(s)|ds \\ &+ L(r_1)r_1 \int_0^\infty |K_{ij}(s)|e^{\alpha s}ds)\end{aligned}$$

where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. It is not difficult to prove that $\Gamma_{ij}(ij = 11, 12, \dots, mn)$ are continuous functions on $[0, \lambda_0]$. Moreover, by (H4) and (H5), we have

$$\Gamma_{ij}(0) = -\underline{a}_{ij} + \sum_{C_{kl}\in N_r(i,j)} \bar{C}^{kl}_{ij}(|f(0)| + L(r_0)r_0 + L(r_1)r_1) \int_0^\infty |K_{ij}(s)|ds < 0$$

Thus, there exists a sufficiently small constant $\lambda > 0$ such that

$$\Gamma_{ij}(\lambda) < 0, \quad ij = 11, 12, \dots, mn. \tag{2.6}$$

Take $\varepsilon > 0$, and set

$$z_{ij}(t) = |x_{ij}(t) - x^*_{ij}(t)|e^{\lambda t}, \quad ij = 11, 12, \dots, mn.$$

It follows that

$$z_{ij}(t) \le M < M + \varepsilon, \quad \forall t \in (-\infty, 0],\ ij = 11, 12, \dots, mn.$$

Now, we claim that

$$z_{ij}(t) \le M + \varepsilon, \quad \forall t > 0,\ ij = 11, 12, \dots, mn. \tag{2.7}$$

If this is not true, then there exists $i_0 \in \{1, 2, \dots, m\}$ and $j_0 \in \{1, 2, \dots, n\}$ such that

$$\{t > 0 | z_{i_0j_0}(t) > M + \varepsilon\} \ne \phi \tag{2.8}$$

Let

$$t_{ij} = \begin{cases} \inf\{t > 0 | z_{ij}(t) > M + \varepsilon\}, & \text{if } \{t > 0 : z_{ij}(t) > M + \varepsilon\} \ne \emptyset \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $t_{ij} > 0$ and

$$z_{ij}(t) \le M + \varepsilon, \quad \forall t \in (-\infty, t_{ij}],\ ij = 11, 12, \dots, mn. \tag{2.9}$$

We denote $t_{pq} = \min_{(i,j)} t_{ij}$, where $p \in \{1, 2, \dots, m\}$ and $q \in \{1, 2, \dots, n\}$. In view of (2.8), we have $0 < t_{pq} < +\infty$. It follow from (2.9), we have

$$z_{ij}(t) \le M + \varepsilon, \quad \forall t \in (-\infty, t_{pq}],\ ij = 11, 12, \dots, mn. \tag{2.10}$$

In addition, noting that $t_{pq} = \inf\{t > 0 : z_{pq}(t) > M + \varepsilon\}$, we obtain

$$z_{pq}(t_{pq}) = M + \varepsilon, \quad D^+z_{pq}(t_{pq}) \ge 0. \tag{2.11}$$

Since $x(t)$ and $x^*(t)$ are solutions of (1.1), we have

$$\begin{aligned}
0 &\le D^+ z_{pq}(t_{pq}) \\
&= D^+[|x_{pq}(t) - x^*_{pq}(t)|e^{\lambda t}]|_{t=t_{pq}} \\
&\le |x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})|\lambda e^{\lambda t_{pq}} - \underline{a}_{pq}|x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})|e^{\lambda t_{pq}} \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}|\int_0^\infty K_{ij}(u)f(x_{kl}(t_{pq}-u))du x_{pq}(t_{pq}) \\
&\quad - \int_0^\infty K_{ij}(u)f(x^*_{kl}(t_{pq}-u))du x^*_{pq}(t_{pq})|e^{\lambda t_{pq}} \\
&= |x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})|\lambda e^{\lambda t_{pq}} - \underline{a}_{pq}|x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})|e^{\lambda t_{pq}} \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}|\int_0^\infty K_{ij}(u)f(x^*_{kl}(t_{pq}-u)du(x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})) \\
&\quad + \int_0^\infty K_{ij}(u)(f(x_{kl}(t_{pq}-u)) - f(x^*_{kl}(t_{pq}-u)))du x_{pq}(t_{pq})|e^{\lambda t_{pq}} \\
&\le (\lambda - \underline{a}_{pq})z_{pq}(t_{pq}) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}\int_0^\infty |K_{ij}(u)||f(x^*_{kl}(t_{pq}-u)|du \\
&\quad \times |x_{pq}(t_{pq}) - x^*_{pq}(t_{pq})|e^{\lambda t_{pq}} \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}\int_0^\infty |K_{ij}(u)||f(x_{kl}(t_{pq}-u)) - f(x^*_{kl}(t_{pq}-u))|du \\
&\quad \times |x_{pq}(t_{pq})|e^{\lambda t_{pq}}.
\end{aligned}$$

