Electronic Journal of Differential Equations, Vol. 2009(2009), No. 76, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

HOPF BIFURCATION FOR SIMPLE FOOD CHAIN MODEL WITH DELAY

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ABSTRACT. In this article we consider a chemostat-like model for a simple food chain where there is a well stirred nutrient substance that serves as food for a prey population of microorganisms, which in turn, is the food for a predator population of microorganisms. The nutrient-uptake of each microorganism is of Holling type I (or Lotka-Volterra) form. We show the existence of a global attractor for solutions of this system. Also we show that the positive globally asymptotically stable equilibrium point of the system undergoes a Hopf bifurcation when the dynamics of the microorganisms at the bottom of the chain depends on the history of the prey population by means of a distributed delay that takes an average of the microorganism in the middle of the chain.

1. INTRODUCTION

We consider the food chain model

$$
\begin{aligned}
S'(t) &= (S^0 - S(t))D - \frac{b}{\gamma} S(t)X(t), \\
X'(t) &= X(t)(bS(t) - D - \frac{d}{\eta} Y(t)), \\
Y'(t) &= Y(t)(dX(t) - D),
\end{aligned}
\tag{1.1}
$$

where $S(0) = S_0 \geq 0$, $X(0) = X_0 \geq 0$, $Y(0) = Y_0 \geq 0$. These equations are in the form of the chemostat model [7]. The meaning of the variables and parameters is as follows: $S(t)$ denotes the substrate concentration, $X(t)$ is the concentration of a prey microorganism population that grows eating the substrate, and $Y(t)$ the concentration of a predator microorganism population that eats the prey microorganism. The functional responses of the species $X(t)$ and $Y(t)$ are of the so called Holling type I (or Lotka-Volterra) form (see [5]), where the parameters b and d denote the per capita growth rate of the prey and the predator respectively; γ and η represent the growth yield constants of the microorganisms $X(t)$ and $Y(t)$ respectively. The parameter S^0 is the concentration in the feed bottle and D denotes the input rate from the feed bottle and the washout rate from the growth chamber. This model may be considered as a specialization of some of the systems given in [1], we shall perform a wide description of the global dynamic of this model in order

2000 *Mathematics Subject Classification.* 34D99.
Key words and phrases. Simple food chain model; Hopf bifurcation; Holling type I; attractor.

Submitted May 11, 2009. Published June 16, 2009.

to point out the differences between the dynamic of this system and the one corresponding to a system with a distributed delay that we propose in order to model the case when the existence of a significative time lag in the growth of the predator microorganism is considered. The recognition of time delays in the growth response of a population to changes in the environment has led to extensive theoretical and experimental studies, however there has been little emphasis in distributed delays in chemostat models, [8] is very important reference in this direction. Thus, we are assuming in a more realistic fashion that the growth of the predator is influenced by the amount of prey in the past. More precisely, we suppose as for example in [3] or [8], that the predator grows up depending on the weight average over the past by mean of the the function $Z(t)$ given by the following integral

$$Z(t) := \int_{-\infty}^{t} dX(\tau)Y(\tau)e^{-D(t-\tau)}(\alpha e^{-\alpha(t-\tau)})d\tau, \quad \alpha > 0, \tag{1.2}$$

therefore, we have the integro-differential system

$$\begin{aligned} S'(t) &= (S^0 - S(t))D - \frac{b}{\gamma}S(t)X(t), \\ X'(t) &= X(t)(bS(t) - D - \frac{d}{\eta}Y(t)), \\ Y'(t) &= \int_{-\infty}^{t} dX(\tau)Y(\tau)e^{-D(t-\tau)}(\alpha e^{-\alpha(t-\tau)})d\tau - DY(t), \end{aligned} \tag{1.3}$$

$S(0) = S_0 \geq 0$, $X(0) = X_0 \geq 0$, $Y(0) = Y_0(t) = \varphi(t) \geq 0$ $(t \leq 0)$. Clearly these assumptions imply that the influence of the past is fading away exponentially and the number $\frac{1}{\alpha}$ could be interpreted as the measure of the influence of the past. So, to smaller $\alpha > 0$, the interval in the past in which the values of X are taken, is bigger ([3, 4, 6, 8]). Also we assume that initial function φ is in BC_+, the Banach space of the bounded and continuous functions from $(-\infty, 0]$ to $\mathbb{R}_+$.

