

**GENERALIZED FIRST-ORDER NONLINEAR EVOLUTION  
EQUATIONS AND GENERALIZED YOSIDA APPROXIMATIONS  
BASED ON H-MAXIMAL MONOTONICITY FRAMEWORKS**

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ABSTRACT. First a general framework for the Yosida approximation is introduced based on the relative  $H$ -maximal monotonicity model, and then it is applied to the solvability of a general class of first-order nonlinear evolution equations. The obtained results generalize and unify a wide range of results to the context of the solvability of first-order nonlinear evolution equations in several settings.

1. PRELIMINARIES

Let  $X$  be a real Hilbert space with the norm  $\|\cdot\|$  and the inner product  $\langle \cdot, \cdot \rangle$ . We consider a class of first-order nonlinear evolution equations of the form

$$\begin{aligned}u'(t) + Mu(t) &= 0, \quad 0 < t < \infty \\ u(0) &= u_0,\end{aligned}\tag{1.1}$$

where  $M : \text{dom}(M) \subseteq X \rightarrow X$  is a single-valued mapping on  $X$ ,  $u : [0, \infty) \rightarrow X$  is a continuous function such that (1.1) holds, and the derivative  $u'(t)$  exists in the sense of weak convergence if and only if

$$\frac{u(t+h) - u(t)}{h} \rightharpoonup u'(t) \in X \quad \text{as } h \rightarrow 0.$$

We note that in a Hilbert space setting, we have the fundamental equivalence:

$M$  is maximal accretive if and only if  $M$  is maximal monotone

This equivalence provides a close connection among nonexpansive semigroups, first-order evolutions, and the theory of monotone mappings. It is observed that the solution set of (1.1) coincides with that of the Yosida approximate evolution equation

$$\begin{aligned}u'_\rho(t) + M_\rho u_\rho(t) &= 0, \quad 0 < t < \infty \\ u_\rho(0) &= u_0,\end{aligned}\tag{1.2}$$

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where  $M_\rho = \rho^{-1}(I - (I + \rho M)^{-1})$ , the Yosida approximation for  $M$  with parameter  $\rho > 0$ . Moreover, as far as the solvability of (1.1) is concerned, it is easier to work with (1.2). As  $M_\rho$  is Lipschitz continuous, the Yosida approximate equation can be easily solved by using the Picard-Lindelöf theorem [10, Theorem 3.A].

Now we state the theorem by Komura [3] on the solvability of (1.1) based on the Yosida approximation

**Theorem 1.1.** *Let  $M : \text{dom}(M) \subseteq X \rightarrow X$  be a mapping on a real Hilbert space  $H$  such that:*

- (i)  $M$  is monotone.
- (ii)  $R(I+M)=X$ .

*Then, for each  $u_0 \in \text{dom}(M)$ , there exists exactly one continuous function  $u : [0, \infty) \rightarrow X$  such that (1.1) holds for all  $t \in (0, \infty)$ , where the derivative  $u'(t)$  is in the sense of weak convergence.*

Note that (i) and (ii) imply that  $M$  is maximal accretive if and only if  $M$  is maximal monotone.

**Remark 1.2.** Note that the unique solution in Theorem 1.1 has the following significant properties:

- (a)  $u(t) \in D(M)$  for all  $t \geq 0$ .
- (b)  $u(\cdot)$  is Lipschitz continuous on  $[0, \infty)$ .
- (c) For almost all  $t \in (0, \infty)$ , the derivative  $u'(t)$  exists in the usual sense and satisfies equation (1.1). Furthermore, we have  $\|u'(t)\| \leq \|Mu_0\|$ .
- (d) The function  $t \mapsto u'(t)$  is the generalized derivative of the function  $t \mapsto u(t)$  on  $(0, \infty)$ . Besides,  $u' \in C_w([0, \infty), X)$ , that is,  $u'(\cdot) : [0, \infty) \rightarrow X$  is weakly continuous.
- (e) For all  $t \geq 0$ , there exists a derivative  $u'_+(t)$  from the right and

$$u'_+(t) + Mu(t) = 0, \quad u(0) = u_0.$$

Next, we describe the connection of the solution to (1.1) with nonexpansive semigroups.

**Theorem 1.3** ([18, Corollary 31.1]). *Let  $u = u(t)$  be the solution of (1.1). We set  $S(t)u_0$  by*

$$S(t)u_0 = u(t) \quad \forall t \geq 0, \quad u_0 \in \text{dom}(M). \quad (1.3)$$

*Then  $\{S(t)\}$  is a nonexpansive semigroup on  $\text{dom}(M)$  that can be uniquely extended to a nonexpansive semigroup on  $\overline{\text{dom}(M)}$ , where the generator of  $\{S(t)\}$  on  $\overline{\text{dom}(M)}$  is  $-M$ .*

It is worth mentioning the following convergence result on the maximal monotonicity ([18, Proposition 31.6]) which quite useful in many ways.

**Lemma 1.4.** *Let  $M : \text{dom}(M) \subseteq X \rightarrow X$  be a mapping on a real Hilbert space  $X$  be maximal monotone. Then we have:*

$$\begin{aligned} Mu_n &\rightarrow b \quad \text{as } n \rightarrow \infty, \\ u_n &\rightarrow u \quad \text{as } n \rightarrow \infty, \end{aligned}$$

or

$$Mu_n \rightarrow b \quad \text{as } n \rightarrow \infty,$$

$$u_n \rightharpoonup u \quad \text{as } n \rightarrow \infty,$$

then  $Mu = b$ .

We intend in this communication to generalize Theorem 1.1 to the case of the  $H$ -maximal accretivity based on the generalized Yosida approximations. Unlike to the case of the maximal accretivity, the generalized Yosida approximation turns out to be Lipschitz continuous, while we explored the best Lipschitz continuity constant as well. The obtained results seem to be application-enhanced to problems arising from other fields, including optimization and control theory, variational inequality and variational inclusion problems, and unify a large class of results relating to nonlinear first-order evolution equations. There are also some detailed results that are investigated on the generalized Yosida approximations empowered by the  $H$ -maximal monotonicity frameworks. Furthermore, the results are general in nature and offer more unifying to other fields. For more details, we refer the reader to the references in this article.