Now, combining the above inequality, (2.10), (2.11), (H2) and (H3), we deduce

$$\begin{aligned}
0 &\le D^+ z_{pq}(t_{pq}) \\
&\le (\lambda - \underline{a}_{pq})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}(|f(0)| + L(r_0)r_0)\int_0^\infty |K_{ij}(u)|du \cdot z_{pq}(t_{pq}) \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}L(r_1)\int_0^\infty |K_{ij}(u)||x_{kl}(t_{pq}-u) - x^*_{kl}(t_{pq}-u)| \\
&\quad \times e^{\lambda(t_{pq}-u)}e^{\lambda u}du \cdot r_1 \\
&\le (\lambda - \underline{a}_{pq})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}(|f(0)| + L(r_0)r_0) \cdot (M+\varepsilon) \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}L(r_1)r_1\int_0^\infty |K_{ij}(u)|e^{\lambda u}du(M+\varepsilon) \\
&\le (\lambda - \underline{a}_{pq})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}(|f(0)| \\
&\quad + L(r_0)r_0)\int_0^\infty |K_{ij}(u)|du \cdot (M+\varepsilon) \\
&\quad + \sum_{C_{kl}\in N_r(p,q)} \bar{C}^{kl}_{pq}L(r_1)r_1\int_0^\infty |K_{ij}(u)|e^{\lambda u}du \cdot (M+\varepsilon)\,.
\end{aligned}$$

It follow that

$$\begin{aligned}&\lambda - \underline{a}_{pq} + \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \cdot (|f(0)| \\ &+ L(r_0)r_0 \int_0^\infty |K_{ij}(u)|du + L(r_1)r_1 \int_0^\infty |K_{ij}(u)|e^{\lambda u} du) \geq 0\end{aligned}$$

which contradicts with (2.6). Hence, (2.7) holds; i.e.

$$|x_{ij}(t) - x_{ij}^*(t)|e^{\lambda t} = z_{ij}(t) \leq M + \varepsilon, \quad \forall t > 0,\ i = 1, 2, \dots, m,\ j = 1, 2, \dots, n.$$

Therefore,

$$\|x(t) - x^*(t)\| = \max_{(i,j)} |x_{ij}(t) - x_{ij}^*(t)| \leq (M + \varepsilon)e^{-\lambda t}, \quad \forall t > 0.$$

Letting $\varepsilon \to 0^+$, we obtain

$$\|x(t) - x^*(t)\| \leq Me^{-\lambda t}, \quad \forall t > 0.$$

The proof is complete. $\square$

Remark 2.4. Let $x^*(t) = (x_{11}^*(t), x_{12}^*(t), \dots, x_{mn}^*(t))^T$ is the T-anti-periodic solution of (1.1). It follows from Lemma 2.3 that $x^*(t)$ is globally exponentially stable.

3. Main Results

Theorem 3.1. *Assume* (H1)-(H5). *Then* (1.1) *has at least one T-anti-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is locally exponentially stable.*

Proof. . Let $v(t) = (v_{11}(t), v_{12}(t), \dots, v_{mn}(t))^T$ be a solution of (1.1) with initial conditions

$$v_{ij}(s) = \varphi_i^v(s), \quad |\varphi_{ij}^v(s)| < r_0, \quad s \in (-\infty, 0],\ ij = 11, 12, \dots, mn. \tag{3.1}$$

By Lemma 2.1, the solution $v(t)$ is bounded and

$$|v_{ij}(t)| < r_0, \quad \text{for all } \ t \in \mathbb{R},\ ij = 11, 12, \dots, mn. \tag{3.2}$$