To make the model in more treatable way we perform the change of variables:

$$\begin{gathered} \bar{t} = tD, \quad \overline{S} = \frac{S}{S^0}, \quad \overline{X} = \frac{X}{\gamma S^0}, \quad \overline{Y} = \frac{Y}{\eta\gamma S^0}, \\ \overline{b} = \frac{bS^0}{D}, \quad \overline{d} = \frac{\gamma d S^0}{D}, \quad \overline{\alpha} = \frac{\alpha}{D}. \end{gathered}$$

Omitting the bars, the nondimensional version of models (1.1) and (1.3) can be rewritten, respectively, as

$$\begin{aligned} S'(t) &= 1 - S(t) - bS(t)X(t), \\ X'(t) &= X(t)(bS(t) - 1 - dY(t)), \\ Y'(t) &= Y(t)(dX(t) - 1), \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} S'(t) &= 1 - S(t) - bS(t)X(t), \\ X'(t) &= X(t)(bS(t) - 1 - dY(t)), \\ Y'(t) &= \int_{-\infty}^{t} dX(\tau)Y(\tau)\alpha e^{-(\alpha+1)(t-\tau)}d\tau - Y(t). \end{aligned} \tag{1.5}$$

In this article we show the existence of the global attractor for the solutions of the foregoing systems. We also show that the positive globally asymptotically stable equilibrium point of (1.4) loses its stability when we model the food chain by (1.5). In this case the equilibrium of positive coordinates undergoes a Hopf bifurcation and more realistic periodic solutions gain the stability.

2. A simple food chain without delay

Here we show some properties of system (1.4).

Lemma 2.1. *The positive cone, $\mathbb{R}^3_+$, is positively invariant with respect to* (1.4).

Proof. As we can see, if $S(t^*) = 0$ for some $t^* \geq 0$ then $S(t) \geq 0$ for all $t \geq t^*$. The positiveness of the functions $X(t)$ and $Y(t)$ are straightforward checked once the corresponding equations are considered. □

Note that by adding the three equations of (1.4) and defining $W(t) = 1 - S(t) - X(t) - Y(t)$, we obtain a single equation

$$W'(t) = -W(t)$$

with $W(0) > 0$. It is easy to verify that $\lim_{t\to\infty} W(t) = 0$ and that the convergence is exponential. This implies that (1.4) has the property of pointwise dissipativity in the sense that there exists a bounded set B to which the solutions eventually enter and remain. Thus we have shown the following

Lemma 2.2 (Dissipativity)**.** *System* (1.4) *is pointwise dissipative. Moreover, the attractors of the solutions are located on the manifold*

$$\Sigma = \{(S, X, Y) \in \mathbb{R}^3_+ : S + X + Y = 1\}. \tag{2.1}$$

The pointwise dissipative property implies the existence of a unique global attractor of (1.4) which must lie in the manifold Σ.

Lemma 2.3. *If $d \leq 1$, the predator population $Y(t)$ dies out.*

Proof. By taking into account the equation for $Y(t)$ in (1.4) and applying comparative arguments the result follows. □

By virtue of the previous lemma we shall assume for the rest of this article that

$$d > 1. \tag{2.2}$$

Another important conclusion of the Dissipativity lemma is that the system can be simplified by eliminating one variable. In fact by taking

$$S(t) = 1 - X(t) - Y(t),$$

we obtain the following system of two ordinary differential equations

$$\begin{gathered} X'(t) = (b-1)X(t) - bX^2(t) - (b+d)X(t)Y(t), \\ Y'(t) = Y(t)(dX(t) - 1). \end{gathered} \tag{2.3}$$

Figure 2 shows some numerical examples for the above equation with several values of b and d.