The content of this research is organized as follows: Section 1 deals, as usual, with introductory and preliminary materials on first-order nonlinear evolution equations based on the Yosida approximation. In Section 2, the  $H$ -maximal monotonicity/maximal accretivity, and related auxiliary results are discussed, while in Section 3 the Yosida approximation is generalized to case of the resolvent operator based on  $H$ -maximal monotonicity models with several results presented, especially for the solvability of the generalized first-order nonlinear evolution equations. Section 4, deals with the main result on the solvability of (4.1) along with some auxiliary results relating to Yosida approximations.

## 2. $H$ -MAXIMAL MONOTONICITY RESULTS

In this section we discuss some results based on the basic properties of the relative  $H$ -maximal monotonicity.

**Definition 2.1.** Let  $A : D(A) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(A) \cap D(M) \neq \emptyset$ . The map  $M$  is said to be:

(i) Monotone if

$$\langle M(u) - M(v), u - v \rangle \geq 0 \quad \forall u, v \in D(M).$$

(ii) ( $r$ )-strongly monotone if there exists a positive constant  $r$  such that

$$\langle M(u) - M(v), u - v \rangle \geq r\|u - v\|^2 \quad \forall u, v \in D(M).$$

(iii) ( $m$ )-relaxed monotone if there exists a positive constant  $m$  such that

$$\langle M(u) - M(v), u - v \rangle \geq (-m)\|u - v\|^2 \quad \forall u, v \in D(M).$$

(iv) Cocoercive if

$$\langle M(u) - M(v), u - v \rangle \geq \|M(u) - M(v)\|^2 \quad \forall u, v \in D(M).$$

(v) ( $c$ )-cocoercive if there exists a positive constant  $c$  such that

$$\langle M(u) - M(v), u - v \rangle \geq c\|M(u) - M(v)\|^2 \quad \forall u, v \in D(M).$$

(vi) Monotone with respect to  $A$  if

$$\langle M(u) - M(v), A(u) - A(v) \rangle \geq 0 \quad \forall u, v \in D(A) \cap D(M).$$

- (vii) ( $r$ )-strongly monotone with respect to  $A$  if there exists a positive constant  $r$  such that

$$\langle M(u) - M(v), A(u) - A(v) \rangle \geq r \|u - v\|^2 \quad \forall u, v \in D(A) \cap D(M).$$

- (viii) ( $m$ )-relaxed monotone with respect to  $A$  if there exists a positive constant  $m$  such that

$$\langle M(u) - M(v), A(u) - A(v) \rangle \geq (-m) \|u - v\|^2 \quad \forall u, v \in D(A) \cap D(M).$$

- (ix) Cocoercive with respect to  $A$  if

$$\langle M(u) - M(v), A(u) - A(v) \rangle \geq \|M(u) - M(v)\|^2 \quad \forall u, v \in D(A) \cap D(M).$$

- (x) ( $c$ )-cocoercive with respect to  $A$

$$\langle M(u) - M(v), A(u) - A(v) \rangle \geq c \|M(u) - M(v)\|^2 \quad \forall u, v \in D(A) \cap D(M).$$

As an example consider  $X = (-\infty, +\infty)$ ,  $M(x) = -x$  and  $H(x) = -\frac{1}{2}x$  for all  $x \in X$ . Then  $M$  is monotone with respect  $H$  but not monotone.

Note that the monotonicity of  $M$  with respect to  $H$  is also referred to as the *hyper monotonicity* in the literature.

**Definition 2.2.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . The map  $M : D(M) \subseteq X \rightarrow X$  is said to be  $H$ -maximal monotone relative to  $H$  if

- (i)  $M$  is monotone with respect to  $H$ ; that is,

$$\langle M(u) - M(v), H(u) - H(v) \rangle \geq 0,$$

- (ii)  $R(H + \rho M) = X$  for  $\rho > 0$ .

**Definition 2.3** ([2]). Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . The map  $M : D(M) \subseteq X \rightarrow X$  is said to be  $H$ -maximal monotone if

- (i)  $M$  is monotone, that is,  $\langle M(u) - M(v), u - v \rangle \geq 0$ ;  
(ii)  $R(H + \rho M) = X$  for  $\rho > 0$ .

**Definition 2.4.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . The map  $M : D(M) \subseteq X \rightarrow X$  is said to be  $H$ -accretive (or accretive with respect to  $H$ ) if and only if  $H + \rho M$  is injective and  $(H + \rho M)^{-1}$  is Lipschitz continuous for all  $\rho > 0$ .

**Definition 2.5.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . The map  $M : D(M) \subseteq X \rightarrow X$  is said to be  $H$ -maximal accretive relative to  $H$  if and only if  $M$  is  $H$ -accretive and  $(H + \rho M)^{-1}$  exists on  $X$  for all  $\rho > 0$ .

For  $H = I$  in Definitions 2.4 and 2.5, we have Definitions 2.6 and 2.7.

**Definition 2.6.** Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. The map  $M : D(M) \subseteq X \rightarrow X$  is said to be accretive if and only if  $I + \rho M$  is injective and  $(I + \rho M)^{-1}$  is nonexpansive for all  $\rho > 0$ .

**Definition 2.7.** Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. The map  $M : D(M) \subseteq X \rightarrow X$  is said to be maximal accretive (or m-accretive) if and only if  $M$  is accretive and  $(I + \rho M)^{-1}$  exists on  $X$  for all  $\rho > 0$ .

**Definition 2.8.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $M$  be an  $H$ -maximal monotone mapping. Then the generalized resolvent operator  $J_{\rho,H}^M : X \rightarrow D(H + \rho M)$  is defined by

$$J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X.$$

**Definition 2.9.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Then the generalized relative resolvent operator  $R_{\rho,H}^M : X \rightarrow D(H + \rho M)$  is defined by

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X.$$

**Proposition 2.10.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Then the generalized resolvent operator associated with  $M$  and defined by

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X$$

is single-valued.

**Proposition 2.11.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Then the generalized resolvent operator associated with  $M$  and defined by

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X$$

is  $(\frac{1}{r})$ -Lipschitz continuous.

