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SOME REMARKS ON CONTROLLABILITY OF EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. In almost all papers in the literature, the results on exact controllability hold only for finite dimensional Banach spaces, since compactness of the semigroup and the bounded invertibility of an operator implies finite dimensional. In this note we show that the existence theory on controllability in the literature, can trivially be adjusted to include the infinite dimensional space setting, if we replace the compactness of operators with the complete continuity of the nonlinearity.

1. Introduction & Preliminaries

In a long list of papers in the literature on exact controllability of abstract control systems (e.g. [1, 3, 4]) the compactness of the linear operators $T(t)$, $t > 0$, with other hypothesis guarantee that the Banach space X is finite dimensional. This has been pointed out by a number of authors (see [6, 12]). However it is easy to consider the case when X is infinite dimensional. Simply replace the compactness of $T(t)$, $t > 0$ with the complete continuity of the nonlinearity. As a result the case when X is infinite dimensional is trivially extended (the proof is almost exactly the same as in the literature). For simplicity we will consider the case in [1] (the other papers [3]-[5], [7]-[8], [10], use exactly the same ideas). Consider the first order semilinear controllability problem of the form

$$y'(t) = Ay(t) + f(t, y(t)) + (\mathcal{B}u)(t), \quad t \in J := [0, b], \tag{1.1}$$

$$y(0) = y_0. \tag{1.2}$$

Here $J = [0, b]$, $b > 0$, $f : J \times X \to X$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$, $y_0 \in X$, and X a real Banach space with norm $|\cdot|$. Also the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. Finally $\mathcal{B}$ is a bounded linear operator from U to X.

Definition 1.1. A function $y \in C(J, X)$ is said to be a mild solution of (1.1)–(1.2) if $y(0) = y_0$ and

$$y(t) = T(t)y_0 + \int_0^t T(t-s)[(\mathcal{B}u)(s) + f(s, y(s))]ds.$$

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Definition 1.2. The system (1.1)–(1.2) is said to be controllable on the interval J, if for every $y_0, y_1 \in X$ there exists a control $u \in L^2(J, U)$, such that there exists a mild solution $y(t)$ of (1.1)–(1.2) satisfying $y(b) = x_1$.

In almost all papers in the literature, including our papers [1], [3]-[5], [7]-[8], [10], the study of controllability is based on the compactness of the operator T and the bounded invertibility on the operator W, i.e.:

(HT) A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ on X, which is compact for $t > 0$.

(HW) $\mathcal{B}$ is a continuous operator from U to X and the linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b T(b-s)\mathcal{B}u(s)\,ds,$$

has a bounded invertible operator $W^{-1} : X \to L^2(J, U)$ such that $\|\mathcal{B}\| \leq M_1$ and $\|W^{-1}\| \leq M_2$, for some positive constants M_1, M_2.

We remark (see [6, 12]) that the above two hypotheses are valid if X is finite dimensional. In this note we show that the existence theory in the literature can trivially be adjusted to include the infinite dimensional space setting if we replace the compactness of the linear operators with the complete continuity of the nonlinearity.

2. First Order Abstract Semilinear Differential Equations

Using hypothesis (HW) for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\Big[y_1 - T(b)y_0 - \int_0^b T(b-s)f(s, y(s))\,ds\Big](t).$$

To prove the controllability of the problem (1.1)–(1.2), we must show that when using this control, the operator $K_1 : C([0,b], X) \to C([0,b], X)$ defined by

$$K_1 y(t) = T(t)y_0 + \int_0^t T(t-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]ds, \quad t \in [0, b]$$

has a fixed point.