Lemma 2.4. *If $b \leq 1$ in* (2.3)*, the prey $X(t)$ and predator populations $Y(t)$ die out.*

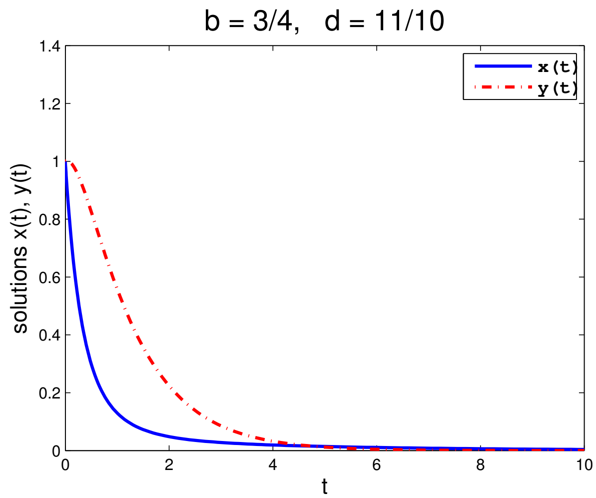

FIGURE 1. $b < 1$ and both species extinguish

The proof runs in the same fashion as in Lemma 2.3. As a consequence of the previous result we will assume in the sequel that $b > 1$.

System (2.3) has three equilibrium points given by

$$E_0 = (0,0), \quad E_1 = \big(\frac{b-1}{b}, 0\big), \quad E_2 = \big(\frac{1}{d}, \frac{d(b-1)-b}{d(b+d)}\big).$$

The stability properties of these points are summarized as follows.

Theorem 2.5. (i) *If $b < 1$, then E_0 is the unique equilibrium point of* (2.3) *in the positive cone and is globally asymptotically stable.*

(ii) *If $b = 1$, the point E_0 undergoes a node-saddle bifurcation. And for $b > 1$ the equilibrium point E_1 shows up, and it is globally asymptotically stable for $1 < b < \frac{d}{d-1}$.*

(iii) *If $b = \frac{d}{d-1}$, the point E_1 undergoes a node-saddle bifurcation. And for $b = \frac{d}{d-1}$ the equilibrium point E_2 shows up, and it is globally asymptotically stable for*

$$b > \frac{d}{d-1}. \tag{2.4}$$

Proof. Parts (i) and (ii) follow immediately from a linear analysis of the equilibria solutions E_0 and E_1. To show (iii) we use Dulac's Criterion [4]. Let $f_1(X,Y)$ and $f_2(X,Y)$ be the corresponding functions in the right hand side of (2.3) for $X'(t)$ and $Y'(t)$ respectively. In our case we look for a function of the form $h(X,Y) = X^\alpha Y^\delta$ such that the expression $\frac{\partial h f_1}{\partial X} + \frac{\partial h f_2}{\partial Y}$ is not zero and does not change its sign while $X > 0$ and $Y > 0$. In doing so, we see that

$$\begin{aligned}
&\frac{\partial(hf_1)(X,Y)}{\partial X} + \frac{\partial(hf_2)(X,Y)}{\partial Y} \\
&= [(\alpha+1)(b-1) - (\delta+1)]X^\alpha Y^\delta + [(\delta+1)d - b(\alpha+2)]X^{\alpha+1}Y^\delta \\
&\quad - (b+d)(\alpha+1)X^\alpha Y^{\delta+1}.
\end{aligned} \tag{2.5}$$

Therefore, while $X > 0$ and $Y > 0$, (2.5) will be negative if we can find values of α and δ such that

$$(\alpha + 1) - \frac{\delta + 1}{b - 1} \le 0,$$
$$(\alpha + 2) - \frac{d}{b}(\delta + 1) \ge 0.$$

But it is an easy task to guarantee the existence of such a values α^* and δ^* for which the previous inequalities hold, and therefore the function $h(X, Y) = X^{\alpha^*} Y^{\delta^*}$ satisfies the conditions we were looking for. Hence by applying the Dulac's Criterion to this function we conclude that (2.3) has no periodic orbits and the Poincaré-Bendixson theory implies that equilibrium point E_2 is globally asymptotically stable. □

