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UNIFORM BOUNDEDNESS OF SOLUTIONS FOR A CLASS OF LIÉNARD EQUATIONS

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Abstract. In this article, we study a class of Liénard equations

$$x''(t) + f(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t)) = e(t).$$

Under some suitable conditions, we ensure that all solutions of the above Liénard equations are uniformly bounded. Our assumptions are less restrictive than those in [9]; thus we extend some previous results.

1. Introduction

As it is we all know, Liénard equations appears in a number of physical models and is important in describing fluid mechanical and nonlinear elastic mechanical phenomena. Thus, there has been great interest for many mathematicians to study the dynamical behavior of all kinds of Liénard equations (cf. [1, 3, 6, 8, 9, 10, 11, 12, 4, 5] and references therein). Especially, several authors have contributed to the study on boundedness of solutions to Liénard equations (cf. [6, 8, 10, 9, 12] and references therein). For example, in 1998, the authors in [6] discussed the bounded solutions of the Liénard equation

$$x''(t) + f(x)x' + g(x) = e(t).$$

Recently, the authors in [10] studied the boundedness of solutions to the following Liénard equation with a deviating argument:

$$x''(t) + f(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t))) = e(t), \tag{1.1}$$

where f, g_1 and g_2 are continuous functions on $\mathbb{R}$, $\tau(t) \geq 0$ is a bounded continuous function on $\mathbb{R}$, and $e(t)$ is a bounded continuous function on $\mathbb{R}^+ = [0, +\infty)$.

The authors in [10] established a theorem which ensure that all solutions of (1.1) are uniformly bounded, under the following two assumptions:

(C1) There exists a constant $d > 1$ such that $d|u| \leq \operatorname{sgn}(u)\varphi(u)$ for all $u \in \mathbb{R}$, where

$$\varphi(u) = \int_0^u [f(x) - 1]dx.$$

(C2) There exist nonnegative constants L_1, L_2, q_1, q_2 such that $L_1 + L_2 < 1$ and

$$|g_1(u) - \varphi(u)| \leq L_1|u| + q_1, \quad |g_2(u)| \leq L_2|u| + q_2, \quad \forall u \in \mathbb{R}.$$

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In this article, we will make further study on this problem. As one will see, under weaker assumptions than (C1) and (C2), we also get the same conclusion to [10]. Next, let us recall some notations and basic results.

Throughout this paper, we denote

$$\varphi(x) = \int_0^x [f(u) - 1]du, \quad y = \frac{dx}{dt} + \varphi(x).$$

Then (1.1) is transformed into the system

$$\begin{gathered} \frac{dx(t)}{dt} = -\varphi(x(t)) + y(t), \\ \frac{dy(t)}{dt} = -y(t) - [g_1(x(t)) - \varphi(x(t))] - g_2(x(t-\tau(t)) + e(t). \end{gathered} \tag{1.2}$$

Let $h = \sup_{t\in\mathbb{R}} \tau(t) \geq 0$. $C([-h, 0], \mathbb{R})$ denotes the space of continuous functions $\phi : [-h, 0] \to \mathbb{R}$ with the supremum norm $\|\cdot\|$. It is well known (cf. [2, 7]) that for any given continuous initial function $\phi \in C([-h, 0], \mathbb{R})$ and a number y_0, there exists a solution of (1.2) on an interval $[0, T)$ satisfying the initial conditions and (1.2) on $[0, T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $x(t) = x(t, \phi, y_0)$, $y(t) = y(t, \phi, y_0)$.

Definition 1.1 ([10])**.** Solutions of (1.2) are called uniformly bounded if for each $B_1 > 0$ there is a $B_2 > 0$ such that $(\phi, y_0) \in C([-h, 0], \mathbb{R}) \times \mathbb{R}$ and $\|\phi\| + |y_0| \leq B_1$ implies that $|x(t, \phi, y_0)| + |y(t, \phi, y_0)| \leq B_2$ for all $t \in \mathbb{R}^+$.

2. Main results

For our convenience, we list the following assumptions:

(A1) $|u| < \operatorname{sgn}(u)\varphi(u)$ for all $u \in \mathbb{R}$.

