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EXISTENCE OF SOLUTIONS FOR AN ABSTRACT SECOND-ORDER DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. We discuss the existence of mild solutions for abstract second-order differential equation with nonlocal conditions. Also we consider some application of our results.

1. Introduction

In this paper we study the existence of mild solutions for the abstract second order differential system

$$\frac{d^2}{dt^2}\big(x(t) - g(t, x(t))\big) = Ax(t) + f(t, x(t)), \quad t \in I := [0, a], \tag{1.1}$$

$$x(0) = P(x_0, x), \tag{1.2}$$

$$\frac{d}{dt}(x(t) - g(t, x(t)))\big|_{t=0} = Q(y_0, x), \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space $(X, \|\cdot\|)$, $x_0, y_0 \in X$ and $f, g : I \times X \to X$, $P, Q : X \times C(I, X) \to X$ are appropriate functions.

The study of initial-value problems with nonlocal conditions arises to deal specially with some situations in physics. For the importance of nonlocal conditions in different fields we refer to [1, 3] and the references contained therein. There exists a extensive literature treating the problem of the existence of solutions for first and second order differential equations with nonlocal conditions. Concerning first order differential systems we cite the pioneers Byszewski works [1, 3] and [4, 2] between some contributions. In the case of second order differential equations with nonlocal, we mention [16, 17, 18, 19] for systems described on finite dimensional spaces and [6, 7, 8, 9, 10, 13, 14] for systems defined on abstract Banach spaces.

To the best of our knowledge, the existence of solutions for differential systems with nonlocal conditions described in the abstract form (1.1)-(1.3) is a untreated topic in the literature, and this fact, is the main motivation of the present paper. We also remark that the ideas, results and the general technical framework introduced

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in this paper can be used in the study of second order abstract neutral differential equations, which is an additional motivation.

First, we review some basic concepts, notation and properties needed to establish our results. Throughout this paper, $(X, \|\cdot\|)$ is a abstract Banach space and A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on X. We denote by $(S(t))_{t\in\mathbb{R}}$ the associated sine function which is defined by $S(t)x = \int_0^t C(s)xds$, for $(t,x) \in \mathbb{R}\times X$. In addition, N and $\tilde{N}$ are positive constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$ for every $t \in I$.

In this paper, $[D(A)]$ represents the domain of A endowed with the graph norm given by $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$, while E stands for the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. We know from [12], that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le a} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E\times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A)\times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This assertion is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \Big[\int_0^t S(t-s)x(s)\,ds, \quad \int_0^t C(t-s)x(s)\,ds\Big]^T$$

defines an $E\times X$-valued continuous function. In addition, it follows from the definition of the norm in E that a function $u : I \to E$ is continuous if, and only if, it is continuous with respect to the norm in X and the set of functions $\{AS(t)u(\cdot) : t \in [0,1]\}$ is an equicontinuous subset of $C(I,X)$.

The existence of solutions for the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in I, \tag{1.4}$$

$$x(0) = w, \quad x'(0) = z, \tag{1.5}$$

where $h : I \to X$ is an integrable function, is studied in [21]. Similarly, the existence of solutions of semi-linear second-order abstract Cauchy problems has been treated in [20]. We mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)ds, \quad t \in I, \tag{1.6}$$

is called a mild solution of (1.4)-(1.5). If $w \in E$, then the function $x(\cdot)$ is of class C^1 on I and

$$x'(t) = AS(t)w + C(t)z + \int_0^t C(t-s)h(s)\,ds, \; t \in I. \tag{1.7}$$

For additional details on the cosine function theory, we cite [5, 20, 21, 22].

Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z,W)$ stands for the Banach space of bounded linear operators from Z into

W endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(Z,W)}$. In addition, $B_r(x,Z)$ represents the closed ball with center at x and radius $r>0$ in Z.

This article has three sections. In the next section we discuss the existence of mild solutions for the system (1.1)-(1.3). In Section 3, some applications are considered.

2. Existence Results

In this section we study the existence of mild solutions for the system (1.1)-(1.3). From the theory of cosine functions of operators, we introduce the next definition.