3. A simple food chain with delay

Now we considered the delayed model given by (1.5). If we take $Z(t)$ in (1.2) as a change of variable, then following system shows up.

$$\begin{aligned} S'(t) &= 1 - S(t) - bS(t)X(t), \\ X'(t) &= X(t)(bS(t) - 1 - dY(t)), \\ Y'(t) &= Z(t) - Y(t), \\ Z'(t) &= \alpha d X(t)Y(t) - (\alpha + 1)Z(t), \end{aligned} \tag{3.1}$$

with $S(0) = S_0 \ge 0$, $X(0) = X_0 \ge 0$, $Y(0) = Y_0 \ge 0$, $Z(0) = \varphi(0) \ge 0$. The relations between the solutions of this system and those of (1.5) are as in the corresponding description given in [2]. The properties of positiveness and pointwise dissipativeness hold as in the non-delay model and consequently similar results can be stated and proved.

Lemma 3.1. *The positive cone, $\mathbb{R}^4_+$, is positively invariant with respect to* (3.1).

Now we set $U(t) = 1 - S(t) - X(t) - Y(t) - \frac{Z(t)}{\alpha}$, and obtain single equation,

$$U'(t) = -U(t)$$

with $U(0) > 0$. From here, $\lim_{t\to\infty} U(t) = 0$ and the convergence is exponential, and again as before (3.1) has the property of pointwise dissipativity.

Lemma 3.2 (Dissipativity)**.** *The system* (3.1) *is pointwise dissipative. Moreover the attractors of the system are located on the manifold*

$$\Lambda = \{(S, X, Y, Z) \in \mathbb{R}^4_+ : S + X + Y + \frac{Z}{\alpha} = 1\}. \tag{3.2}$$

The pointwise dissipative property implies the existence of a unique global attractor of (3.1) which must lie in the manifold Λ.

As before the system can be simplified by the elimination of one variable. In this case we take

$$S(t) = 1 - X(t) - Y(t) - \frac{Z(t)}{\alpha}$$

and obtain a system of three ordinary differential equations,

$$\begin{gathered} X'(t) = (b-1)X(t) - bX^2(t) - (b+d)X(t)Y(t) - \frac{b}{\alpha}X(t)Z(t), \\ Y'(t) = Z(t) - Y(t), \\ Z'(t) = \alpha d X(t)Y(t) - (\alpha+1)Z(t). \end{gathered} \tag{3.3}$$

See illustration in Figure 3.

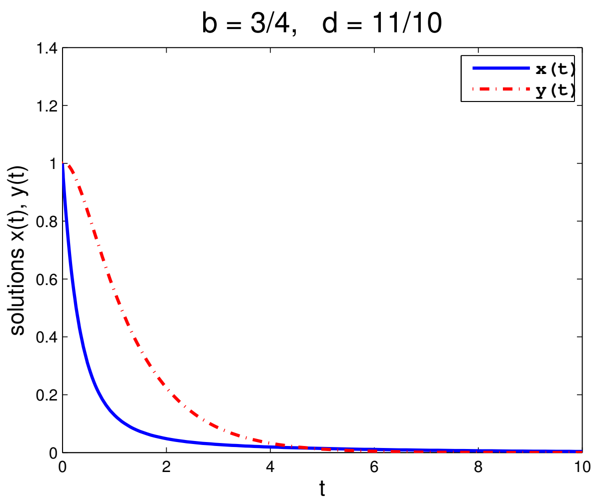

FIGURE 2. $b < 1$ and all species extinguish

This system has three equilibrium points:

$$P_0 = (0,0,0), \quad P_1 = (\frac{b-1}{b}, 0, 0), \quad P_2 = (X^*, Y^*, Y^*),$$

where

$$X^* = \frac{\alpha+1}{\alpha d}, \quad Y^* = \frac{\alpha d(b-1) - b(\alpha+1)}{d(b(\alpha+1)+\alpha d)}.$$

The expression for P_2 makes sense only when $\alpha d(b-1) - b(\alpha+1) > 0$, inequality that is equivalent to

$$X^* < \frac{b-1}{b}\,. \tag{3.4}$$

For $b > 1$, the right hand side of 3.4 implies $X^* < 1$. The stability properties of these equilibrium points are summarized as follows.