In the following we use the notation:

$$M = \sup\{\|T(t)\| : t \in [0, b]\},$$
$$B_r = \{y \in C([0, b], X) : \|y\| \leq r\}.$$

Also it is clear that for $y \in B_r$ we have

$$\|(\mathcal{B}u_y)(s)\| \leq M_1 M_2 \Big[|y_1 + M|y_0| + \int_0^b h_\rho(s)\,ds\Big] := G_0.$$

Remark 2.1. In what follows, without loss of generality, we take $y_0 = 0$, since if $y_0 \neq 0$ and $y(t) = x(t) + T(t)y_0$ it is easy to see that x satisfies

$$x'(t) = \int_0^t T(t-s)f(s, x(s) + T(s)y_0)\,ds + (\mathcal{B}u)(t), \quad t \in J$$
$$x(0) = 0.$$

We concentrate only in proving in detail the complete continuity of the operator $K_1 y(t)$, since all the other steps in [1] remain unchanged. Thus we have the following result.

Lemma 2.1. *Suppose* (HW) *holds. In addition assume the following conditions are satisfied:*

(HT) $T(\cdot)$ *is strongly continuous.*

(Hf1) $f : [0, b] \times X \to X$ *is an* L^1*-Carátheodory function, i.e.*
- (i) $t \mapsto f(t, u)$ *is measurable for each* $u \in X$*;*
- (ii) $u \mapsto f(t, u)$ *is continuous on* X *for almost all* $t \in J$*;*
- (iii) *For each* $\rho > 0$*, there exists* $h_\rho \in L^1(J, \mathbb{R}_+)$ *such that for a.e.* $t \in J$
$$\sup_{|u|\le\rho} \|f(t, u)\| \le h_\rho(t).$$

(Hf2) $f : [0, b] \times X \to X$ *is completely continuous.*

Then the operator
$$Ky(t) = \int_0^t T(t-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]ds, \quad t \in [0, b]$$
is completely continuous.

Proof. (I) The set $\{Ky(t) : y \in B_r\}$ is precompact in X, for every $t \in [0, b]$. It follows from the strong continuity of $T(\cdot)$ and conditions (Hf1), (Hf2) that the set $\{T(t-s)f(s, y) : t, s \in [0, b], y \in B_r\}$ is relatively compact in X. Moreover, for $y \in B_r$, from the mean value theorem for the Bochner integral, we obtain
$$Ky(t) \in t\,\overline{\text{conv}}\{T(t-s)f(s, y) : s \in [0, t], y \in B_r\},$$
for all $t \in [0, b]$. As a result we conclude that the set $\{Ky(t) : y \in B_r\}$ is precompact in X, for every $t \in [0, b]$.

(II) The set $\{Ky(t) : y \in B_r\}$ is equicontinuous on $[0, b]$. We just do the case $0 < t \le b$. A similar argument will work if $t = 0$. Let $\epsilon > 0$. From (I) $(KB_r)(t)$ is relatively compact for each t and by the strong continuity of $(T(t))_{t\ge0}$ we can choose $0 < \delta \le b - t$ with
$$\|T(h)y - y\| < \epsilon \quad \text{for } y \in (KB_r)(t) \text{ when } 0 < h < \delta. \tag{2.1}$$
For $y \in B_r$ we have
$$\begin{aligned} Ky(t+h) - Ky(t) &= \int_t^{t+h} T(t+h-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]\,ds \\ &\quad + \int_0^t [T(t+h-s) - T(t-s)][f(s, y(s)) + (\mathcal{B}u_y)(s)]\,ds \\ &= \int_t^{t+h} T(t+h-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]\,ds \\ &\quad + (T(h) - I)\int_0^t T(t-s)f(s, y(s))\,ds \\ &= \int_t^{t+h} T(t+h-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]\,ds \\ &\quad + (T(h) - I)Ky(t), \end{aligned}$$

so

$$\begin{aligned}\|Ky(t+h)-Ky(t)\| &\le M\int_t^{t+h}[\|f(s,y(s))\|+M_1G_0]ds+\|[T(h)-I]Ky(t)\| \\ &\le M\int_t^{t+h}[h_r(s)+M_1G_0]ds+\|[T(h)-I]Ky(t)\|.\end{aligned}$$

The equicontinuity follows from (2.1). □

Remark 2.2. We can apply the same idea to establish existence results for the initial value problem (1.1)–(1.2), when $\mathcal{B}=0$. As a result we obtain alternative results (i.e. we replace the compactness of the operators with the complete continuity of the nonlinearity) to those in [2]-[3], [7], [9]-[11]. We have:

Lemma 2.2. *Assume that the conditions* (HT), (Hf1), (Hf2) *hold. Then the operator*

$$K_2y(t)=\int_0^t T(t-s)f(s,y(s))ds, \quad t\in[0,b]$$

is completely continuous.