Definition 2.1. A function $x \in C(I,X)$ is a mild solution of (1.1)-(1.3) if $x(0) = P(x_0,x)$ and

$$\begin{aligned} x(t) &= C(t)(P(x_0,x) - g(0,P(x_0,x))) + S(t)Q(y_0,x) + g(t,x(t)) \\ &\quad + \int_0^t AS(t-s)g(s,x(s))ds + \int_0^t S(t-s)f(s,x(s))ds, \quad t \in I. \end{aligned} \tag{2.1}$$

In the rest of this article, we assume the next hypotheses:

(H1) There exists a Banach space $(Y,\|\cdot\|_Y)$ continuously included in X such that $AS(t) \in \mathcal{L}(Y,X)$ for all $t \in I$, and $AS(\cdot)x \in C(I,X)$ for all $x \in Y$. Let N_Y, $\widetilde{N}_1$ be constants such that $\|x\| \le N_Y\|x\|_Y$ for all $x \in X$, and $\|AS(t)\|_{\mathcal{L}(Y,X)} \le \widetilde{N}_1$ for all $t \in I$.

(H2) The cosine function $(C(t))_{t\in\mathbb{R}}$ is such that the range of $(C(t)-I)$ is closed and $\dim\ker(C(t)-I) < \infty$ for all $0 < t \le a$.

We now introduce some assumptions for the functions f, g, P and Q.

(H3) The function $f(\cdot,y)$ is strongly measurable for every $y \in X$, $f(t,\cdot)$ is continuous $a.e$ for $t \in I$ and $g \in C(I\times X,Y)$. There are positive constants c_1, c_2, an integrable function $m_f : I \to [0,\infty)$ and a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that $\|f(t,y)\| \le m_f(t)W_f(\|y\|)$ and $\max\{\|g(t,y)\|_Y, \|g(t,y)\|\} \le c_1\|y\| + c_2$ for all $(t,y) \in I\times X$.

(H4) The function $f(\cdot,y) : I \to X$ strongly measurable for every $y \in X$, $g \in C(I\times X,Y)$ and there are positive numbers L_f, L_g such that

$$\begin{aligned} \|g(t,y_1) - g(t,y_2)\|_Y &\le L_g\|y_1 - y_2\|, \quad y_1,y_2 \in X, \\ \|f(t,y_1) - f(t,y_2)\| &\le L_f\|y_1 - y_2\|, \quad y_1,y_2 \in X. \end{aligned}$$

(H5) The functions $P(x_0,\cdot), Q(y_0,\cdot) : C(I,X) \to X$ are continuous and there are positive constants L_P, L_Q such that

$$\begin{aligned} \|P(x_0,u) - P(x_0,v)\| &\le L_P\|u - v\|, \quad u,v \in C(I,X), \\ \|Q(y_0,u) - Q(y_0,v)\| &\le L_Q\|u - v\|, \quad u,v \in C(I,X). \end{aligned}$$

(H6) The functions $P(x_0,\cdot), Q(y_0,\cdot) : C(I,X) \to X$ are continuous, locally bounded and P is completely continuous. Let $N_Q^r = \sup\{\|Q(y_0,u)\| : u \in B_r(0,C(I,X))\}$ and $N_P^r = \sup\{\|P(x_0,u)\| : u \in B_r(0,C(I,X))\}$.

We consider important to make some observations about the above conditions.

Remark 2.2. The assumption (H1) and the properties of g in condition (H3), are linked to the integrability of the function $s \to AS(t-s)g(s,x(s))$. We observe that, except for trivial cases, the operator function $s \to AS(s)$ is not integrable over $[0,b]$ for $b>0$. In fact, if we assume that $AS(\cdot) \in L^1([0,b])$, then from the relation

$C(t)x - C(s)x = A\int_s^t S(\tau)x d\tau$, which is valid for $s \le t \le b$ and $x \in X$ (see [20]), it follows that

$$C(t)x - x = A\int_0^t S(\tau)x\,d\tau = \int_0^t AS(s)x\,ds, \tag{2.2}$$

which implies that the $C(\cdot)$ is uniformly continuous on $[0,b]$ and, as consequence, that A is a bounded linear operator, see [21] for details.

On the another hand, if (H1) and (H3) hold, then from Bochner's criterion for integrable functions and the estimate

$$\|AS(t-s)g(s,x(s))\| \le \widetilde{N}_1(c_1\|x(s)\| + c_2),$$

we infer that the function $s \mapsto AS(t-s)g(s,x(s))$ is integrable on $[0,t)$, for all $t \in I$.