Theorem 3.3. (i) *If $b < 1$, then P_0 is the unique equilibrium point of* (3.3) *in the positive cone and it is globally asymptotically stable.*

(ii) *If $b = 1$, P_0 undergoes a node-saddle bifurcation. For $b > 1$, P_1 appears and it is globally asymptotically stable for $b(\alpha+1) - \alpha d(b-1) \geq 0$, or equivalently*

$$1 < d \leq \frac{(\alpha+1)b}{\alpha(b-1)}.$$

(iii) *If* $d = \frac{(\alpha+1)b}{\alpha(b-1)}$, E_1 *undergoes a node-saddle bifurcation. For*

$$d > \frac{(\alpha+1)b}{\alpha(b-1)} \tag{3.5}$$

the point P_2 *appears and has positive coordinates.*

The proof of the above theorem follows from a linear analysis of the equilibria solutions. Stability properties of the equilibrium point E_2 are given in the following result.

Theorem 3.4. *There exists a value* d^* *satisfying* (3.5) *and if* $d < d^*$, E_2 *is locally asymptotically stable. If* $d = d^*$ *then* E_2 *undergoes a supercritical Hopf bifurcation.*

Proof. The Jacobian matrix of (3.3) at the equilibrium E_2 is given by

$$A = \begin{bmatrix} -bX^* & -(b+d)X^* & -\frac{b}{\alpha}X^* \\ 0 & -1 & 1 \\ \alpha dY^* & \alpha+1 & -(\alpha+1) \end{bmatrix}$$

and the corresponding characteristic polynomial is

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{3.6}$$

where

$$a_2 = (\alpha + 2 + bX^*), \quad a_1 = bX^*(\alpha + 2 + dY^*), \quad a_0 = (\alpha+1)(b-1-bX^*).$$

To apply the Routh-Hurwitz Criterion we note that a_0, a_1, a_2 are positive and check the sign of

$$\Phi = a_2a_1 - a_0.$$

But the sign of Φ is the same of Ψ, where

$$\Psi(d) = \frac{\alpha^2d^2(1+bX^*)}{X^*}(a_2a_1 - a_0)\,.$$

The above expression can be written as

$$\Psi(d) = c_3d^3 + c_2d^2 + c_1d + d_0,$$

where

$$\begin{gathered} c_3 = -\alpha^3(b-1), \\ c_2 = \alpha^2b((\alpha+2)b + (\alpha+1)(\alpha+3)), \\ c_1 = \alpha(\alpha+1)b^2(b + (\alpha+1)(\alpha+3)), \\ c_0 = (\alpha+1)^3b^3. \end{gathered}$$

Here are some properties of Ψ and Ψ':

$$\begin{gathered} \lim_{d\to-\infty}\Psi(d) = +\infty, \quad \Psi(0) > 0, \quad \lim_{d\to+\infty}\Psi(d) = -\infty, \\ \lim_{d\to-\infty}\Psi'(d) = -\infty, \quad \Psi'(0) > 0, \quad \lim_{d\to+\infty}\Psi'(d) = -\infty\,. \end{gathered}$$

Therefore, there exists a unique value d_1 such that $\Psi'(d) > 0$ for $0 < d < d_1$, $\Psi'(d_1) = 0$ and $\Psi'(d) < 0$ for $d > d_1$. This means that $\Psi(d)$ increases for $0 \le d < d_1$ and decreases for $d > d_1$. But the we can guarantee the existence of a unique value

$d^* > d_1$, such that $\Psi(d) > 0$ for $0 < d < d^*$, $\Psi(d^*) = 0$, and $\Psi(d) < 0$ for $d > d^*$. Moreover, since $c_2 > 2\alpha^2(\alpha+1)$,

$$d^* > d_1 > \frac{(\alpha+1)b}{\alpha(b-1)} > 1,$$

so Routh-Hurwitz implies that E_2 is locally asymptotically stable for $d < d^*$ and unstable for $d > d^*$. For $d = d^*$ the equilibrium undergoes a Hopf bifurcation. In fact,

$$\lambda^3 + a_2(d^*)\lambda^2 + a_1(d^*)\lambda + a_0(d^*) = (\lambda^2 + a_1(d^*))(\lambda + a_2(d^*)).$$