*Proof.* For any  $u, v \in X$ , it follows from the definition of the resolvent operator  $R_{\rho,H}^M$  that

$$\begin{aligned} \rho^{-1}(u - H(R_{\rho,H}^M(u))) &= M(R_{\rho,H}^M(u)), \\ \rho^{-1}(v - H(R_{\rho,H}^M(v))) &= M(R_{\rho,H}^M(v)). \end{aligned}$$

Since  $M$  is monotone relative to  $H$  and  $H$  is  $(r)$ -strongly monotone, we have

$$\begin{aligned} &\rho^{-1}\langle u - H(R_{\rho,H}^M(u)) - (v - H(R_{\rho,H}^M(v))), H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v)) \rangle \\ &= \rho^{-1}\langle u - v - [H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v))], H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v)) \rangle \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\langle u - v, H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v)) \rangle \\ &\geq \langle H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v)), H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v)) \rangle \\ &= \|H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v))\|^2. \end{aligned}$$

It follows that

$$\|u - v\| \|H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v))\| \geq \|H(R_{\rho,H}^M(u)) - H(R_{\rho,H}^M(v))\|^2$$

which completes the proof.  $\square$

**Proposition 2.12.** Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone. Then  $M$  is monotone (with respect to  $H$ ) if and only if  $M$  is accretive (with respect to  $H$ ).

*Proof.* Assume that  $M$  is  $H$ -accretive. Then, for all  $u, v \in D(H) \cap D(M)$  and  $\rho > 0$ , we have

$$\begin{aligned} & \|(H(u) + \rho M(u)) - (H(v) + \rho M(v))\|^2 \\ &= \|H(u) - H(v)\|^2 + 2\rho \langle H(u) - H(v), M(u) - M(v) \rangle + \rho^2 \|M(u) - M(v)\|^2. \end{aligned}$$

Therefore,

$$\|(H(u) + \rho M(u)) - (H(v) + \rho M(v))\|^2 \geq r^2 \|u - v\|^2 \quad \forall \rho > 0$$

if and only if

$$\langle M(u) - M(v), H(u) - H(v) \rangle \geq 0.$$

□

**Proposition 2.13** ([18]). *Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. Then  $M$  is monotone if and only if  $M$  is accretive.*

**Proposition 2.14.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone and  $(s)$ -Lipschitz continuous. Then the following statements are equivalent:*

- (i)  $M$  is monotone relative to  $H$  and  $R(H + \lambda M) = X$ .
- (ii)  $M$  is  $H$ -maximal accretive relative to  $H$ .
- (iii)  $M$  is  $H$ -maximal monotone relative to  $H$ .

*Proof.* We just prove implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii). Let us begin with (i)  $\Rightarrow$  (ii). From Proposition 2.12, it follows that  $M$  is accretive relative to  $H$ , and hence  $R_{\lambda, H}^M = (H + \lambda M)^{-1}$  is  $(\frac{1}{r})$ -Lipschitz continuous. Next all we need is to show for the existence of  $R_{\lambda, H}^M$  that  $H + \lambda M$  is one-one and onto. It would be sufficient to establish  $R(H + \rho M) = X$  for all  $\rho > 0$ . We start with equation

$$H(u) + \rho M(u) = w \quad \text{for } u \in X, \tag{2.1}$$

which is equivalent to

$$u = L_w(u) \quad \text{for } u \in X, \tag{2.2}$$

where

$$L_w(u) = R_{\lambda, H}^M [(1 - \rho^{-1}\lambda)H(u) + \rho^{-1}\lambda w].$$

Furthermore, we observe that for  $\rho > \lambda s(r + s)$ , we have  $|1 - \rho^{-1}\lambda| < r/s$  and

$$\|L_w(u) - L_w(v)\| \leq \frac{s}{r} |1 - \rho^{-1}\lambda| \|u - v\| \quad \forall u, v \in X.$$

Then by the Banach fixed point theorem [16, Theorem 1.A], Equation (2.2) has a unique solution; that is,

$$R(H + \rho M) = X \quad \text{for all } \rho > \frac{\lambda s}{r + s}.$$

Hence, based on an  $n$ -fold repetitions of the argument, we end up with

$$R(H + \rho M) = X \quad \text{for all } \rho > \frac{\lambda s^n}{(r + s)^n} \text{ and all } n.$$

To prove (ii)  $\Rightarrow$  (iii), we begin with  $M$  as  $H$ -maximal accretive relative to  $H$ . Since  $M$  is monotone relative to  $H$  in light of Proposition 2.12, all we need is to show that  $R(H + \rho M) = X$ . As  $M$  is  $H$ -maximal accretive relative to  $H$ , it implies that  $(H + \rho M)^{-1}$  exists and is  $(\frac{1}{r})$ -Lipschitz continuous. It further follows that  $R(H + \rho M) = X$ . This completes the proof. □

For  $\rho = 1$  in Proposition 2.14, we have the following result.

**Proposition 2.15.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone. Then the following statements are equivalent:*

- (i)  $M$  is monotone relative to  $H$  and  $R(H + M) = X$ .
- (ii)  $M$  is  $H$ -maximal accretive relative to  $H$ .
- (iii)  $M$  is  $H$ -maximal monotone relative to  $H$ .

**Proposition 2.16** ([18, Proposition 31.5]). *Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. Then the following statements are equivalent:*

- (i)  $M$  is monotone and  $R(I + M) = X$ .
- (ii)  $M$  is maximal accretive.
- (iii)  $M$  is maximal monotone.

### 3. GENERALIZED YOSIDA APPROXIMATIONS

Based on Proposition 2.11, we define the generalized Yosida approximation  $M_\rho = \rho^{-1}(H - HoJ_{\rho,H}^M oH)$ , where  $H : X \rightarrow X$  is an  $(r)$ -strongly monotone mapping on  $X$ , represents the generalized Yosida approximation of  $M$  for  $\rho > 0$ , which reduces to the Yosida approximation of  $M$  for  $H = I$ :

$$M_\rho = \rho^{-1}(I - R_\rho^M),$$

where  $I$  is the identity and  $R_\rho^M = (I + \rho M)^{-1}$ .

**Proposition 3.1.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone and  $(s)$ -Lipschitz continuous, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Then the generalized Yosida approximation  $M_\rho$  of  $M$  defined by*

$$M_\rho = \rho^{-1}(H - HoR_{\rho,H}^M oH),$$

where

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X,$$

is  $(\frac{s(r+s)}{\rho r})$ -Lipschitz continuous.

