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REGULARITY ESTIMATES IN BESOV SPACES FOR INITIAL-VALUE PROBLEMS OF GENERAL PARABOLIC EQUATIONS

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Abstract. In this paper we give regularity estimates for solutions to initial-value problem of general parabolic equations with data in adapted Besov spaces characterized by heat kernels.

1. Introduction

The purpose of this paper is to consider the regularity estimates for solutions of parabolic equations with data in adapted Besov spaces. The general parabolic equation is defined by

$$\partial_t u + Lu = f, \quad t \in [0,T] \tag{1.1}$$

$$u|_{t=0} = u_0, \tag{1.2}$$

where $u_0 \in \mathcal{D}(\mathbb{R}^n)$, $f \in C([0,T];\mathcal{D}(\mathbb{R}^n))(T>0)$, the elliptic operator L is defined by

$$L = -\operatorname{div} A\nabla \tag{1.3}$$

where $A = (a_{i,j})_{n\times n}$ is a matrix of complex-valued, measurable functions satisfying the elliptic conditions

(1) $\lambda|\xi|^2 \le \operatorname{Re}\sum_{i,j} a_{i,j}(x)\xi_i\bar{\xi}_j = \operatorname{Re}\langle A\xi,\xi\rangle$;
(2) $|\langle A\xi,\eta\rangle| \le \Lambda|\xi||\eta|$,

where $0<\lambda\le\Lambda<\infty$ and $\xi,\eta\in\mathbb{C}^n$.

In the classical theory, if L denotes the Laplace operator, the regularity estimates of the solution of the heat equations with data in the conventional Besov spaces have been given by using Fourier analysis methods(see[4]). Naturally there is one question, if L is defined by (1.3), what about the regularity estimates of the solution of the corresponding parabolic equations in Besov spaces? The purpose of the paper is to answer such question. The main difficulty here is that the Fourier analysis methods can't be used, since the heat kernel of e^{-tL} for the divergence elliptic operator L is not convolutional now. As we know, the regularity estimates of the solution of the parabolic equations depend on the bound of the heat semigroup e^{-tL}. In [1], Auscher and Tchamitchian studied bounds for the heat kernel $p_t(x,y)$

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of the semigroup $e^{-tL}(t > 0)$ for such divergence elliptic operator L as given in (1.3). These will be described in the next section in detail. In this paper we firstly define a kind of adapted Besov space associated to the parabolic equations by using the heat kernel, then we apply some other harmonic analysis methods in place of the Fourier transform methods to get our regularity estimates of the solution. From this point of view, our results are new.

The paper is organized as follows: in section 2, we give two assumptions and the definition of adapted Besov spaces, then give the main theorem; in section 3, we give the proof of the main theorem.

Through the paper, the constant "C" and "c" may be different somewhere, but it is not essential.

2. Two assumptions and main theorem

In this paper, we use two assumptions

Assumption (a) The holomorphic semigroup e^{-zL}, $|\arg(z)| < \pi/2 - \theta$ is represented by the kernel $a_z(x, y)$ which satisfies the so-called Poisson bound; that is, for all $\nu > \theta$,

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for all $x, y \in \mathbb{R}^n$ and $|\arg(z)| < \pi/2 - \nu$, where $h_t(x, y) = \frac{s(|x-y|^2/t)}{|B(x,t^{1/2})|}$, in which $B(x, t^{1/2})$ denotes any ball with center $x \in \mathbb{R}^n$ and radius $t^{1/2} > 0$, and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\kappa} s(r^2) = 0,$$

for some $\kappa > 0$. Denote $P_t = e^{-tL}$ and its kernel is $p_t(x, y)$, and also assume that $p_t(x, y)$ satisfies the Hölder continuity estimates

$$|p_t(x+h, y) - p_t(x, y)| + |p_t(y, x+h) - p_t(y, x)| \le c\Big(\frac{|h|}{t^{1/2} + |x - y|}\Big)^\mu h_t(x, y), \quad (2.1)$$

where $0 < \mu \le 1$ and $|h| \le \frac{1}{2}(t^{1/2} + |x - y|)$.

Remark 2.1. Auscher and Tchamitchian [1], found that if A was real, symmetric valued, the heat kernel $p_t(x, y)$ of the semigroup $e^{-tL}(t > 0)$ satisfied the upper Gaussian bounds and the Hölder continuity estimates. More details about the bound of e^{-tL} for elliptic operators can be found in [1]. In fact here some other elliptic operators have the above similar properties, such as Schrödinger operators or degenerate elliptic operators, related details can be found in [7] or [2].

In the following, we assume that $p_t(x, y)$ is the kernel of $P_t = e^{-tL}$ which can be seen as an approximation to identity. Set $Q_t = tLe^{-tL}$,then it can be proved that its kernel $v_t(x, y)$ also satisfies the Poisson bound and Hölder continuity estimates by using the Cauchy formula with the previous assumption.