Remark 2.3. If assumption (H1) holds, then Y is continuously included in E. To prove this claim, we note that for $x \in Y$

$$C(t)x - x = A\int_0^t S(\tau)xd\tau = \int_0^t AS(s)x\,ds,$$

which implies that $C(\cdot)x$ is of class C^1 and $Y \subseteq E$. Moreover, since

$$\|x\|_E = \|x\| + \sup_{0\le t\le a}\|AS(t)x\| \le (N_Y + \widetilde{N}_1)\|x\|_Y$$

when $a \ge 1$, we obtain that the inclusion $\iota : Y \to E$ is continuous in this case. A similar argument using the properties of the sine function shows that $\iota : Y \to E$ is also continuous for $0 < a < 1$. To complete this remark, we note that $[D(A)]$ and E satisfy (H1).

Our main result is proved using a point fixed criterion for condensing operators. The assumption (H2) will be useful to this objective.

Lemma 2.4. Let condition (H2) be holds. If $B \subseteq Y$ is bounded in X and the set $\{AS(t)x : t \in I, x \in B\}$ is relatively compact in X, then B is relatively compact in X.

Proof. Let $x \in B$. From (2.2) and the mean value theorem for the Bochner integral (see [15, Lemma 2.1.3]) it follows that $C(t)x - x \in t\,\overline{c(\{AS(s)x : s \in [0,t]\})}$, where $c(\cdot)$ denotes the convex hull of a set. Now, the assertion is a consequence of the fact that $\overline{c(\{AS(s)y : s \in [0,t], y \in B\})}$ is compact and the properties of the operators $C(t) - I$. $\square$

We now establish our first existence result.

Theorem 2.5. *Let assumptions* (H1)-(H3), (H6) *hold, and assume that the following two conditions hold*

(a) *For every $t \in I$ and all $r > 0$, the set*

$$U_r^t = \{S(t)[f(s,y) + P(x_0,u)] : s \in I, y \in B_r(0,X), u \in B_r(0,C(I,X)\}$$

is relatively compact in X.

(b) *For each $r > 0$ and all $t \in I$, the sets $V_1^r = \{AS(s)g(s,y) : s \in I,\ y \in B_r(0,X)\}$ and $V_2^r = \{S(t)Q(y_0,u) : u \in B_r(0,C(I,X))\}$ are relatively compact in X, and the set of functions $\{t \mapsto g(\cdot,u(\cdot)) : u \in B_r(0,C(I,X))\}$ is equicontinuous on I.*

If

$$\limsup_{r\to\infty}\Big[\frac{NN_P^r+\tilde{N}N_Q^r+W_f(r)\tilde{N}a}{r}\Big]+c_1\Big(1+N+\int_0^a\|AS(t)\|_{\mathcal{L}(Y,X)}dt\Big)<1, \quad (2.3)$$

then there exists a mild solution of (1.1)-(1.3).

Proof. Let $\Gamma : C(I,X)\to C(I,X)$ be the map defined by

$$\begin{aligned}\Gamma u(t) &= C(t)(P(x_0,u)-g(0,u(0)))+S(t)Q(y_0,u)+g(t,u(t))\\ &\quad+\int_0^t AS(t-s)g(s,u(s))ds+\int_0^t S(t-s)f(s,u(s))ds, \quad t\in I,\end{aligned}$$

and consider the decomposition $\Gamma=\Gamma_1+\Gamma_2$ where

$$\begin{aligned}\Gamma_1 u(t) &= C(t)(P(x_0,u)-g(0,u(0)))+S(t)Q(y_0,u)+g(t,u(t)), \quad t\in I,\\ \Gamma_1 u(t) &= \int_0^t AS(t-s)g(s,u(s))ds+\int_0^t S(t-s)f(s,u(s))ds, \quad t\in I.\end{aligned}$$

From Remark 2.2 and the properties of the functions f,g,P,Q, it is easy to see that $\Gamma_i u\in C(I,X)$ for $i=1,2$.