*Proof.* Applying Proposition 2.11, for any  $u, v \in X$ , we have

$$\begin{aligned} \|M_\rho(u) - M_\rho(v)\| &= \rho^{-1} \|(H - HoR_{\rho,H}^M oH)(u) - (H - HoR_{\rho,H}^M oH)(v)\| \\ &= \rho^{-1} [\|H(u) - H(v)\| + \|(HoR_{\rho,H}^M oH)(u) - (HoR_{\rho,H}^M oH)(v)\|] \\ &\leq \rho^{-1} [s\|u - v\| + s\|(R_{\rho,H}^M oH)(u) - (R_{\rho,H}^M oH)(v)\|] \\ &\leq \rho^{-1} [s\|u - v\| + \frac{s}{r}\|H(u) - H(v)\|] \\ &\leq \rho^{-1} [s + \frac{s^2}{r}]\|u - v\|. \end{aligned}$$

□

For  $H = I$ , Proposition 3.1, reduces to the following statement, [18, Lemma 31.7],

**Proposition 3.2.** *Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. Let  $M$  be a maximal monotone mapping. Then the Yosida approximation  $M_\rho$  of  $M$  defined by*

$$M_\rho = \rho^{-1}(I - R_\rho^M),$$

where

$$R_\rho^M(u) = (I + \rho M)^{-1}(u) \quad \forall u \in X,$$

is  $(2/\rho)$ -Lipschitz continuous.

**Proposition 3.3.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone and  $(s)$ -Lipschitz continuous, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Furthermore, if  $HoR_{\rho,H}^M \circ H$  is cocoercive with respect to  $H$ , then the generalized Yosida approximation  $M_\rho$  of  $M$  defined by*

$$M_\rho = \rho^{-1}(H - HoR_{\rho,H}^M \circ H),$$

where

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X,$$

is  $(s/\rho)$ -Lipschitz continuous.

*Proof.* Since, for any  $u, v \in D(H) \cap D(M)$ ,

$$H(u) - H(v) = \rho(M_\rho(u) - M_\rho(v)) + (HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v),$$

we have

$$\begin{aligned} & \langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \\ &= \langle M_\rho(u) - M_\rho(v), \rho(M_\rho(u) - M_\rho(v)) \\ & \quad + (HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v) \rangle \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2 + \langle M_\rho(u) - M_\rho(v), (HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v) \rangle \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2 + \langle H(u) - H(v), (HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v) \rangle \\ & \quad - \|(HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v)\|^2 \\ &\geq \rho \|M_\rho(u) - M_\rho(v)\|^2 + \|(HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v)\|^2 \\ & \quad - \|(HoR_{\rho,H}^M \circ H)(u) - (HoR_{\rho,H}^M \circ H)(v)\|^2 \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2. \end{aligned}$$

Thus,

$$\langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \geq \rho \|M_\rho(u) - M_\rho(v)\|^2.$$

□

**Remark 3.4.** Note that the Lipschitz continuity constant  $\frac{s}{\rho}$  is more application-enhanced than that of  $\rho^{-1}[s + \frac{s^2}{r}]$  in Proposition 3.1.

**Proposition 3.5.** *Let  $M : D(M) \subseteq X \rightarrow X$  be a single-valued mapping. Let  $M$  be a maximal monotone mapping. Then the Yosida approximation  $M_\rho$  of  $M$  defined by*

$$M_\rho = \rho^{-1}(I - R_\rho^M),$$

where  $R_\rho^M(u) = (I + \rho M)^{-1}(u)$  for all  $u \in X$ , is  $(\frac{1}{\rho})$ -Lipschitz continuous.



*Proof.* We include the proof for the sake of the completeness. It is well-known that the resolvent operator  $R_\rho^M$  is cocoercive as well as nonexpansive. Since, for any  $u, v \in D(M)$ ,

$$u - v = \rho(M_\rho(u) - M_\rho(v)) + R_\rho^M(u) - R_\rho^M(v),$$

we have

$$\begin{aligned} & \langle M_\rho(u) - M_\rho(v), u - v \rangle \\ &= \langle M_\rho(u) - M_\rho(v), \rho(M_\rho(u) - M_\rho(v)) + R_\rho^M(u) - R_\rho^M(v) \rangle \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2 + \langle M_\rho(u) - M_\rho(v), R_\rho^M(u) - R_\rho^M(v) \rangle \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2 + \langle u - v, R_\rho^M(u) - R_\rho^M(v) \rangle - \|R_\rho^M(u) - R_\rho^M(v)\|^2 \\ &\geq \rho \|M_\rho(u) - M_\rho(v)\|^2 + \|R_\rho^M(u) - R_\rho^M(v)\|^2 - \|R_\rho^M(u) - R_\rho^M(v)\|^2 \\ &= \rho \|M_\rho(u) - M_\rho(v)\|^2. \end{aligned}$$

Hence, we have

$$\langle M_\rho(u) - M_\rho(v), u - v \rangle \geq \rho \|M_\rho(u) - M_\rho(v)\|^2.$$

□

**Lemma 3.6.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone and  $(s)$ -Lipschitz continuous, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Furthermore, if  $HoR_{\rho,H}^M oH$  is cocoercive with respect to  $H$ , then the generalized Yosida approximation  $M_\rho$  of  $M$  defined by*

$$M_\rho = \rho^{-1}(H - HoR_{\rho,H}^M oH),$$

where  $R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u)$  for all  $u \in X$ , satisfies the following conditions:

(i) For all  $\rho > 0$  and for all  $u, v \in X$ , we have

$$\rho^{-1}(H - HoR_{\rho,H}^M oH) = MR_{\rho,H}^M(H(u)).$$

(ii)  $M_\rho$  is  $(\rho)$ -cocoercive with respect to  $H$ ; that is,

$$\langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \geq \rho \|M_\rho(u) - M_\rho(v)\|^2.$$

*Proof.* The proof of (i) follows from the definition of the resolvent operator, while the proof for (ii) is derived from the proof of Proposition 3.3 as follows:

$$\langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \geq \rho \|M_\rho(u) - M_\rho(v)\|^2.$$

□

Since  $H$  is  $(s)$ -Lipschitz continuous (and hence,  $I - H$  is monotone), we have the following result in light of Proposition 3.3.