Moreover it is easy to see that $e^{-tL}(1) = 1$ and $Le^{-tL}(1) = 0$. We also notice that the adjoint operator L^* has the similar properties to L. Set $\tilde{Q}_t = \sqrt{tL}e^{-tL}$,which kernel also satisfies the Poisson bound and Hölder continuity estimates due to the representation of $L^{1/2}$. Here it's obvious that $\tilde{Q}_t(1) = 0$ and $\tilde{Q}_t^*(1) = 0$, where $\tilde{Q}_t^*$ is the adjoint operator of $\tilde{Q}_t$. Here for simplicity, we assume that L is a self-adjoint operator.

Assumption (b) The operator L has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$. About the definition and related properties of H_∞-calculus, readers can refer to [5][1].

Remark 2.2. By the previous assumptions ,the following equality holds in the sense of the norm of $W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$

$$L^{1/2}f = \frac{1}{\pi^{-1/2}} \int_0^\infty e^{-tL} Lf \frac{dt}{\sqrt{t}}. \tag{2.2}$$

Moreover if matrix A is real, symmetric valued, the above equality holds in the norm of $W^{1,p}$. Here we mention that following representation formula also holds in the sense of the norm of L^p for $1 < p < \infty$ on basis of previous two assumptions,

$$I = c \int_0^\infty Q_t Q_t \frac{dt}{t}. \tag{2.3}$$

Related details can be found in [5][1].

Next we give the adapted Besov space for the corresponding parabolic equations.

Definition 2.3. Let $1 < p,\, q < \infty$ and $\alpha \in [-\mu, \mu]$. For $u \in L^p(\mathbb{R}^n)$

$$\|u\|_{B_p^{\alpha,q}} = \|u\|_{L^p} + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|Q_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q} < \infty.$$

For $T > 0$, $1 \le \rho \le \infty$,

$$\|u\|_{\tilde{L}_T^\rho(B_p^{\alpha,q})} = \|u\|_{L_T^\rho(L^p)} + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|Q_s(u)\|_{L_T^\rho(L^p)}^q \frac{ds}{s} \Big\}^{1/q} < \infty,$$

where for any $u \in L_T^\rho(L^p)(\mathbb{R}^n)$,

$$\|u\|_{L_T^\rho(L^p)} = \Big\{ \int_0^T \|u\|_{L^p}^\rho dt \Big\}^{1/\rho} < \infty.$$

Remark 2.4. Note that for any $u \in L_T^\rho(W^{k,p})(\mathbb{R}^n)$, similar definition can be given in the following form

$$\|u\|_{L_T^\rho(W^{k,p})} = \{ \int_0^T \|u\|_{W^{k,p}}^\rho dt \}^{1/\rho} < \infty,$$

where $W^{k,p}(\mathbb{R}^n)$ is Sobolev space and $k \in \mathbb{Z}^+$. Here for $p, q, \rho = \infty$, definitions for the above spaces can be given conventionally. Moreover by the coordinate transform, we have

$$\|u\|_{B_p^{\alpha,q}} \sim \|u\|_{L^p} + \Big\{ \int_0^2 s^{-\frac{1}{2}\alpha q} \|Q_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q} < \infty.$$

Now we give the main theorem in the paper.

Theorem 2.5. *Suppose that L satisfies assumptions (a) and (b). Let $\alpha \in (-\mu, \mu)$ $(0 < \mu \le 1)$ and $1 < p, q, \rho < \infty$. Let $u_0 \in B_p^{\alpha,q}$ and $f \in \tilde{L}_T^\rho(B_p^{\alpha-2+\frac{2}{\rho},q}) \cap L_T^\infty(L^p)$. Then the initial-value problem of the parabolic equation has a solution $u \in \tilde{L}_T^\rho(B_p^{\alpha+\frac{2}{\rho},q}) \cap L_T^1(W^{2,p})$ and there exists a constant $C > 0$ depending only on n and such that*

$$\|u\|_{\tilde{L}_T^\rho(B_p^{\alpha+\frac{2}{\rho_1},q})} \le C\big((1 + T^{1/\rho_1})\|u_0\|_{B_p^{\alpha,q}} + (1 + T^{1+\frac{1}{\rho_1}-\frac{1}{\rho}})\|f\|_{\tilde{L}_T^\rho(B_p^{\alpha-2+\frac{2}{\rho},q})}\big),$$

where $\rho_1 \in (\rho, +\infty)$ satisfying $|\alpha + \frac{2}{\rho_1}| \le \mu$ and $|\alpha - 2 + \frac{2}{\rho}| \le \mu$.

3. The proof of main theorem

Before we prove the main theorem, we need the following lemmas.