Now, we prove that Γ is completely continuous. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence in $C(I,X)$ and $u\in C(I,X)$ such that $u^n\to u$. Let $(u^{n_j})_{j\in\mathbb{N}}$ be a sub-sequence of $(u^n)_{n\in\mathbb{N}}$. From the condition **(b)** and Lemma 2.4, it is easy to see that the set of functions $\{g(\cdot,u^{n_j}(\cdot)) : j\in\mathbb{N}\}$ is relatively compact in $C(I,X)$. Then, there exists a sub-sequence $(u^{n_{j_p}})_{p\in\mathbb{N}}$ of $(u^{n_j})_{j\in\mathbb{N}}$ such that $g(s,u^{n_{j_p}}(s))\to g(s,u(s))$ uniformly for $s\in I$ as $p\to\infty$, from which we obtain that $\Gamma_1 u^{n_{j_p}}\to\Gamma_1 u$ in $C(I,X)$ as $p\to\infty$. Moreover, an standard application of the Lebesgue dominated convergence Theorem permit to prove that $\Gamma_2(u^{n_{j_p}})\to\Gamma_2 u$ in $C(I,X)$ as $p\to\infty$ which implies that $\Gamma(u^{n_{j_p}})\to\Gamma u$ in $C(I,X)$ as $p\to\infty$. Since the $(u^{n_j})_{j\in\mathbb{N}}$ is an arbitrary subsequence of $(u^n)_{n\in\mathbb{N}}$, we can conclude that $\Gamma u^n\to\Gamma u$ in $C(I,X)$. Thus, Γ is continuous.

From (2.3), there exists a positive number r such that $\Gamma(B_r(0,C(I,X)))\subset B_r(0,C(I,X))$. In fact, let $r>0$ be such that

$$\frac{1}{r}\big[NN_P^r+\tilde{N}N_Q^r+W_f(r)\tilde{N}a\big]+(c_1+\frac{c_2}{r})(1+N+\int_0^a\|AS(t)\|_{\mathcal{L}(Y,X)}dt)<1. \quad (2.4)$$

Then, for $t\in[0,a]$ and $u\in B_r(0,C([0,a],X))$ we see that

$$\begin{aligned}\|\Gamma u(t)\| &\le N(N_P^r+c_1r+c_2)+\tilde{N}N_Q^r+c_1r+c_2\\ &\quad+\int_0^t\|AS(t-s)\|_{\mathcal{L}(Y,X)}\|g(s,u(s))\|ds+\widetilde{N}\int_0^t m_f(s)W(\|u(s)\|)ds\\ &\le\big[NN_P^r+\tilde{N}N_Q^r+W_f(r)\tilde{N}\int_0^a m_f(s)ds\big]\\ &\quad+(c_1r+c_2)(1+N+\int_0^a\|AS(t)\|_{\mathcal{L}(Y,X)}dt),\end{aligned}$$

which from (2.4) implies that $\Gamma u\in B_r(0,C(I,X))$. Thus, $\Gamma(B_r(0,C(I,X)))\subset B_r(0,C(I,X))$.

From Lemma 2.4, the assumptions (H6) and (b), it is easy to see that Γ_1 is completely continuous. Moreover, from [11, Lemma 3.1] we infer that Γ_2 is also completely continuous, which complete the proof that Γ is completely continuous.

Now, from the Schauder's point fixed Theorem we obtain a mild solution for (1.1)-(1.3). □

Proposition 2.6. *Assume that the assumptions in Theorem 2.5 be hold. If $x(\cdot)$ is a mild solution of (1.1)-(1.3), $P(x_0, x) \in Y$ and $\frac{d}{dt}C(t)g(0, x(0))\big|_{t=0} = 0$, then $\frac{d}{dt}(x(t) - g(t, x(t)))\big|_{t=0} = Q(y_0, x)$.*

Proof. At first, we note that from the inequality

$$\|\frac{1}{t}\int_0^t S(t-s)f(s, x(s))\,ds\| \leq N\int_0^t \|f(s, x(s))\|\,ds$$

it follows that $\frac{1}{t}\int_0^t S(t-s)f(s, x(s))ds \to 0$ as $t \to 0$. In addition, for $\delta > 0$, we can write

$$\begin{aligned}\int_0^t AS(t-s)g(s, x(s))ds = (I - \frac{1}{\delta}S(\delta))\int_0^t AS(t-s)g(s, x(s))\,ds \\ + \frac{1}{\delta}\int_0^t S(t-s)AS(\delta)g(s, x(s))\,ds.\end{aligned} \tag{2.5}$$