To check the Hopf bifurcation, we set $\lambda_1(d^*)$ as the root of (3.6) that assume the value $i\omega$, $\omega^2 = a_1(d^*)$, at d^* and by

$$F(\lambda, d) = \lambda^3 + a_2(d)\lambda^2 + a_1(d)\lambda + a_0(d)$$

the characteristic polynomial (3.6) as a function of d. Thus the derivative of the implicit function λ_1 at d^* is

$$\lambda_1'(d^*) = -\frac{F_d'(i\omega, d^*)}{F_\lambda'(i\omega, d^*)} = -\frac{a_2'(d^*)(i\omega)^2 + a_1'(d^*)(i\omega) + a_0'(d^*)}{3(i\omega)^2 + 2a_2(d^*)(i\omega) + a_1'(d^*)},$$

and

$$\begin{aligned}
(\mathrm{Re}(\lambda_1(d^*))_d' &= \mathrm{Re}((\lambda_1)_d'(d^*)) \\
&= -\frac{(a_1(d^*)a_2(d^*) - a_0(d^*))_d'}{a_1(d^*)(1 + a_2^2(d^*))} \\
&= -\frac{a_2(d^*)b\,[X_d^{*\prime}(\alpha + 2 + dY^*) + X^*(Y^* + dY_d^{*\prime})]}{a_1(d^*)(1 + a_2^2(d^*))},
\end{aligned}$$

after some boring calculations we can see that the quantity between brackets is negative, therefore $(\mathrm{Re}(\lambda_1(d^*))_d' > 0$, and so the transversality condition required for the Hopf bifurcation holds.

To verify the properties of stability of the periodic orbit we need to translate (3.3) and locate its origin at the equilibrium point (X^*, Y^*, Y^*). In this case the new system is

$$\begin{gathered}
x' = -bX^*x - (b+d)X^*y - \frac{b}{\alpha}X^*z - bx^2 - (b+d)xy - \frac{b}{\alpha}xz, \\
y' = -y + z, \\
z' = \alpha dY^*x + \alpha dX^*y - (\alpha+1)z + \alpha dxy.
\end{gathered} \tag{3.7}$$

We perform the following change of variables to transform the above system in a normal form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where T is the 3×3 matrix whose first and second columns are the real and imaginary part of the eigenvector associated with the eigenvalue $\lambda_1(d^*)$ and the third column is the eigenvector associated to the eigenvalue $\lambda_0(d^*) = -a_2(d^*)$. Indeed

$$T = \begin{bmatrix} A & B & C \\ \frac{1}{\omega} & 1 & 1 \\ \frac{1}{\omega} & 0 & D \end{bmatrix}$$

where

$$A=-\frac{bX^*(\omega^2+(\alpha d+b(\alpha+1))X^*)}{\omega^2+bX^*},\quad B=\frac{1}{\alpha}+\frac{\omega(\omega^2+(\alpha d+b(\alpha+1))X^*)}{\omega^2+bX^*},$$
$$C=-\frac{b+d}{b}+(\frac{1}{\alpha}-\frac{\alpha+2+bX^*}{bX^*})(\alpha+1+bX^*),\quad D=-(\alpha+1+bX^*).$$

Therefore, the system (3.7) takes the form

$$\begin{aligned} u' &= \omega v + G_1(u,v,w),\\ v' &= -\omega u + G_2(u,v,w),\\ w' &= -a_2(d^*)w + G_3(u,v,w), \end{aligned} \tag{3.8}$$