Lemma 3.1. *Let $k_{s,t}(x,y)$ be the kernel of $P_t(Q_s)$. If $t \geq s$, there exists a constant $c > 0$ such that*

$$|k_{s,t}(x,y)| \leq c\big(\sqrt{s/t}\big)^{\mu} h_s(x,y). \tag{3.1}$$

If $t \leq s$, there exists a constant $c > 0$ such that

$$|k_{s,t}(x,y)| \leq c h_t(x,y). \tag{3.2}$$

Proof. The proof is similar to [3, Lemma B.1], which uses the vanishing conditions of the kernel of Q_s and Hölder continuity conditions. □

Remark 3.2. Here the kernels of $Q_t(Q_s)$ and $Q_t(\tilde{Q}_s)$ have better properties due to the vanishing moment conditions; that is, let $K_{s,t}(x,y)$ be the kernel of $Q_t(Q_s)$, then if $t \geq s$, there exists a constant $c > 0$ such that

$$|K_{s,t}(x,y)| \leq c\big(\sqrt{s/t}\big)^{\mu} h_s(x,y); \tag{3.3}$$

Also if $t \leq s$, there exists a constant $c > 0$ such that

$$|K_{s,t}(x,y)| \leq c\big(\sqrt{t/s}\big)^{\mu} h_t(x,y). \tag{3.4}$$

Lemma 3.3. *Let $1 < p,\ q < \infty$ and $\alpha \in (-\mu, \mu)$. For $u \in L^p(\mathbb{R}^n)$,*

$$\|u\|_{B_p^{\alpha,q}} \sim \|u\|_{L^p} + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|\tilde{Q}_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q} < \infty.$$

Proof. Firstly we prove there exists a constant $c > 0$ such that

$$\|u\|_{B_p^{\alpha,q}} \leq c(\|u\|_{L^p} + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|\tilde{Q}_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q}) < \infty.$$

Here we need to verify that

$$\Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|Q_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q} \leq c\Big(\|u\|_{L^p} + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|\tilde{Q}_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q}\Big). \tag{3.5}$$

Note that the following identity holds in L^p for $1 < p < \infty$,

$$I = \int_0^\infty Q_t \frac{dt}{t} = c \int_0^\infty Q_{2t} \frac{dt}{t}.$$

Also note that $Q_{2t} = \tilde{Q}_t \tilde{Q}_t$, then

$$I = c \int_0^\infty \tilde{Q}_t \tilde{Q}_t \frac{dt}{t}.$$

Next

$$\begin{aligned}
\Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \|Q_s(u)\|_p^q \frac{ds}{s} \Big\}^{1/q} &= \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \| \int_0^\infty Q_s \tilde{Q}_t \tilde{Q}_t(u) \frac{dt}{t} \|_p^q \frac{ds}{s} \Big\}^{1/q} \\
&= \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \| \int_0^s Q_s \tilde{Q}_t \tilde{Q}_t(u) \frac{dt}{t} \|_p^q \frac{ds}{s} \Big\}^{1/q} \\
&\quad + \Big\{ \int_0^1 s^{-\frac{1}{2}\alpha q} \| \int_s^\infty Q_s \tilde{Q}_t \tilde{Q}_t(u) \frac{dt}{t} \|_p^q \frac{ds}{s} \Big\}^{1/q} \\
&= I + II.
\end{aligned}$$

Using Lemma 3.1, remark 3.1 and standard harmonic analysis technique, we can obtain

$$I \leq c\Big\{ \int_0^1 \Big(\int_t^1 s^{-\frac{1}{2}\alpha}(\sqrt{\frac{t}{s}})^{\mu} \|\tilde{Q}_t(u)\|_p \frac{ds}{s}\Big)^q \frac{dt}{t}\Big\}^{1/q} \leq c\Big\{ \int_0^1 t^{-\frac{1}{2}\alpha q} \|\tilde{Q}_t(u)\|_p^q \frac{dt}{t}\Big\}^{1/q}.$$

Similarly we can also get

$$\begin{aligned} II &\leq c\Big\{ \int_0^1 \Big(\int_0^t s^{-\frac{1}{2}\alpha}(\sqrt{s/t})^{\mu} \|\tilde{Q}_t(u)\|_p \frac{ds}{s}\Big)^q \frac{dt}{t}\Big\}^{1/q} \\ &\leq c\Big\{ \int_0^1 t^{-\frac{1}{2}\alpha q} \|\tilde{Q}_t(u)\|_p^q \frac{dt}{t}\Big\}^{1/q}. \end{aligned}$$

Thus we have end the proof of (3.5). The proof of the reverse inequality of (3.5) depends on (2.3) and Lemma 3.1, which proof is similar to the above one. We omit the details, the proof is end. □

Now we present the proof of main theorem.