Let $r > 0$ be such that $\|x(s)\| \leq r$ for every $s \in I$. Since $AS(t-s)g(s, x(s)) \in V_1^r$, it follows from the mean value theorem for the Bochner integral [15, Lemma 2.1.3] that $\int_0^t AS(t-s)g(s, x(s))ds \in t\overline{c(V_1^r)}$, so that

$$(I - \frac{1}{\delta}S(\delta))\frac{1}{t}\int_0^t AS(t-s)g(s, x(s))\,ds \in (I - \frac{1}{\delta}S(\delta))\overline{c(V_1^r)}.$$

In view of the fact that $(I - \frac{1}{\delta}S(\delta))x \to 0$, as $\delta \to 0$, for each $x \in X$ and $\overline{c(V_1^r)}$ is a compact, we can affirm that $(I - \frac{1}{\delta}S(\delta))x \to 0$, as $\delta \to 0$, uniformly for $x \in \overline{c(V_1^r)}$, which implies that the first term of the right hand side of (2.5) converge to zero as $\delta \to 0$. Moreover, if $c_\delta > 0$ is such that $\|AS(\delta)g(s, x(s))\| \leq c_\delta$ for all $s \in I$, then

$$\Big\|\frac{1}{\delta}\int_0^t S(t-s)AS(\delta)g(s, x(s))\,ds\Big\| \leq \frac{N}{\delta}\int_0^t (t-s)c_\delta\,ds \leq \frac{Nc_\delta}{2\delta}t^2.$$

From the above remarks, we infer that $\frac{1}{t}\int_0^t AS(t-s)g(s, x(s))ds \to 0$, as $t \to 0^+$. Finally, by using that $P(x_0, x) \in Y$ and $\frac{d}{dt}C(t)g(0, x(0))\big|_{t=0} = 0$, we obtain

$$\begin{aligned}&\frac{d}{dt}(x(t) - g(t, x(t)))\big|_{t=0} \\ &= \lim_{t\to 0^+}\frac{1}{t}\big(C(t)P(x_0, x) - P(x_0, x)\big) - \frac{1}{t}[C(t)g(0, x(0)) - g(0, x(0))] \\ &\quad + \frac{S(t)}{t}Q(y_0, x) + \lim_{t\to 0^+}\frac{1}{t}\int_0^t AS(t-s)g(s, x(s))ds \\ &\quad + \lim_{t\to 0^+}\frac{1}{t}\int_0^t S(t-s)f(s, x(s))\,ds \\ &= \lim_{t\to 0^+}\frac{1}{t}\int_0^t AS(s)P(x_0, x)ds + \frac{S(t)}{t}Q(y_0, x) = Q(y_0, x),\end{aligned}$$

which completes the proof. □

The following result is a consequence of Theorem 2.5 and Proposition 2.6.

Corollary 2.7. *Let conditions* (H1)-(H3), (H6) *hold with* $m_f \in L^\infty(I)$*. Assume* $S(t)$ *is compact, for all* $t \geq 0$ *and*

- (b*) *For* $r > 0$*, the set* $V_1^r = \{AS(s)g(s,y) : s \in I,\ y \in B_r(0,X)\}$ *is relatively compact in* X*, and the set of functions* $\{g(\cdot,u(\cdot)) : u \in B_r(0,C(I,X))\}$ *is equicontinuous on* I*.*

If (2.3) *is valid, then there exists a mild solution of* $x(\cdot)$ *of* (1.1)-(1.3)*. Moreover, if* $P(x_0,x) \in Y$ *and* $\frac{d}{dt}C(t)g(0,x(0))\big|_{t=0} = 0$ *then* $\frac{d}{dt}(x(t) - g(t,x(t)))\big|_{t=0} = Q(y_0,x)$*.*

The proof of the next result is an standard application of the contraction mapping principle. We omit it.