**Lemma 3.7.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . Let  $H$  be  $(r)$ -strongly monotone and  $(s)$ -Lipschitz continuous, and let  $M$  be an  $H$ -maximal monotone mapping relative to  $H$ . Suppose that  $HoR_{\rho,H}^M oH$  is cocoercive with respect to  $H$ . Furthermore, if the generalized Yosida approximation  $M_\rho$  is cocoercive with respect to  $I - H$ , then  $M_\rho$  defined by*

$$M_\rho = \rho^{-1}(H - HoR_{\rho,H}^M oH),$$

where  $R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u)$  for all  $u \in X$ , is monotone, that is,

$$\langle M_\rho(u) - M_\rho(v), u - v \rangle \geq 0$$

and

$$\langle M_\rho(u) - M_\rho(v), u - v \rangle \geq \langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle.$$

*Proof.* Since  $M_\rho$  is monotone with respect to  $H$  from Lemma 3.6, and under assumptions it is cocoercive with respect to  $I - H$ , which is strongly monotone from the (s)-Lipschitz continuity of  $H$ , we have

$$\begin{aligned} & \langle M_\rho(u) - M_\rho(v), u - v \rangle - \langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \\ &= \langle M_\rho(u) - M_\rho(v), (I - H)(u) - (I - H)(v) \rangle \\ &\geq \|M_\rho(u) - M_\rho(v)\|^2. \end{aligned}$$

Hence, we have

$$\langle M_\rho(u) - M_\rho(v), u - v \rangle - \langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \geq 0.$$

It follows that

$$\langle M_\rho(u) - M_\rho(v), u - v \rangle \geq \langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle.$$

□

#### 4. GENERALIZED FIRST-ORDER EVOLUTION EQUATIONS

Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ . In this section, we consider the solvability of first-order nonlinear evolution equations of the form

$$\begin{aligned} u'(t) + Mu(t) &= 0, \quad 0 < t < \infty \\ u(0) &= u_0, \end{aligned} \tag{4.1}$$

where  $M$  is  $H$ -maximal monotone relative to  $H$  and the Yosida approximation of  $M$  is defined by

$$M_\rho = \rho^{-1}(H - HoR_{\rho,H}^M oH),$$

where  $o$  denotes composition of functions. We generalize the theorem of Komura [3] to the case of the  $H$ -maximal monotonicity framework in the context of generalized Yosida approximations.

**Theorem 4.1.** *Let  $H : D(H) \subseteq X \rightarrow X$  and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(H) \cap D(M) \neq \emptyset$ , where  $X$  is a real Hilbert space. Let  $H$  be ( $r$ )-strongly monotone and (s)-Lipschitz continuous, and let  $M$  be  $H$ -maximal monotone relative to  $H$ . Suppose that  $HoR_{\rho,H}^M oH$  is cocoercive with respect to  $H$ , and  $M$  and the generalized Yosida approximation  $M_\rho$  are cocoercive with respect to  $I - H$ , where  $I$  is the identity mapping. Then, for each  $u_0 \in D(M)$ , there exists exactly one continuous function  $u : [0, \infty) \rightarrow X$  such that equation (4.1) holds for all  $t \in (0, \infty)$ , where the derivative  $u'(t)$  is in the sense of weak convergence, that is,*

$$\frac{u(t+h) - u(t)}{h} \rightharpoonup u'(t) \quad \text{in } X \text{ as } h \rightarrow 0.$$

*Proof.* We start the proof with the uniqueness of the solution to (4.1). Suppose that  $u : [0, \infty) \rightarrow X$  is a solution to (4.1), where  $u$  is continuous and  $u'(t)$  exists for all  $t \in (0, \infty)$  in the sense of the weak convergence. It is well-known that

$$\frac{d}{dt} \langle u(t), u(t) \rangle = \frac{d}{dt} \|u(t)\|^2 = 2 \langle u'(t), u(t) \rangle \quad \forall t \in (0, \infty).$$

If we assume that  $v$  is another solution to (4.1), then, for  $t \in (0, \infty)$ , the monotonicity of  $M$  relative to  $H$ , and the cocoercivity of  $M$  with respect to  $I - H$  imply that

$$\begin{aligned} \frac{d}{dt} \|u(t) - v(t)\|^2 &= 2 \langle u'(t) - v'(t), u(t) - v(t) \rangle \\ &= - \langle M(u(t)) - M(v(t)), u(t) - v(t) \rangle \\ &\leq - \langle M(u(t)) - M(v(t)), H(u(t)) - H(v(t)) \rangle \leq 0. \end{aligned}$$

It follows that

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| \quad \forall t \geq 0.$$

Since  $u(0) = v(0)$ , we have  $u(t) = v(t)$  for all  $t \geq 0$ .

Next, we prove the existence of a solution to (4.1). We begin with the generalized resolvent operator  $R_{\rho, H}^M = (H + \rho M)^{-1}$  for  $\rho > 0$ , and conclude that  $M$  is  $H$ -maximal accretive relative to  $H$  in light of Proposition 2.14. Therefore,  $R_{\rho, H}^M$  exists for all  $\rho > 0$  and

$$R_{\rho, H}^M : X \rightarrow D(H + \rho M)$$

is bijective (that is,  $H + \rho M$  is one-one and onto), while  $R_{\rho, H}^M$  is  $(\frac{1}{r})$ -Lipschitz continuous. Under the assumptions of the theorem, it is sufficient to show that  $H + \rho M$  is injective, that is, if we assume  $u \neq v$  for  $u, v \in D(H) \cap D(M)$ , and  $(H + \rho M)(u) = (H + \rho M)(v)$ , we have

$$\begin{aligned} &\langle (H + \rho M)(u) - (H + \rho M)(v), H(u) - H(v) \rangle \\ &= \langle H(u) - H(v), H(u) - H(v) \rangle + \rho \langle M(u) - M(v), H(u) - H(v) \rangle \\ &\geq r \|u - v\|^2 + \rho \langle M(u) - M(v), H(u) - H(v) \rangle \\ &\geq r \|u - v\|^2. \end{aligned}$$

This implies  $u = v$ , a contradiction.