From (1.1) and (1.2), we have

$$\begin{aligned}&((-1)^{k+1} v_{ij}(t + (k+1)T))' \\ &= (-1)^{k+1} v_{ij}'(t + (k+1)T) \\ &= (-1)^{k+1}\{-a_{ij}(t + (k+1)T))v_{ij}(t + (k+1)T) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t + (k+1)T)) \\ &\quad \times \int_0^\infty K_{ij}(u) f(v_{kl}(t + (k+1)T - u)) du v_{ij}(t + (k+1)T) + L_{ij}(t + (k+1)T)\} \\ &= -a_{ij}(t))(-1)^{k+1} v_{ij}(t + (k+1)T) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)) \int_0^\infty K_{ij}(u) \\ &\quad \times f((-1)^{k+1} v_{kl}(t + (k+1)T - u)) du (-1)^{k+1} v_{ij}(t + (k+1)T) + L_{ij}(t),\end{aligned} \tag{3.3}$$

where $ij = 11, 12, \dots, mn$. Thus, for any natural number k, $(-1)^{k+1}v(t+(k+1)T)$ are the solutions of (1.1). Then, by Lemma 2.3, we get

$$\begin{aligned} &|(-1)^{k+1}v_{ij}(t+(k+1)T) - (-1)^k v_{ij}(t+kT)| \\ &\le e^{-\lambda(t+kT)} \sup_{-\infty \le s \le 0} \max_{(i,j)} |v_{ij}(s+T) + v_{ij}(s)| \\ &\le e^{-\lambda(t+kT)} 2r_0, \quad \forall t+kT > 0,\ ij = 11, 12, \dots, mn. \end{aligned} \tag{3.4}$$

Thus, for any natural number p, we obtain

$$\begin{aligned} &(-1)^{p+1}v_{ij}(t+(p+1)T) \\ &= v_{ij}(t) + \sum_{k=0}^{p} [(-1)^{k+1}v_{ij}(t+(k+1)T) - (-1)^k v_{ij}(t+kT)]. \end{aligned} \tag{3.5}$$

Then

$$\begin{aligned} &|(-1)^{p+1}v_{ij}(t+(p+1)T)| \\ &\le |v_{ij}(t)| + \sum_{k=0}^{p} |(-1)^{k+1}v_{ij}(t+(k+1)T) - (-1)^k v_{ij}(t+kT)|, \end{aligned} \tag{3.6}$$

where $ij = 11, 12, \dots, mn$.

In view of (3.4), we can choose a sufficiently large constant $N > 0$ and a positive constant α such that

$$|(-1)^{k+1}v_{ij}(t+(k+1)T) - (-1)^k v_{ij}(t+kT)| \le \alpha(e^{-\lambda T})^k, \tag{3.7}$$

for all $k > N$, $i = 1, 2, \dots, n$, on any compact set of $\mathbb{R}$. It follows from (3.6) and (3.7) that $\{(-1)^p v(t+pT)\}$ uniformly converges to a continuous function $x^*(t) = (x^*_{11}(t), x^*_{12}(t), \dots, x^*_{mn}(t))^T$ on any compact set of $\mathbb{R}$.

Now we show that $x^*(t)$ is T-anti-periodic solution of (1.1). First, $x^*(t)$ is T-anti-periodic, since

$$x^*(t+T) = \lim_{p\to\infty} (-1)^p v(t+T+pT) = -\lim_{(p+1)\to\infty} (-1)^{p+1} v(t+(p+1)T) = -x^*(t).$$

Next, we prove that $x^*(t)$ is a solution of (1.1). In fact, together with the continuity of the right side of (1.1), (3.3) implies that $\{((-1)^{p+1}v(t+(p+1)T))'\}$ uniformly converges to a continuous function on any compact set of $\mathbb{R}$. Thus, letting $p \to \infty$, we obtain

$$\begin{aligned} \frac{d}{dt}\{x^*_{ij}(t)\} &= -a_{ij}(t)x^*_{ij}(t) \\ &\quad - \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij}(t) \int_0^\infty K_{ij}(u) f(x^*_{kl}(t-u))du\, x^*_{ij}(t) + L_{ij}(t). \end{aligned} \tag{3.8}$$

Therefore, $x^*(t)$ is a solution of (1.1). Finally, by Lemma 2.3, we can prove that $x^*(t)$ is locally exponentially stable. This completes the proof. $\square$

4. An Example

In this section, we give an example to illustrate the results obtained in previous sections. Consider the following SICNNs with continuously distributed delays:

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij}(t)\int_0^{\infty} K_{ij}(u)f(x_{kl}(t-u))du\,x_{ij}(t) + L_{ij}(t), \tag{4.1}$$

where $i = 1,2,3$, $j = 1,2,3$, $K_{ij}(u) = (\sin u)e^{-u}$, $f(x) = \frac{x^4+1}{6}$,