Theorem 2.8. *Let* (H1), (H4), (H5) *hold. If*

$$\big[L_g(N_Y + a\widetilde{N}_1) + aL_f\tilde{N} + NL_P + \tilde{N}L_Q\big] < 1,$$

then there exists a unique mild solution $x(\cdot)$ *of* (1.1)-(1.3)*. Moreover,* $P(x_0,x) \in Y$ *and* $\frac{d}{dt}C(t)g(0,x(0))\big|_{t=0} = 0$ *then* $\frac{d}{dt}(x(t) - g(t,x(t)))\big|_{t=0} = Q(y_0,x)$*.*

3. Applications

In this section, we consider some applications of our abstract results. At first, we discuss briefly the particular case in which X is finite dimensional. In this case, the operator A is a matrix of order $n \times n$ which generates the uniformly continuous cosine function $C(t) = \cosh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}A^n$, with associated sine function $S(t) = A^{-\frac{1}{2}}\sinh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n+2}}{(2n+1)!}A^n$ (here, the expressions $\cosh(tA^{1/2})$ and $\sinh(tA^{1/2})$ are purely symbolic and do not assume the existence of the square roots of A). We note that the condition (H1) is automatically satisfied with $Y = X$, the operators $C(t)$, $S(t)$, $AS(t)$ are compact for all $t \in \mathbb{R}$, and $\|C(t)\| \leq \cosh(t\|A\|^{1/2})$ and $\|S(t)\| \leq \|A\|^{-1/2}\sinh(t\|A\|^{1/2})$ for all $t \in \mathbb{R}$.

The next proposition is a re-formulation of Theorem 2.5. In this result, $\gamma = \cosh(a\|A\|^{1/2}) + \|A\|^{-1/2}\sinh(a\|A\|^{1/2})$ and (H3*) is the condition,

- (H3*) The function $f(\cdot,y)$ is strongly measurable for every $y \in X$, $f(t,\cdot)$ is continuous a.e. for $t \in I$ and $g \in C(I \times X, X)$. There are positive constants c_1, c_2, an integrable function $m_f : I \to [0,\infty)$ and a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that $\|g(t,y)\| \leq c_1\|y\| + c_2$ and $\|f(t,y)\| \leq m_f(t)W_f(\|y\|)$ for all $(t,y) \in I \times X$.

Proposition 3.1. *Assume* (H3*), (H6) *hold, and for all* $r > 0$ *the set of functions* $\{g(\cdot,u(\cdot)) : u \in B_r(0,C(I,\mathbb{R}^n))\}$ *is equicontinuous on* I *and*

$$\limsup_{r\to\infty} \frac{\gamma}{r}[N_P^r + N_Q^r + W_f(r)a] + c_1(1 + \gamma(1 + a\|A\|)) < 1.$$

Then there exists a mild solution $x(\cdot)$ *of* (1.1)-(1.3)*. If, in addition,*

$$\frac{d}{dt}C(t)g(0,x(0))\big|_{t=0} = 0,$$

then $\frac{d}{dt}(x(t) - g(t,x(t)))\big|_{t=0} = Q(y_0,x)$*.*

To complete this section, we apply our abstract results on an concrete second order partial differential equation. Consider the differential system

$$\frac{\partial^2}{\partial t^2}\big[u(t,\tau)-\int_0^\pi b(\omega,\tau)u(t,\omega)d\omega\big]=\frac{\partial^2}{\partial \tau^2}u(t,\tau)+F\big(t,u(t,\tau)\big), \tag{3.1}$$

$$u(t,0)=u(t,\pi)=0, \tag{3.2}$$

$$u(0,\tau)=x_0(\tau)+\int_0^a p(u(s,\tau))ds, \tag{3.3}$$

$$\frac{\partial}{\partial t}u(0,\tau)=y_0(\tau)+\int_0^a q(u(s,\tau))ds, \tag{3.4}$$

for $(t,\tau)\in I\times J=[0,a]\times[0,\pi]$.

To study this system we chose the space $X=L^2([0,\pi])$, and we assume $x_0,y_0\in X$. In addition, we consider the operator $A:D(A)\subseteq X\to X$ by $Ax=x''$, where $D(A)=\{x\in X:x''\in X, x(0)=x(\pi)=0\}$. It is well-known that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, for $n\in\mathbb{N}$, with corresponding eigenvectors $z_n(\tau)=\big(\frac{2}{\pi}\big)^{1/2}\sin(n\tau)$, the set of functions $\{z_n:n\in\mathbb{N}\}$ is an orthonormal basis of X and the following properties hold.