Now we look back at Section 3 and examine some of the properties of the generalized Yosida approximation  $M_\rho = \rho^{-1}(H - H \circ R_{\rho, H}^M \circ H)$  as follows:

(i)

$$\rho^{-1}(H - H \circ R_{\rho, H}^M \circ H) = M R_{\rho, H}^M(H(u)) \quad \forall u \in X.$$

(ii)  $M_\rho$  is  $(\frac{s(r+s)}{\rho r})$ -Lipschitz continuous for all  $u \in X$ .

(iii)  $M_\rho$  is monotone with respect to  $H$ , that is,

$$\langle M_\rho(u) - M_\rho(v), H(u) - H(v) \rangle \geq 0 \quad \forall u \in X.$$

(iv)  $\|M_\rho(u)\| \leq \frac{s}{r} \|M(u)\|$  for all  $u \in D(M)$ .

Most of these properties follow from the definition of  $M_\rho = \rho^{-1}(H - H \circ R_{\rho, H}^M \circ H)$ , but consider (iv) as follows: Using the definition of  $M_\rho$  and the  $(\frac{1}{r})$ -Lipschitz continuity of  $R_{\rho, H}^M$ , we have

$$\begin{aligned} \|M_\rho(u)\| &= \rho^{-1} \|(H - H \circ R_{\rho, H}^M \circ H)(u)\| \\ &= \rho^{-1} \|H(R_{\rho, H}^M(H + \rho M))(u) - H(R_{\rho, H}^M H)(u)\| \end{aligned}$$

$$\begin{aligned}
&\leq \rho^{-1}s\|(R_{\rho,H}^M(H + \rho M)(u) - (R_{\rho,H}^M H)(u))\| \\
&\leq \frac{s}{\rho r}\|\rho M(u)\| \\
&= \frac{s}{r}\|M(u)\|.
\end{aligned}$$

At the crucial stage of the proof, we extend the map  $M : D(M) \subseteq X \rightarrow X$  to the Hilbert space  $Z$  by defining a map  $M^* : D(M^*) \subseteq Z \rightarrow Z$ , where  $Z = L_2(0, T; X)$  for fixed  $T > 0$ . We set

$$(M^*u)(t) = Mu(t) \quad \text{for almost all } t \in [0, T], \quad (4.2)$$

and define the domain  $D(M^*)$  as the set of all  $u \in Z$  such that  $u(t) \in D(M)$  holds for almost all  $t \in [0, T]$  and  $t \mapsto Mu(t)$  belongs to  $Z$ . We observe that the  $H$ -maximal accretivity of  $M : D(M) \subseteq X \rightarrow X$  relative to  $H$  implies that  $M^* : D(M^*) \subseteq Z \rightarrow Z$  is  $H$ -maximal accretive and  $H$ -maximal monotone relative to  $H$  in light Proposition 2.14. Let  $Z = L_2(0, T; X)$  for fixed  $T > 0$ . Then  $M^*$  is monotone relative to  $H$ . Furthermore, for all  $u, v \in D(M^*)$ , the monotonicity of  $M$  relative to  $H$  and the cocoercivity of  $M$  (from the hypotheses of the theorem) imply

$$\begin{aligned}
\langle M^*(u) - M^*(v), u - v \rangle_Z &= \int_0^T \langle Mu(t) - Mv(t), u(t) - v(t) \rangle dt \\
&\geq \int_0^T \langle Mu(t) - Mv(t), Hu(t) - Hv(t) \rangle dt \geq 0.
\end{aligned}$$

To this context, we need show that  $R(H + M^*) = Z$ . Suppose that  $w \in Z$ , and set

$$u(t) = (H + M)^{-1}w(t), \quad u_0 = (H + M)^{-1}(0).$$

We know that  $(H + M)^{-1}$  is  $(\frac{1}{r})$ -Lipschitz continuous, and it implies

$$\|u(t) - u(0)\|^2 = \|(H + M)^{-1}w(t) - (H + M)^{-1}(0)\|^2 \leq \frac{1}{r^2}\|w(t)\|^2.$$

If we integrate over  $[0, T]$ , we find that  $u - u_0 \in Z$ , and hence,  $u \in Z$ . Thus,  $M^*$  is  $H$ -maximal accretive and  $H$ -maximal monotone relative to  $H$  by Proposition 2.14.

To our next leg of the proof, we consider the solvability of the auxiliary problem

$$\begin{aligned}
u'_\rho(t) + M_\rho u_\rho(t) &= 0, \quad 0 < t < \infty \\
u_\rho(0) &= u_0 \in D(M).
\end{aligned} \quad (4.3)$$

$M_\rho$  is Lipschitz continuous with Lipschitz constant  $\frac{s}{\rho}$  from Proposition 3.3 on  $X$ , while the global Picard-Lindelöf theorem [10, Corollary 3.8] implies that (4.3) has exactly one  $C^1$ -solution  $u : R \rightarrow X$ .

To show the uniqueness of the solution to (4.3), like in the beginning of the proof, assume  $u_\rho$  and  $v_\rho$  be two solutions to (4.3). Since, based on Lemma 3.7,  $M_\rho$  is monotone, we have

$$\|u_\rho(t) - v_\rho(t)\| \leq \|u_\rho(0) - v_\rho(0)\| \quad \forall t \geq 0. \quad (4.4)$$

In order for us to prove the convergence of  $u_\rho$  in  $X$ , we need the following inequalities. For all  $t, s \geq 0$  and all  $\rho, \lambda > 0$ , we have

$$\|u'_\rho(t)\| = \|M_\rho u_\rho(t)\| \leq \frac{s}{r}\|M(u_0)\|, \quad (4.5)$$

$$\|u_\rho(t) - u_\rho(s)\| \leq \frac{s}{r} \|M(u_0)\| |t - s|, \quad (4.6)$$

$$\|Hu_\rho(t) - HR_{\rho,H}^M Hu_\rho(t)\| \leq \frac{\rho s}{r} \|M(u_0)\|, \quad (4.7)$$

$$\|u_\rho(t) - u_\lambda(t)\| \leq \frac{2s}{r} \sqrt{\rho + \lambda} (t \|M(u_0)\|). \quad (4.8)$$

Note that since  $H$  is  $(s)$ -Lipschitz continuous, it follows from (4.8) that

$$\|Hu_\rho(t) - Hu_\lambda(t)\| \leq \frac{2s^2}{r} \sqrt{\rho + \lambda} (t \|M(u_0)\|). \quad (4.9)$$