3. Second Order Semilinear Differential Equations

Consider the semilinear second order differential control system

$$y''(t)=Ay(t)+f(t,y(t))+(\mathcal{B}u)(t), \quad t\in J:=[0,b], \tag{3.1}$$

$$y(0)=x_0, \quad y'(0)=\eta \tag{3.2}$$

where $x_0,y_0\in X$, A is the infinitesimal generator of the strongly continuous cosine family $C(t),t\in\mathbb{R}$, of bounded linear operators in X, and $f,\mathcal{B}$ are as in problem (1.1)–(1.2).

Recall that a family $\{C(t)\mid t\in\mathbb{R}\}$ of operators in $B(X)$ is a *strongly continuous cosine family* if

(i) $C(0)=I$,
(ii) $C(t+s)+C(t-s)=2C(t)C(s)$, for all $s,t\in\mathbb{R}$,
(iii) the map $t\mapsto C(t)(x)$ is strongly continuous, for each $x\in X$.

The strongly continuous sine family $\{S(t):t\in\mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t):t\in\mathbb{R}\}$, is defined by

$$S(t)(y)=\int_0^t C(s)(y)\,ds, \quad x\in X,\ t\in\mathbb{R}. \tag{3.3}$$

The infinitesimal generator $A:X\to X$ of a cosine family $\{C(t):t\in\mathbb{R}\}$ is defined by

$$A(y)=\frac{d^2}{dt^2}C(t)(y)\Big|_{t=0}.$$

Definition 3.1. A continuous solution of the integral equation

$$y(t)=C(t)x_0+S(t)\eta+\int_0^t S(t-s)[(\mathcal{B}u)(s)+f(s,y(s))]ds, \quad t\in J$$

is said to be a mild solution of the problem (3.1)-(3.2) on J.

Definition 3.2. The system (3.1)-(3.2) is said to be controllable on the interval J, if for every $x_0,\eta,x_1\in X$ there exists a control $u\in L^2(J,U)$, such that the mild solution $y(t)$ of (3.1)-(3.2) satisfies $y(b)=x_1$.

Assume that

(W) $\mathcal{B}$ is a continuous operator from U to X and the linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b S(b-s)\mathcal{B}u(s)\,ds,$$

has a bounded invertible operator $W^{-1} : X \to L^2(J,U)$ such that $\|\mathcal{B}\| \leq M_1$ and $\|W^{-1}\| \leq M_2$, for some positive constants M_1, M_2.

Using hypothesis (W) for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\Big[y_1 - f(y) - C(b)y_0 - S(b)\eta - \int_0^b S(b-s)f(s,y(s))ds\Big](t).$$

Then we must show that when using this control, the operator $N_1 : C(J,X) \to C(J,X)$ defined by:

$$N_1y(t) = C(t)y_0 + S(t)\eta + \int_0^t S(t-s)[(\mathcal{B}u_y)(s) + f(s,y(s))]ds$$

has a fixed point.

Remark 3.1. We can take $y_0 = 0, \eta = 0$. See Remark 2.1.

The following lemma is proved as in Lemma 2.1.

Lemma 3.1. *Assume* (Hf1), (Hf2), (W) *hold. Then the operator*

$$Ny(t) = \int_0^t S(t-s)[(\mathcal{B}u_y)(s) + f(s,y(s))]ds, \quad t \in J$$

is completely continuous.

Remark 3.2. Similar results to those of Lemmas 2.1 and 3.1 hold for differential inclusions.

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