(A2) There exist two nondecreasing functions G, Φ defined on $\mathbb{R}^+$ such that

$$\begin{gathered} |g_1(u) - \varphi(u)| \leq \Phi(|u|), \quad |g_2(u)| \leq G(|u|), \quad \forall u \in \mathbb{R}, \\ \limsup_{x\to+\infty} [\Phi(x) + G(x) - x + \bar{e}] < 0, \quad \bar{e} = \sup_{t\in\mathbb{R}^+} |e(t)|. \end{gathered}$$

Theorem 2.1. *Suppose that* (A1), (A2) *hold. Then solutions of* (1.2) *are uniformly bounded.*

Proof. Let $x(t) = x(t, \phi, y_0)$, $y(t) = y(t, \phi, y_0)$ be a solution of (1.2). Calculating the upper right derivatives of $|x(s)|$ and $|y(s)|$, in view of (A1) and (A2), we have

$$\begin{aligned} D^+(|x(s)|)|_{s=t} &= \operatorname{sgn}(x(t))\{-\varphi(x(t)) + y(t)\} \\ &< -|x(t)| + |y(t)|, \\ D^+(|y(s)|)|_{s=t} &= \operatorname{sgn}(y(t))\{-y(t) - [g_1(x(t)) - \varphi(x(t))] - g_2(x(t-\tau(t)) + e(t)\} \\ &\leq -|y(t)| + \Phi(|x(t)|) + G(|x(t-\tau(t))|) + \bar{e}. \end{aligned}$$

Let

$$M(t) = \max_{-h\leq s\leq t} \{\max\{|x(s)|, |y(s)|\}\}, \quad t \geq 0.$$

By (A2), there is a constant $M > 0$ such that

$$\Phi(x) + G(x) - x + \bar{e} < 0, \quad x \geq M. \tag{2.1}$$

For any given $t_0 \geq 0$, we consider five cases.

Case (i): $M(t_0) > \max\{|x(t_0)|, |y(t_0)|\}$. It follows from the continuity of $x(t)$ and $y(t)$ that there exists $\delta_1 > 0$ such that

$$\max\{|x(t)|, |y(t)|\} < M(t_0), \quad \forall t \in (t_0, t_0 + \delta_1).$$

Thus, one can conclude $M(t) = M(t_0)$, for all $t \in (t_0, t_0 + \delta_1)$.

Case (ii): $M(t_0) = \max\{|x(t_0)|, |y(t_0)|\} < M$. Also, by the continuity of $x(t)$ and $y(t)$, there exists $\delta_2 > 0$ such that

$$\max\{|x(t)|, |y(t)|\} < M, \quad \forall t \in (t_0, t_0 + \delta_2).$$

Therefore, $M(t) < M$, for all $t \in (t_0, t_0 + \delta_2)$.

Case (iii): $M(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |x(t_0)| \geq M$, and $|x(t_0)| > |y(t_0)|$. Since

$$D^+(|x(s)|)|_{s=t_0} < -|x(t_0)| + |y(t_0)| < 0,$$

there exists $\delta_3 > 0$ such that

$$|x(t)| < |x(t_0)| = M(t_0) \quad \forall t \in (t_0, t_0 + \delta_3).$$

On the other hand, by the continuity of $y(t)$, without loss, one can assume that

$$|y(t)| < |x(t_0)| = M(t_0), \quad \forall t \in (t_0, t_0 + \delta_3).$$

So

$$\max\{|x(t)|, |y(t)|\} < M(t_0), \quad \forall t \in (t_0, t_0 + \delta_3),$$

which implies $M(t) = M(t_0)$, for all $t \in (t_0, t_0 + \delta_3)$.

Case (iv): $M(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |y(t_0)| \geq M$, and $|x(t_0)| < |y(t_0)|$. By (2.1), we have

$$\begin{aligned} D^+(|y(s)|)|_{s=t_0} &\leq -|y(t_0)| + \Phi(|x(t_0)|) + G(|x(t_0 - \tau(t_0))|) + \overline{e} \\ &\leq -M(t_0) + \Phi(M(t_0)) + G(M(t_0)) + \overline{e} < 0, \end{aligned}$$

which yields that there exists $\delta_4 > 0$ such that

$$|y(t)| < |y(t_0)| = M(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

On the other hand, without loss of generality, one can assume that

$$|x(t)| < |y(t_0)| = M(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

So one can conclude

$$\max\{|x(t)|, |y(t)|\} < M(t_0), \quad \forall t \in (t_0, t_0 + \delta_4).$$

Thus $M(t) = M(t_0)$ for all $t \in (t_0, t_0 + \delta_4)$.