Let us start the proof of (4.5) with  $t \mapsto u_\rho(t)$ . The function  $t \mapsto u_\rho(t+h)$  is also a solution to (4.3) with suitable initial values. Applying (4.4), we have

$$\|u_\rho(t) - u_\rho(t+h)\| \leq \|u_\rho(0) - u_\rho(h)\|.$$

Dividing by  $h$  and letting  $h \rightarrow +0$ , it turns out using (iv) in the proof that

$$\|u'_\rho(t)\| \leq \|u'_\rho(0)\| = \|M_\rho(u_0)\| \leq \frac{s}{r} \|M(u_0)\|.$$

The proofs of (4.6) and (4.7) follow easily from (4.5) and the definition of  $M_\rho$ , we move to prove (4.8) by setting

$$\Delta = -\langle M_\rho u_\rho(t) - M_\lambda u_\lambda(t), HR_{\lambda,H}^M Hu_\lambda(t) - Hu_\lambda(t) + Hu_\rho(t) - HR_{\rho,H}^M Hu_\rho(t) \rangle.$$

It follows from applying (4.5) and (4.7) that

$$|\Delta| \leq \frac{2s^2}{r^2} (\rho + \lambda) \|Mu_0\|^2.$$

If we apply (4.3),  $M_\rho = MR_{\rho,H}^M(H(u))$ , the monotonicity of  $M$  with respect to  $H$ , and Lemma 3.7, then we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\rho(t) - u_\lambda(t)\|^2 &= \langle u'_\rho(t) - u'_\lambda(t), u_\rho(t) - u_\lambda(t) \rangle \\ &= -\langle M_\rho u_\rho(t) - M_\lambda u_\lambda(t), u_\rho(t) - u_\lambda(t) \rangle \\ &\leq -\langle M_\rho u_\rho(t) - M_\lambda u_\lambda(t), Hu_\rho(t) - Hu_\lambda(t) \rangle \\ &= -\langle MR_{\rho,H}^M Hu_\rho(t) - MR_{\lambda,H}^M Hu_\lambda(t), HR_{\rho,H}^M Hu_\rho(t) - HR_{\lambda,H}^M Hu_\lambda(t) \rangle + \Delta \\ &\leq \Delta. \end{aligned}$$

Since  $u_\rho(0) - u_\lambda(0) = 0$ , integrating over  $[0, t]$  completes the proof of (4.8).

In next steps, we consider the convergence of  $u_\rho(t)$  in  $X$  as  $\rho \rightarrow +0$ . It follows from (4.8) that  $u_\rho(t)$  converges to a certain  $u(t)$  in  $X$  as  $\rho \rightarrow +0$ . As a matter of fact, it converges uniformly with respect to all compact  $t$ -intervals. Inequality (4.6) yields

$$\|u(t) - u(s)\| \leq \frac{s}{r} \|M(u_0)\| |t - s| \quad \forall t, s \geq 0. \quad (4.10)$$

On the other hand, we examine the convergence in  $Z = L_2(0, T; X)$  as  $\rho \rightarrow +0$ . Indeed, the uniform convergence follows from the preceding step in the following manner

$$u_\rho \rightarrow u \in Z \quad \text{as } \rho \rightarrow +0.$$

Applying (4.10), the function  $t \mapsto u(t)$  is Lipschitz continuous on  $R_+$ . In light of [17, Corollary 23.22], the derivative  $u'(t)$  exists for almost all  $t \in R_+$ , while it follows from (4.10) that

$$\|u'(t)\| \leq \frac{s}{r} \|M(u_0)\| \quad \text{for almost all } t \in R_+.$$

This implies that  $u' \in Z$ . Moreover,  $u'$  is the generalized derivative of  $u$  on each interval  $(0, T)$ . It follows from (4.5) that there exists a constant  $c$  such that

$$\|u'_\rho\|_Z \leq c \quad \forall \rho > 0.$$

Since  $Z$  is a Hilbert space,  $Z$  is reflexive. Therefore, by choosing a suitable subsequence, we obtain

$$u_\rho \rightarrow u \in Z \quad \text{and} \quad u'_\rho \rightarrow w \in Z \quad \text{as } \rho \rightarrow +0.$$

Then, by [17, Proposition 23.19], it follows that  $u' = w$ . Since

$$u'_\rho(t) = -M_\rho u_\rho(t) = -(MR_{\rho, H}^M)u_\rho(t),$$

applying the  $(r)$ -expansiveness of  $H$ ; that is,

$$\|Hu - Hv\| \geq r\|u - v\|,$$

due to (4.7), we have  $R_{\rho, H}^M H u_\rho \rightarrow u \in Z$  as  $\rho \rightarrow +0$ , and

$$-M^* R_{\rho, H}^M H u \rightarrow w \in Z \quad \text{as } \rho \rightarrow +0.$$

The map  $M^*$  is  $H$ -maximal monotone. Therefore,  $u \in D(M^*)$  and

$$-M^* u = w \quad \text{or} \quad -M^* u = u'.$$

It follows that  $u'(t) = -Mu(t)$  for almost all  $t \in R_+$ .

Finally, it turns out that the function  $t \mapsto Mu(t)$  is continuous from the right on  $R_+$ , and as a result, it follows that the function (for each  $w \in X$ )  $t \mapsto \langle Mu(t), w \rangle$  is continuous on  $[0, \infty)$ .  $\square$

## 5. CONCLUDING REMARKS

**Remark 5.1.** If we generalize Definition 2.2 to the case of another single-valued mapping  $A : D(A) \subseteq X \rightarrow X$ , then we could achieve a mild generalization to Theorem 4.1.

**Definition 5.2.** Let  $B : D(B) \subseteq X \rightarrow X$ ,  $H : D(H) \subseteq X \rightarrow X$ , and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(B) \cap D(H) \cap D(M) \neq \emptyset$ . The map  $M : D(M) \subseteq X \rightarrow X$  is said to be  $H$ -maximal monotone relative to  $B$  if

(i)  $M$  is monotone with respect to  $B$ , that is,

$$\langle M(u) - M(v), B(u) - B(v) \rangle \geq 0,$$

(ii)  $R(H + \rho M) = X$  for  $\rho > 0$ .