Proof of Theorem 2.1. Similar to the classical theory of parabolic equations, by using the contraction mapping theorem(Here readers can refer to the Chapter 4 in [6]), for $u_0 \in L^p$ and $f \in L^\infty(L^p)$ for $1 < p < \infty$, there exists a solution $u \in L^1_T(W^{2,p})$ for (1.3) and

$$u(t,x) = e^{-tL}(u_0)(x) + \int_0^t e^{(\tau-t)L}(f)(\tau,x)d\tau. \tag{3.6}$$

More precisely, by using the bound of e^{-tL} and (3.6), for $1 < \rho < \rho_1 < \infty$, we have

$$\begin{aligned} \|u\|_{L^p} &\leq \|e^{-tL}(u_0)(\cdot)\|_{L^p} + \int_0^t \|e^{(\tau-t)L}(f)(\tau,\cdot)\|_{L^p} d\tau \\ &\leq c(\|u_0\|_{L^p} + T^{1-1/\rho}\|f\|_{L^\rho_T(L^p)}), \end{aligned}$$

then

$$\|u\|_{L^{\rho_1}_T(L^p)} \leq c(T^{1/\rho_1}\|u_0\|_{L^p} + T^{1+\frac{1}{\rho_1}-\frac{1}{\rho}}\|f\|_{L^\rho_T(L^p)}). \tag{3.7}$$

Now apply (3.6) to $Q_{2s}(u)$, then

$$Q_{2s}(u) = e^{-tL}(Q_{2s}(u_0)) + \int_0^t e^{(\tau-t)L}(Q_{2s}(f))d\tau.$$

Since $Q_{2s} = \tilde{Q}_s\tilde{Q}_s$, then for $1 < p < \infty$

$$\|Q_{2s}(u)\|_{L^p} \leq c(\|e^{-tL}\tilde{Q}_s\tilde{Q}_s(u_0)\|_{L^p} + \int_0^t \|e^{(\tau-t)L}\tilde{Q}_s\tilde{Q}_s(f)\|_{L^p} d\tau).$$

Next by using Lemma 3.1, we have

$$\begin{aligned} \|Q_{2s}(u)\|_{L^{\rho_1}_T(L^p)} \leq c\Big(&\Big\{ \int_0^T \min\big((\sqrt{s/t})^{\mu}, 1\big)^{\rho_1} dt\Big\}^{1/\rho_1} \|\tilde{Q}_s(u_0)\|_{L^p} \\ &+ \Big\| \int_0^t \min\big((\sqrt{\frac{s}{t-\tau}})^{\mu}, 1\big) \|\tilde{Q}_s(f)\|_{L^p} d\tau \Big\|_{L^{\rho_1}_T}\Big). \end{aligned}$$

For the second term of the above inequality, we use the young inequality, then by some simple calculus,we have

$$\|Q_{2s}(u)\|_{L^{\rho_1}_T(L^p)} \leq c(s^{\frac{1}{2\rho_1}}\|\tilde{Q}_s(u_0)\|_{L^p} + s^{\frac{1}{2\rho_2}}\|\tilde{Q}_s(f)\|_{L^\rho_T(L^p)}),$$

where $\frac{1}{\rho_2} = 1 + \frac{1}{\rho_1} - \frac{1}{\rho}$. By using Definition 2.2 and Lemma 3.2 and (3.7), we obtain the inequality

$$\|u\|_{\tilde{L}^\rho_T(B_p^{\alpha+\frac{2}{\rho_1},q})} \le C\big((1+T^{1/\rho_1})\|u_0\|_{B_p^{\alpha,q}} + (1+T^{1+\frac{1}{\rho_1}-\frac{1}{\rho}})\|f\|_{\tilde{L}^\rho_T(B_p^{\alpha-2+\frac{2}{\rho},q})}\big),$$

This completes the proof of the theorem. □

Remark 3.4. For higher regularity and the cases of critical indexes for Besov spaces, the method in the paper doesn't work because some difficulties arise in the process. We will consider these cases later. For the weighted case, we notice that recently Cruz-Uribe and Rios ([2]) studied the boundedness of the semigroup $e^{-tL_\omega}(t>0)$ for $\omega \in A_2$, where the elliptic operator was defined by

$$L_\omega = -\omega^{-1} \operatorname{div} A\nabla \tag{3.8}$$

where $A = (a_{i,j})_{n\times n}$ was a matrix of complex-valued, measurable functions satisfying some degenerate elliptic conditions. They pointed that if A is real and symmetric valued, the heat kernel $p_t(x,y)$ of the semigroup $e^{-tL_\omega}(t>0)$ also satisfies Gaussian upper bounds. We think that regularity results in adapted weighted Besov spaces for solutions of the corresponding parabolic equations can also be obtained by using similar methods to the ones described in this paper.

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