(a) For $z\in X$, $C(t)z=\sum_{n=1}^\infty \cos(nt)\langle z,z_n\rangle z_n$ and the associated sine function is given by $S(t)z=\sum_{n=1}^\infty \frac{\sin(nt)}{n}\langle z,z_n\rangle z_n$. It follows from the last expression that $S(t)$ is compact for all $t\in\mathbb{R}$ and $\|C(t)\|=\|S(t)\|=1$, for all $t\in\mathbb{R}$. In addition, $Az=-\sum_{n=1}^\infty n^2\langle z,z_n\rangle z_n$, for $z\in D(A)$.

(b) If Φ is the group of translations on X defined by $\Phi(t)x(y_0)=\tilde{x}(y_0+t)$, where $\tilde{x}\cdot$ is the extension of $x\cdot$ with period 2π, then $C(t)=\frac{1}{2}(\Phi(t)+\Phi(-t))$ and $A=B^2$, where B is the infinitesimal generator of Φ and $E=\{x\in H^1(0,\pi):x(0)=x(\pi)=0\}$ (see [5] for details). In particular, we observe that the inclusion $\iota:E\to X$ is compact.

In what the follows, we assume that $x_0\in H^1([0,\pi])$ and the conditions.

(i) The function $b(\cdot)$ is of class C^2 on $I\times J$ and $b(\omega,\pi)=b(\omega,0)=0$ for all $\omega\in I$.

(ii) The function $F:I\times[0,\pi]\to\mathbb{R}$ is continuous and there is $L_f>0$ such that
$$|F(t,\tau_1)-F(t,\tau_2)|\le L_F|\tau_1-\tau_2|,\quad t\in I,\tau_i\in\mathbb{R}.$$

(iii) The function $p,q:\mathbb{R}\to\mathbb{R}$ are continuous and there are positive constants L_p,L_q such that
$$|p(\mu_1)-p(\mu_2)|\le L_p|\mu_1-\mu_2|,\quad \mu_i\in\mathbb{R},$$
$$|q(\mu_1)-q(\mu_2)|\le L_q|\mu_1-\mu_2|,\quad \mu_i\in\mathbb{R}.$$

Let $f,g:X\to X$ and $P,Q:C(I,X)\to X$ be the functions defined by $f(t,x)(\tau)=F(t,x(\tau))$ and

$$g(t,x)(\tau)=\int_0^\pi b(\omega,\tau)x(\omega)d\omega,$$
$$P(u)(\tau)=x_0(\tau)+\int_0^a p(u(s,\tau))ds,$$
$$Q(u)(\tau)=y_0(\tau)+\int_0^a q(u(s,\tau))ds.$$

Under the above conditions, the functions f, P, Q are Lipschitz continuous functions with Lipschitz constants $L_F, L_P a^{3/4}$ and $L_Q a^{3/4}$ respectively. In addition, $g(\cdot)$ is continuous, $g(t,\cdot)$ is a bounded linear operator for all $t \in I$, g is $D(A)$-valued,

$$\sup_{t\in I} \|g(t,\cdot)\|_{\mathcal{L}(X,[D(A)])} \leq L_g = \Big(\int_0^{\pi} \int_0^{\pi} (\frac{\partial^2}{\partial \tau^2} b(\omega,\tau))^2 d\omega d\tau \Big)^{1/2},$$

$N_{[D(A)]} \leq 1$ and $\widetilde{N_1} \leq 1$.

The next result follows directly from Theorem 2.8. We remark that for $z \in X$, $\frac{d}{dt}C(t)g(0,z) = AS(t)g(0,z) = S(t)Ag(0,z)$ so that, $\frac{d}{dt}C(t)g(0,z)\big|_{t=0} = 0$.

Proposition 3.2. *If $\big[L_g(1+a) + aL_f + a^{3/4}(L_P + L_Q)\big] < 1$, then there exists a unique mild solution of* (3.1)-(3.4). *Moreover, $\frac{d}{dt}(x(t) - g(t,x(t)))\big|_{t=0} = Q(y_0,x)$ if $P(x_0,x) \in Y$.*

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