with

$$\begin{aligned}\begin{bmatrix} G_1(u,v,w)\\ G_2(u,v,w)\\ G_3(u,v,w)\end{bmatrix} &= T^{-1}\begin{bmatrix} F_1(Au+Bv+Cw,\frac{u}{\omega}+v+w,\frac{u}{\omega}+Dw)\\ F_2(Au+Bv+Cw,\frac{u}{\omega}+v+w,\frac{u}{\omega}+Dw)\\ F_3(Au+Bv+Cw,\frac{u}{\omega}+v+w,\frac{u}{\omega}+Dw)\end{bmatrix}\\ &= \begin{bmatrix} H_1u^2+H_2v^2+H_3w^2+H_4uv+H_5vw+H_6uw\\ 0\\ \alpha dCw^2+L_1vw+L_2uw+\alpha dBv^2+L_3uv+\alpha dA\frac{u^2}{\omega}\end{bmatrix}\end{aligned}$$

and

$$L_1=\alpha d(B+C),\quad L_2=\alpha d(A+\frac{C}{\omega}),\quad L_3=\alpha d(A+\frac{B}{\omega}),$$
$$H_1=-A(bA+\frac{1}{\omega}(\frac{b}{\alpha}+b+d)),\quad H_2=-B(b+d+bB),$$
$$H_3=-C(b+d+bC+\frac{b}{\alpha}D),\quad H_4=-A(b+d+2bB)-\frac{B}{\omega}(b+d+\frac{b}{\alpha}),$$
$$H_5=-(B(b+d+\frac{b}{\alpha}D)+C(b+d+2bB)),$$
$$H_6=-A(b+d+b(2C+\frac{D}{\alpha}))-\frac{C}{\omega}(b+d+\frac{b}{\alpha}).$$

Now we determine approximately the $d=d^*$-section of the center manifold M which is tangent to the (u,v)- plane at the origin. This is $w=h(u,v)$, $h(0,0)=h'_u(0,0)=h'_v(0,0)=0$, and h sufficiently smooth. Then

$$w=h(u,v)=\frac{1}{2}(h_{11}u^2+2h_{12}uv+h_{22}v^2)+o(|\,(u,v)\,|^2). \tag{3.9}$$

Restricting the system to the center manifold, if $(u(t),v(t),w(t))$ is a solution of (3.8) near the origin with a value on M, then it stays locally in M, i.e.,

$$w(t)\equiv h(u(t),v(t)).$$

But then

$$w'(t)-h'_u(u(t),v(t))u'(t)-h'_v(u(t),v(t))v'(t)\equiv 0, \tag{3.10}$$

so by using (3.8), omitting terms of order at least three and equating the coefficients to zero,

$$
\begin{gathered}
h_{11}=2\frac{-2\alpha dB\omega^3-L_3a_2(d^*)\omega^2+2\alpha dA\omega^2+\alpha dAa_2^2(d^*)}{\omega a_2^3(d^*)},\\
h_{12}=\frac{1}{a_2^2(d^*)}(-2\alpha dA+2\alpha dB\omega+a_2(d^*)L_3),\\
h_{22}=2\frac{2\omega\alpha dA-2\alpha dB\omega^2-L_3a_2(d^*)\omega+a_2^2(d^*)\alpha dB}{a_2^3(d^*)}.
\end{gathered}
$$

To restrict (3.8) to the $d=d^*$-section of the center manifold M, we introduce new coordinates, $y_1=u$, $y_2=v$, $y_3=w-h(u,v)$ and 3.8) becomes

$$
\begin{aligned}
y_1' &= \omega y_2+\frac{1}{2}h_{22}H_5y_2^3+(h_{12}H_5+\frac{1}{2}h_{22}H_6)y_1y_2^2+(\frac{1}{2}H_5h_{11}+H_6h_{12})y_1^2y_2\\
&\quad+\frac{1}{2}h_{11}H_6y_1^3+H_1y_1^2+H_2y_2^2+H_4y_1y_2+O(|\ y\ |^4) \qquad (3.11)\\
y_2' &= -\omega y_1.
\end{aligned}
$$

Applying the Bautin's formula [4, Lemma 7.2.7], we can see that

$$
\frac{4\omega}{3\pi}V'''_{y_1y_1y_1}(0,0)=(3h_{11}h_{12}H_5+h_{22})H_6+\frac{2}{\omega}(H_1+H_2)H_4 \qquad (3.12)
$$

$\square$

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