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{pmatrix} = \begin{pmatrix} 5+|\sin t| & 5+|\sin t| & 9+|\sin t|\\ 6+|\sin t| & 6+|\sin t| & 7+|\sin t|\\ 8+|\sin t| & 8+|\sin t| & 5+|\sin t|\end{pmatrix},$$

$$\begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t)\\ C_{21}(t) & C_{22}(t) & C_{23}(t)\\ C_{31}(t) & C_{32}(t) & C_{33}(t)\end{pmatrix} = \begin{pmatrix} 0.1|\sin t| & 0.3|\sin\sqrt{3}t| & 0.5|\sin t|\\ 0.2|\sin t| & 0.1|\sin t| & 0.2|\sin t|\\ 0.1|\sin t| & 0.2|\sin t| & 0.1|\sin\sqrt{3}t|\end{pmatrix},$$

$$\begin{pmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t)\\ L_{21}(t) & L_{22}(t) & L_{23}(t)\\ L_{31}(t) & L_{32}(t) & L_{33}(t)\end{pmatrix} = \begin{pmatrix} \sin t & \sin t & \cos t\\ \sin t & \cos t & \cos t\\ \cos t & \sin t & \sin t\end{pmatrix}.$$

Obviously, (H1) holds. Set $L(r) = \frac{2}{3}r^3$. Since $f'(x) = \frac{2}{3}x^3$, (H2) holds. Next, let us check (H3). Clearly, we have $f(0) = \frac{1}{6}$,

$$\sum_{C_{kl}\in N_1(1,1)} \bar{C}^{kl}_{11} = 0.7, \quad \sum_{C_{kl}\in N_1(1,2)} \bar{C}^{kl}_{12} = 1.4, \quad \sum_{C_{kl}\in N_1(1,3)} \bar{C}^{kl}_{13} = 1.1,$$

$$\sum_{C_{kl}\in N_1(2,1)} \bar{C}^{kl}_{21} = 1, \quad \sum_{C_{kl}\in N_1(2,2)} \bar{C}^{kl}_{22} = 1.8, \quad \sum_{C_{kl}\in N_1(2,3)} \bar{C}^{kl}_{23} = 1.4,$$

$$\sum_{C_{kl}\in N_1(3,1)} \bar{C}^{kl}_{31} = 0.6, \quad \sum_{C_{kl}\in N_1(3,2)} \bar{C}^{kl}_{32} = 0.9, \quad \sum_{C_{kl}\in N_1(3,3)} \bar{C}^{kl}_{33} = 0.6,$$

$$D = \max_{(i,j)}\Big\{\frac{\sum_{C_{kl}\in N_r(i,j)} \bar{C}^{kl}_{ij}\int_0^{\infty}|K_{ij}(u)|du}{\underline{a}_{ij}}\Big\} \le \max_{(i,j)}\Big\{\frac{\sum_{C_{kl}\in N_r(i,j)} \bar{C}^{kl}_{ij}}{\underline{a}_{ij}}\Big\} \le 0.3,$$

$$L^+ = \max_{(i,j)} \frac{\overline{L_{ij}}}{\underline{a}_{ij}} = 0.2$$

Take $r_0 = 1$. Then

$$D[|f(0)|r_0 + L(r_0)r_0^2] + L = 0.3\cdot(\frac{1}{6}+\frac{2}{3}) + 0.2 = 0.45 < 1 = r_0$$

and

$$D|f(0)| + 2DL(r_0)r_0 = 0.005 + 0.4 = 0.45 < 1.$$

Thus, (H3) holds for $r_0 = 1$.

Take $r_1 = (7/2)^{1/4}$. Then

$$D|f(0)| + DL(r_0)r_0 + \max_{(i,j)}\Big\{\frac{\sum_{C_{kl}\in N_r(i,j)} \bar{C}^{kl}_{ij}}{\underline{a}_{ij}}\Big\}L(r_1)r_1 = 0.25 + 0.7 = 0.95 < 1$$

So (H4) holds. By Theorem 3.1, system (4.1) has a π-anti-periodic solution $x^*(t)$ with initial value $\sup_{t\in[-1,0]}\|\varphi^*\| < 1$. Moreover, all solutions of (4.1) with initial value $\sup_{t\in(-\infty,0]}\|\varphi(t)\| \le (7/2)^{1/4}$ converge exponentially to $x^*(t)$ as $t\to+\infty$.

Remark 4.1. SICNNs (4.1) is a very simple form of shunting inhibitory cellular neural networks with continuously distributed delays. Since $f(x) = \frac{x^4+1}{6}$. One can observe that the condition (T0) is not satisfied. Therefore, the results in [11, 12] and the references therein can not be applicable to (4.1). This implies that our results are essentially new.

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