Case (v): $M(t_0) = \max\{|x(t_0)|, |y(t_0)|\} = |x(t_0)| = |y(t_0)| \geq M$. We have

$$D^+(|x(s)|)|_{s=t_0} < -|x(t_0)| + |y(t_0)| = 0.$$

Also, similar to the proof of Case (iv), one can show that

$$D^+(|y(s)|)|_{s=t_0} < 0.$$

Thus, there exists $\delta_5 > 0$ such that

$$|x(t)| < |x(t_0)| = M(t_0), \quad |y(t)| < |y(t_0)| = M(t_0) \quad \forall t \in (t_0, t_0 + \delta_5).$$

Therefore, $M(t) = M(t_0)$ for all $t \in (t_0, t_0 + \delta_5)$. In summary, for each $t_0 \geq 0$, there exists $\delta > 0$ such that

$$M(t) \leq \max\{M(t_0), M\}, \quad \forall t \in (t_0, t_0 + \delta).$$

Let

$$\alpha = \begin{cases} \inf\{t \geq 0 : M(t) > \max\{M(0), M\}\} \\ \quad \text{if } \{t \geq 0 : M(t) > \max\{M(0),\ M\}\} \neq \emptyset, \\ +\infty \\ \quad \text{if } \{t \geq 0 : M(t) > \max\{M(0), M\}\} = \emptyset. \end{cases}$$

We claim that $\alpha = +\infty$. If $\alpha < +\infty$, then

$$M(t) \leq \max\{M(0), M\}, \quad \forall t \in [0, \alpha]. \tag{2.2}$$

It follows from the above proof that there is a constant $\delta' > 0$ such that

$$M(t) \leq \max\{M(\alpha), M\}, \quad \forall t \in (\alpha, \alpha + \delta'). \tag{2.3}$$

Combing (2.2) and (2.3), we have

$$M(t) \leq \max\{M(0), M\}, \quad \forall t \in [0, \alpha + \delta'),$$

which yields $\alpha \geq \alpha + \delta'$. This is a contradiction. Thus, $\alpha = +\infty$, which implies

$$M(t) \leq \max\{M(0), M\}, \quad \forall t \geq 0.$$

Then, we have

$$|x(t)| \leq \max\{M(0), M\}, \quad |y(t)| \leq \max\{M(0), M\}, \quad \forall t \geq 0.$$

Therefore, solutions of (1.2) are uniformly bounded. $\square$

Remark 2.2. One can easily conclude (A1) and (A2) from the assumptions (C1) and (C2). So Theorem 2.1 is a generalization of [10, Theorem 3.1]. In addition, our assumptions are weaker than (C1) and (C2) in essence (see Remark 2.4).

Next, we give an example to illustrate our results.

Example 2.3. Consider the following Liénard equation:

$$x''(t) + f(x(t))x'(t) + g_1(x(t)) + g_2(x(t - \tau(t)) = e(t), \tag{2.4}$$

where

$$f(x) = \frac{e^{-x} - xe^{-x}}{2} + 2, \quad g_1(x) = \frac{xe^{-x} + 3x + x^{1/3}}{2},$$
$$g_2(x) = x^{1/3}, \quad \tau(t) = \cos^2 t, \quad e(t) = \sin t.$$

Then

$$\varphi(x) = \int_0^x [f(u) - 1]du = \frac{1}{2}xe^{-x} + x,$$

and

$$\operatorname{sgn}(x)\varphi(x) = \big(\frac{1}{2}e^{-x} + 1\big)|x| > |x|, \quad \forall x \in \mathbb{R}.$$

So (A1) holds. In addition, let

$$\Phi(x) = \frac{x + x^{1/3}}{2}, \quad G(x) = x^{1/3}.$$

Then

$$|g_1(u) - \varphi(u)| = \big|\frac{u + u^{1/3}}{2}\big| \leq \Phi(|u|), \quad |g_2(u)| = G(|u|), \quad \forall u \in \mathbb{R},$$

and

$$\limsup_{x\to+\infty}[\Phi(x)+G(x)-x+\overline{e}] = \limsup_{x\to+\infty}\big[\frac{x+x^{1/3}}{2}+x^{1/3}-x+1\big] < 0,$$
$$\overline{e} = \sup_{t\in\mathbb{R}^+}|e(t)| = 1.$$

So (A2) holds. Then Theorem 2.1 shows that solutions of (2.4) are uniformly bounded.

Remark 2.4. In the above example, there is no a constant $d > 1$ such that

$$\operatorname{sgn}(x)\varphi(x) \geq d|x|, \quad \forall x \in \mathbb{R}.$$

So (C1) does not hold. Thus, [10, Theorem 3.1] can not be applied.

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