This clearly reduces to Definition 2.2 when  $B = H$ .

We do have further generalization to Definition 5.2 to the case of the  $A$ -maximal  $(m)$ -relaxed monotonicity as follows:

**Definition 5.3.** Let  $A : D(A) \subseteq X \rightarrow X$ ,  $B : D(B) \subseteq X \rightarrow X$ , and  $M : D(M) \subseteq X \rightarrow X$  be single-valued mappings such that  $D(A) \cap D(B) \cap D(M) \neq \emptyset$ . The map  $M : X \rightarrow 2^X$  is said to be  $A$ -maximal  $(m)$ -relaxed monotone relative to  $B$  if

(i)  $M$  is  $(m)$ -relaxed monotone relative to  $B$  for  $m > 0$ .

(ii)  $R(A + \rho M) = X$  for  $\rho > 0$ .

**Remark 5.4.** We consider a class of first order evolution inclusions of the form

$$\begin{aligned} u'(t) + M(u(t)) \ni 0 & \quad \text{for } 0 < t < \infty, \\ u(0) & = u_0 \end{aligned} \tag{5.1}$$

where  $M : X \rightarrow 2^X$  is  $A$ -maximal  $(m)$ -relaxed monotone [9],  $u : [0, \infty) \rightarrow X$  is such that (5.1) holds, and the derivative  $u'(t)$  exists in the sense of the weak convergence. Furthermore, the  $A$ -maximal  $(m)$ -relaxed monotone mapping  $M : X \rightarrow 2^X$  is defined as follows.

**Definition 5.5.** Let  $A : X \rightarrow X$  be a single-valued mapping, and let  $M : X \rightarrow 2^X$  be a set-valued mappings on  $X$ . The map  $M : X \rightarrow 2^X$  is said to be  $A$ -maximal  $(m)$ -relaxed monotone if

- (i)  $M$  is  $(m)$ -relaxed monotone for  $m > 0$ .
- (ii)  $R(A + \rho M) = X$  for  $\rho > 0$ .

Based on Theorem 4.1, we can define  $M_\rho = \rho^{-1}(A - A \circ R_{\rho, A}^M \circ A)$ , where  $A : X \rightarrow X$  is an  $(r)$ -strongly monotone mapping on  $X$ , represents the generalized Yosida regularization of  $M$  for  $\rho > 0$ , that reduces to the Yosida regularization of  $M$  for  $A = I$ . Theory of  $A$ -maximal  $(m)$ -relaxed monotone mappings generalizes most of the existing notions on maximal monotone mappings to Hilbert as well as Banach space settings, and its applications range from nonlinear variational inequalities, equilibrium problems, optimization and control theory, management and decision sciences, and mathematical programming to engineering sciences.

In a subsequent communication on the solvability of the differential inclusions of the form (5.1), based on the generalized Yosida regularization/approximation, is planned, but the real problem could arise due to the presence of the relaxed monotonicity achieving the uniqueness of the solution.

#### REFERENCES

- [1] N. U. Ahmed; *Optimization and Identification of Systems Governed by Evolution Equations on Banach Spaces*, Pitman Research Notes in Mathematics, Volume 184, Longman Scientific and Technical, 1988.
- [2] Y. P. Fang, and N. J. Huang;  $H$ -monotone operators and system of variational inclusions, *Communications on Applied Nonlinear Analysis* 11 (1) (2004), 93–101.
- [3] Y. Komura; Nonlinear semigroups in Hilbert space, *Journal of Mathematical Society of Japan* 19 (1967), 493–507.
- [4] K. Nakajo, and W. Takahashi; Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *Journal of Mathematical Analysis and Applications* 279 (2003), 372–379.
- [5] S. M. Robinson; Composition duality and maximal monotonicity, *Mathematical Programming* 85 (1999), 1–13.
- [6] R. T. Rockafellar; Augmented Lagrangians and applications of the proximal point algorithm in convex programming, *Mathematics of Operations Research* 1 (1976), 97–116.
- [7] R. T. Rockafellar, and R. J-B. Wets; *Variational Analysis*, Springer, New York, New York, 2004.
- [8] R. U. Verma;  $A$ -monotonicity and applications to nonlinear inclusion problems, *Journal of Applied Mathematics and Stochastic Analysis* 17 (2004), 193–195.
- [9] R. U. Verma;  $A$ -monotonicity and its role in nonlinear variational inclusions, *Journal of Optimization Theory and Applications* 129(3)(2006), 457–467.
- [10] R. U. Verma; Generalized over-relaxed proximal point algorithm based on  $A$ -maximal monotonicity framework and applications inclusion problems, *Mathematical and Computer Modelling* 49 (2009), 1587–1594.

- [11] R. U. Verma; General system of  $A$ -monotone nonlinear variational inclusions with applications, *Journal of Optimization Theory and Applications* 131 (1) (2006), 151–157.
- [12] R. U. Verma; Approximation solvability of a class of nonlinear set-valued inclusions involving  $(A, \eta)$ -monotone mappings, *Journal of Mathematical Analysis and Applications* 337 (2008), 969–975.
- [13] R. U. Verma; The over-relaxed  $A$ -proximal point algorithm and applications to nonlinear variational inclusions in Banach spaces, *Fixed Point Theory* 10(1) (2009), 185–195.
- [14] K. Yosida; On differentiability and the representation of one-parameter semigroups of linear operators, *Journal of Mathematical Society of Japan* 1 (1948), 15–21.
- [15] K. Yosida; *Functional Analysis*, Springer-Verlag, Berlin, 1965.
- [16] E. Zeidler; *Nonlinear Functional Analysis and its Applications* I, Springer-Verlag, New York, New York, 1984.
- [17] E. Zeidler; *Nonlinear Functional Analysis and its Applications* II/A, Springer-Verlag, New York, New York, 1990.
- [18] E. Zeidler; *Nonlinear Functional Analysis and its Applications* II/B, Springer-Verlag, New York, New York, 1990.
- [19] E. Zeidler; *Nonlinear Functional Analysis and its Applications* III, Springer-Verlag, New York, New York, 1985.
- [20] E. Zeidler; *Nonlinear Functional Analysis and its Applications* IV, Springer-Verlag, New York, New York, 1986.

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